

SOLUTIONS TO THREE PROBLEMATIC PROBLEMS ABOUT RINGS AND MODULES

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About a month ago (see the date below for a meaningless reference) I was asked three questions on a midterm examination. Unfortunately, my solutions ended up being a “jumbled mess”. I am not quite sure why, but that morning these problems were *incredibly* confusing to me. In this short document, I will give solutions to these problems. I am not exactly sure why I am posting this, but chances are it is just to make myself feel better.



Problem 1. Let R be an integral domain, we say that two non-zero elements $a, b \in R$ are coprime if

$$(a) + (b) = R.$$

Suppose that a and b are coprime and let $a \mid bc$. Prove that $a \mid c$.

Solution. Since $(a) + (b) = R$, there exist elements $x, y \in R$ such that

$$1 = ax + by.$$

Multiply both sides of the above by c , to obtain

$$c = x(ac) + y(bc).$$

Since $a \mid bc$, we see from the above that $a \mid x$. □

Problem 2. Let R and S be rings and denote by $R \oplus S$ the Cartesian product $R \times S$ endowed with the “coordinate-wise” operations inherited from the two rings R and S :

$$(r, s) + (r', s') := (r + r', s + s'), \quad (r, s) \cdot (r', s') := (rr', ss').$$

Prove that every left ideal of $R \oplus S$ is of the form $I \times J$ for left ideals $I \triangleleft R$ and $J \triangleleft S$.

Solution. Let $T \triangleleft R \oplus S$ be a non-empty left ideal and define

$$I := \{r \in R : \exists s \in S \text{ such that } (r, s) \in T\}.$$

Clearly, I is closed under addition. Also, if $r \in I$ then $(r, s) \in T$ for some $s \in S$ so that

$$T \ni (r'r, 1 \cdot s) = (r', 1) \cdot (r, s).$$

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This means that $r'r \in I$ and hence I is a left ideal of R . Defining J as

$$J := \{s \in S : \exists r \in R \text{ such that } (r, s) \in T\}.$$

we obtain (by the same argument) a left ideal of S . Obviously, $T \subseteq I \times J$. For the reverse inclusion, let $(r, s) \in I \times J$ be given. For a suitable choice of $r_0 \in R$ and $s_0 \in S$, one has

$$(r, s_0) \in T \quad \text{and} \quad (r_0, s) \in T.$$

Now,

$$(r, s) = (1, 0) \cdot (r, s_0) + (0, 1) \cdot (r_0, s) \in T$$

since T is closed under left multiplication and addition. This proves that $T = I \times J$ as was required. \square

Problem 3. *Let R be an integral domain and suppose that M is a finitely generated free module over R . Prove that M has finite rank.*

Solution. Let $\{x_1, \dots, x_n\}$ be a finite set of generators for M over R and let $\{b_\alpha\}_{\alpha \in I}$ be a basis for M over R . Now, for every index $1 \leq i \leq n$ we may choose a finite set $A_i \subseteq I$ such that

$$x_i = \sum_{j \in A_i} r_j b_j, \quad r_j \in R.$$

Now, the *finite* set

$$\mathcal{F} := \bigcup_{i=1}^n \{b_j : j \in A_i\}$$

therefore spans the module M . As a subset of the original family $\{b_\alpha\}_{\alpha \in I}$, the collection \mathcal{F} is linearly independent. All this together shows that

$$M \cong R^{|\mathcal{F}|}.$$

This completes the proof. \square