Weakly Monotone Decreasing Solutions to Elliptic Schrödinger Systems

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Objectives

We seek to understand the asymptotic behaviour of positive solutions to a system of differential equations arising in physics, particularly in nonlinear optics and the modeling of Bose-Einstein double condensates.

Introduction

We derive decay estimates and non-existence criteria regarding positive super-solutions to an elliptic Schrödinger system. The aim is to study the decay rates of positive measurable functions on $\mathbb{R}^n$ that satisfy the generalized integral system

$$\begin{cases} u(x) \geq \displaystyle\int_{\mathbb{R}^n} \frac{v(y)^q u(y)^r + \Gamma_1(y,u,v)}{|x-y|^{n-\alpha}\,|y|^{\sigma_1}}\,\mathrm{d}y, \\ v(x) \geq \displaystyle\int_{\mathbb{R}^n} \frac{v(y)^s u(y)^p + \Gamma_2(y,u,v)}{|x-y|^{n-\alpha}\,|y|^{\sigma_2}}\,\mathrm{d}y \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $\Gamma_1, \Gamma_2 \geq 0$, $n \geq 3$, $\alpha \in (0,n)$, $\sigma_{1,2} \in (-\infty, \alpha)$, $p, q \geq 0$, $r, s \in [0,1]$ and $pq > (1-r)(1-s)$.

Definition

A function $f : \mathbb{R}^n \to \mathbb{R}^+$ is said to be weakly monotone decreasing (or simply WMD) if there exists $C, R > 0$ such that $f(x) \leq C f(y)$ whenever $|x| \geq |y| \geq R$.

System (1) is closely related (and often equivalent) to the coupled system:

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) \equiv (v(x)^q u(x)^r + \Gamma_1(x,u,v))\,|x|^{-\sigma_1}, \\ (-\Delta)^{\alpha/2} v(x) \equiv (u(x)^p v(x)^s + \Gamma_2(x,u,v))\,|x|^{-\sigma_2} \end{cases}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. WMD functions are a natural generalization of *decay functions*, which are f such that $f \simeq |x|^{-\theta}$ for some $\theta > 0$.

Theorem 1

System (1) admits no positive solutions if $pq = 0$ or

$$\sigma_1 \leq \alpha - (q+r)(n-\alpha) \quad \text{or} \quad \sigma_2 \leq \alpha - (p+s)(n-\alpha).$$

There are no WMD solutions if we assume $pq \leq (1-r)(1-s)$.

Decay Estimates

The following two constants play a fundamental role in our asymptotic analysis of positve solutions:

$$r_0 := \frac{p(\alpha-\sigma_1) + (\alpha-\sigma_2)(1-r)}{pq - (1-s)(1-r)},$$

$$s_0 := \frac{q(\alpha-\sigma_2) + (\alpha-\sigma_1)(1-s)}{pq - (1-s)(1-r)}$$

When $r = s = 0$ and $\sigma_{1,2} \geq 0$ Villavert [1] established a weaker version of the following theorem. The lower and upper-bounds below are known to be optimal in several cases (see [3]).

Theorem 2

Let (u,v) be positive solutions to (1), then

$$u(x) \gtrsim \begin{cases} (1+|x|)^{-\min\{n-\alpha,(q+r)(n-\alpha)-(\alpha-\sigma_1)\}}, & (q+r)(n-\alpha) \neq n-\sigma_1, \\ (1+|x|)^{-(n-\alpha)} \ln(1+|x|), & (q+r)(n-\alpha) = n-\sigma_1 \end{cases}$$

$$v(x) \gtrsim \begin{cases} (1+|x|)^{-\min\{n-\alpha,(p+s)(n-\alpha)-(\alpha-\sigma_2)\}}, & (p+s)(n-\alpha) \neq n-\sigma_2, \\ (1+|x|)^{-(n-\alpha)} \ln(1+|x|), & (p+s)(n-\alpha) = n-\sigma_2 \end{cases}.$$

Suppose, in addition, that u and v are weakly monotone decreasing with $p, q > 0$. Then $u(x) \lesssim |x|^{-s_0}$ and $v(x) \lesssim |x|^{-r_0}$.

Sketch of Proof of Theorem 1

For the first part, one can assume that $q = 0$. It is possible to show that $u(x) \gtrsim |x|^{-b_0}$ for $b_0 := n - \alpha > 0$. By restricting the domain of integration in (1) to a set of the form

$$B_x := \left\{ y \in \mathbb{R}^n : \frac{|x|}{2} < |y| < |x| \right\} \tag{2}$$

one can show that $u(x) \gtrsim |x|^{-b_k}$ where

$$b_k := r b_{k-1} + \sigma_1 - \alpha, \quad \forall k \in \mathbb{N}.$$

A direct calculation verifies that for large k one has $b_k < 0$. For any such k one can use $u(x) \gtrsim |x|^{-b_k}$ to show that for all $R > 0$ large and $|x| < R$:

$$u(x) \geq c \int_{|y| \geq R} |y|^{-rb_k + \alpha - n - \sigma_1}\,\mathrm{d}y = \infty$$

which is a contradiction. Similarly, if $\sigma_1 \leq \alpha - (q+r)(n-\alpha)$, it can be shown that $u = \infty$, and likewise, $v = \infty$. The same recursive argument can be applied to both u and v to show that no positive solutions exist if $pq \leq (1-r)(1-s)$.

Sketch of Proof of Theorem 2

Assume u and v are WMD. For x large, we integrate over the domain in equation (2) to find that for suitable $C > 0$:

$$u(x) \geq C\,|x|^{\alpha-n-\sigma_1} \int_{B_x} u(y)^r v(y)^q\,\mathrm{d}y \geq C' u(x)^r v(x)^q\,|x|^{\alpha-\sigma_1}$$

where in the last inequality we have used that both u and v are WMD. Similarly, $v(x) \gtrsim u(x)^p v(x)^s\,|x|^{\alpha-\sigma_2}$. These inequalities combined can be used to show that the upper-bounds hold for u and v.

To prove the lower-bounds, one can first show that for $R > 0$ large and $|x| \geq 2R$:

$$u(x) \geq (R+|x|)^{\alpha-n} \int_{\{R-1 \leq |y| \leq R\}} \frac{u(y)^r v(y)^q}{|y|^{\sigma_1}}\,\mathrm{d}y$$

which yields $u(x) \geq C(1+|x|)^{\alpha-n}$ as $|x| \to \infty$. An identical estimate holds for v in all cases. Using these two estimates and integrating over a sequence of annuli, one can establish the lower bounds.

References

[1] Villavert, John. Qualitative propertie of solutions for an integral system related to the Hardy-Sobolev inequality. J. Differential Equations 258 (2015) no. 5, 1685-1714.

[2] Vétois, Jérôme. Decay Estimates and Symmetry of Finite Energy Solutions to Elliptic Systems in $\mathbb{R}^n$. Indiana Univ. Math. J. To appear (2017).

[3] Y. Li, Asymptotic behavior of positive solutions of equation $\Delta u + K(x)u^p = 0$ in $\mathbb{R}^n$. J. Differential Equations 95 (1992) no. 2, 304-330.