True Online Temporal-Difference Learning
(and Dutch Traces)

Harm van Seijen
Research Scientist, Maluuba

Maluuba
A Microsoft company

joint work with

Rupam Mahmood

Patrick Pilarski

Marlos Machado

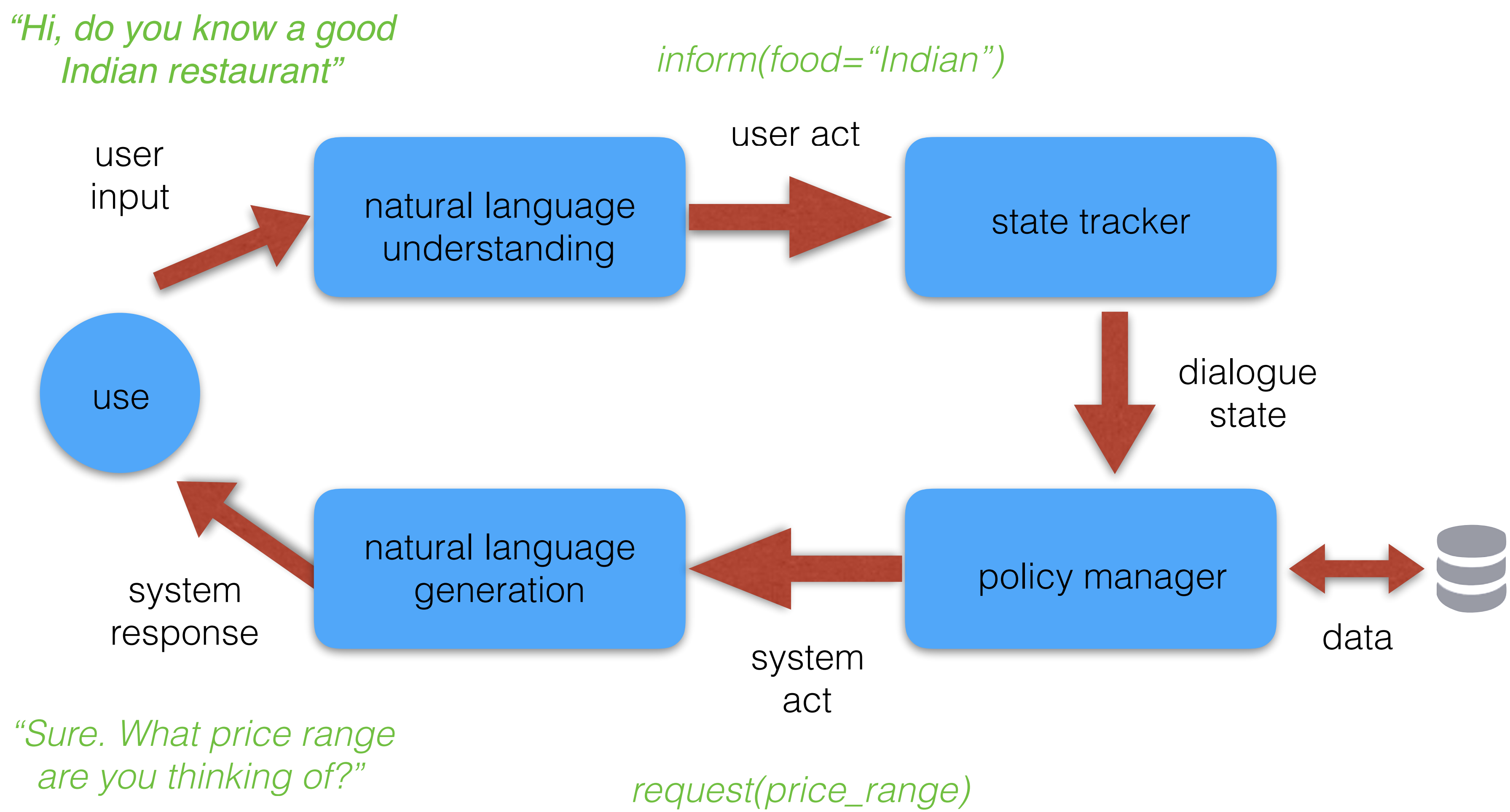
Rich Sutton

UNIVERSITY OF ALBERTA

Outline

- part 1: why reinforcement learning?
- part 2: true online temporal-difference learning
- part 3: effective multi-step learning for non-linear FA

motivating example for RL



The central question: how to train the policy manager?

what is RL

Reinforcement Learning is a data-driven approach towards learning behaviour.

machine learning

unsupervised learning	supervised learning	reinforcement learning
+	+	+
deep learning	deep learning	deep learning
		=
		deep reinforcement learning

RL vs supervised learning

behaviour: function that maps environment states to actions

supervised learning

- hard to specify function
- easy to identify correct output

example: recognizing cats in images

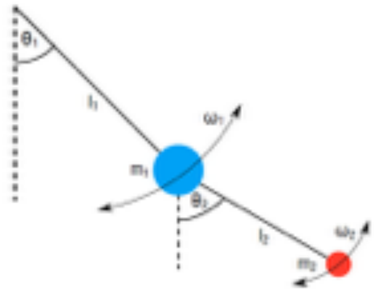
RL vs supervised learning

behaviour: function that maps environment states to actions

reinforcement learning:

- hard to specify function
- hard to identify correct output
- easy to specify behaviour goal

example: double inverted pendulum



state: θ1, θ2, ω1, ω2
action: clockwise/counter-clockwise torque on top joint
goal: balance pendulum upright

advantages RL

- does not require knowledge of good policy
- does not require labelled data
- online learning: adaptation to environment changes

challenges RL

- requires lots of data
- sample distribution changes during learning
- samples are not i.i.d.

Outline

- part 1: why reinforcement learning?
- **part 2: true online temporal-difference learning**
- part 3: effective multi-step learning for non-linear FA

Markov Decision Processes

A *Markov decision process* (MDP) can be described by 5-tuple: $\langle \mathcal{S}, \mathcal{A}, p, r, \gamma \rangle$, with

- $\mathcal{S}$: the set of all states
- $\mathcal{A}$: the set of all actions
- $p(s'|s,a)$: the transition probability function
- $r(s,a,s')$: the reward function
- γ: the discount factor

policy $\pi : \mathcal{S} \times \mathcal{A} \to [0,1]$, function giving the selection probability for each action conditioned on the state

The *return* at time t: $G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \ldots = \sum_{i=1}^{\infty} \gamma^{i-1} R_{t+i}$

state-value function: $v^{\pi}(s) = \mathbb{E}\{G_t \mid S_t = s, \pi\}$

action-value function: $q^{\pi}(s,a) = \mathbb{E}\{G_t \mid S_t = s, A_t, = a, \pi\}$

Estimating the value function

Find a weight vector $\boldsymbol{\theta} \in \mathbb{R}^n$ such that $\hat{V}(s|\boldsymbol{\theta})$ accurately approximates $v_\pi(s)$ for relevant states s.

Error function: $$E(\boldsymbol{\theta}) := \frac{1}{2}\sum_i d_\pi(s_i)\left[v_\pi(s_i) - \hat{V}(s_i|\boldsymbol{\theta})\right]^2$$

where d_π is the stationary distribution induced by π.

Stochastic gradient descent (sampling from the stationary distribution):

$$\begin{aligned}\boldsymbol{\theta}_{t+1} &= \boldsymbol{\theta}_t - \alpha\frac{1}{2}\nabla_\theta\left[v_\pi(S_t) - \hat{V}(S_t|\boldsymbol{\theta}_t)\right]^2 \\ &= \boldsymbol{\theta}_t + \alpha\Big(v_\pi(S_t) - \hat{V}(S_t|\boldsymbol{\theta}_t)\Big)\nabla_\theta\hat{V}(S_t|\boldsymbol{\theta}_t)\end{aligned}$$

Because $v_\pi(S_t)$ is unknown:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \alpha\Big(U_t - \hat{V}(S_t|\boldsymbol{\theta}_t)\Big)\nabla_\theta\hat{V}(S_t|\boldsymbol{\theta}_t).$$

where U_t is an estimate of $v_\pi(S_t)$ that we will call the update target.

With linear function approximation: $\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \alpha\big(U_t - \boldsymbol{\theta}_t^\top\boldsymbol{\phi}_t\big)\boldsymbol{\phi}_t$

unbiased estimate of $v_\pi(S_t)$: $U_t = G_t$

Temporal-difference Learning

- Temporal-Difference (TD) learning exploits knowledge about structure of v_π .
- Bellman Equation:

$$v_\pi(s) = \sum_a \pi(s,a) \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma v_\pi(s')\right] \qquad \text{for all } s$$

$$v_\pi(s) = \mathbb{E}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s, A_t \sim \pi(S_t, \cdot)\right]$$

- TD(0) update target (1-step update target):

$$U_t = R_{t+1} + \gamma\, \boldsymbol{\theta}^\top \boldsymbol{\phi}_{t+1}$$

- 3-step update target:

$$U_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 \boldsymbol{\theta}^\top \boldsymbol{\phi}_{t+3}$$

TD(λ)

- update equations for linear function approximation:

$$\begin{aligned}
\delta_t &= R_{t+1} + \gamma \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_{t+1} - \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t \,, \\
\boldsymbol{e}_t &= \gamma \lambda \boldsymbol{e}_{t-1} + \boldsymbol{\phi}_t \,, \\
\boldsymbol{\theta}_{t+1} &= \boldsymbol{\theta}_t + \alpha \delta_t \, \boldsymbol{e}_t \,,
\end{aligned}$$

- TD(λ) is a multi-step method, even though the update target looks like a 1-step update target.

- This update is different from the general TD update rule.

the traditional forward view of TD(λ)

- the λ-return algorithm:

$$\boldsymbol{\theta}_{t+1} = \boldsymbol{\theta}_t + \alpha\left(G_t^\lambda - \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t\right)\boldsymbol{\phi}_t$$

where G_t^λ is the λ-return, defined as:

$$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

with $G_t^{(n)} = \sum_{k=1}^{n} \gamma^{k-1} R_{t+k} + \gamma^n \boldsymbol{\theta}^\top \boldsymbol{\phi}_{t+n}$

note:

$\lambda = 0 \quad : \quad G_t^\lambda = G_t^{(1)}$

$\lambda = 1 \quad : \quad G_t^\lambda = G_t$

How to set λ?

- λ controls a trade-off between variance and bias of the update target, in general the best value of λ will differ from domain to domain.

- λ not only influences the speed of convergence, but in case of function approximation it also influences the asymptotic performance.

- theoretical results for TD(λ) (Peter Dayan, 1992):
  - for $\lambda = 1$: convergence to LMS solution
  - for $\lambda < 1$: convergence to a different fixed point

online vs offline methods

- **online method:** the value of each visited state is updated at the time step immediately after the visit.
- **offline method:** the value of each visited state is updated at the end of an episode.

- TD(λ) is an online method; the traditional λ-return algorithm is an offline method.

*Is it possible to construct an online version of the λ-return algorithm that approximates TD(λ) at **all** time steps?*

the challenge of an online forward view

- To compute θ_t , no data beyond time t should be used.
- At the same time, we want to have multi-step update targets that look many time steps ahead.

the trick:

Use update targets that grow with the data-horizon.

interim update target

- normal update targets:

$$\phi_t \quad \rightarrow \quad U_t$$

- interim update targets:

$$\phi_t \quad \rightarrow \quad U_t^1,\ U_t^2,\ U_t^3, \ldots,\ U_t^h$$

data-horizon: time step up to which data is observed

interim λ-return

- λ-return: $$G_t^\lambda = (1-\lambda)\sum_{n=1}^{\infty} \lambda^{n-1} G_t^{(n)}$$

- interim λ-return: replace all *n*-step returns with $n > h\text{-}t$ with the $(h\text{-}t)$-step return

$$\begin{aligned}
G_t^{\lambda|h} &= (1-\lambda)\sum_{n=1}^{h-t-1} \lambda^{n-1} G_t^{(n)} + (1-\lambda)\sum_{n=h-t}^{\infty} \lambda^{n-1} G_t^{(h-t)} \\
&= (1-\lambda)\sum_{n=1}^{h-t-1} \lambda^{n-1} G_t^{(n)} + G_t^{(h-t)} \cdot \left[(1-\lambda)\sum_{n=h-t}^{\infty} \lambda^{n-1}\right] \\
&= (1-\lambda)\sum_{n=1}^{h-t-1} \lambda^{n-1} G_t^{(n)} + G_t^{(h-t)} \cdot \left[\lambda^{h-t-1}(1-\lambda)\sum_{k=0}^{\infty} \lambda^{k}\right] \\
&= (1-\lambda)\sum_{n=1}^{h-t-1} \lambda^{n-1} G_t^{(n)} + \lambda^{h-t-1} G_t^{(h-t)}
\end{aligned}$$

update sequences

$$t = 1: \quad \boldsymbol{\theta}_1^1 = \boldsymbol{\theta}_0^1 + \alpha\big(G_0^{\lambda|1} - (\boldsymbol{\theta}_0^1)^\top \phi_0\big)\phi_0$$

$$\begin{aligned} t = 2: \quad \boldsymbol{\theta}_1^2 &= \boldsymbol{\theta}_0^2 + \alpha\big(G_0^{\lambda|2} - (\boldsymbol{\theta}_0^2)^\top \phi_0\big)\phi_0 \\ \boldsymbol{\theta}_2^2 &= \boldsymbol{\theta}_1^2 + \alpha\big(G_1^{\lambda|2} - (\boldsymbol{\theta}_1^2)^\top \phi_1\big)\phi_1 \end{aligned}$$

$$\begin{aligned} t = 3: \quad \boldsymbol{\theta}_1^3 &= \boldsymbol{\theta}_0^3 + \alpha\big(G_0^{\lambda|3} - (\boldsymbol{\theta}_0^3)^\top \phi_0\big)\phi_0 \\ \boldsymbol{\theta}_2^3 &= \boldsymbol{\theta}_1^3 + \alpha\big(G_1^{\lambda|3} - (\boldsymbol{\theta}_1^3)^\top \phi_1\big)\phi_1 \\ \boldsymbol{\theta}_3^3 &= \boldsymbol{\theta}_2^3 + \alpha\big(G_2^{\lambda|3} - (\boldsymbol{\theta}_2^3)^\top \phi_2\big)\phi_2 \end{aligned}$$

with $\boldsymbol{\theta}_0^t := \boldsymbol{\theta}_{init}$

online lambda-return algorithm.

$$\boldsymbol{\theta}_t := \boldsymbol{\theta}_t^t$$

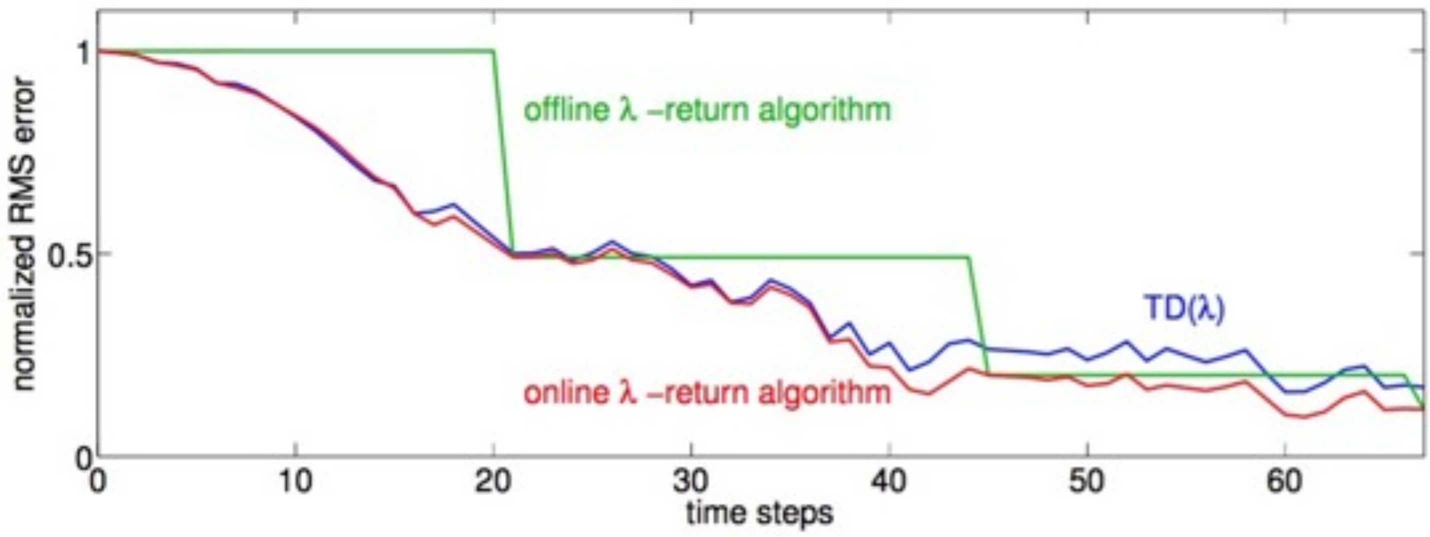
$$\boldsymbol{\theta}_{k+1}^t := \boldsymbol{\theta}_k^t + \alpha\Big(G_k^{\lambda|t} - (\boldsymbol{\theta}_k^t)^\top \boldsymbol{\phi}_k\Big)\boldsymbol{\phi}_k\,, \qquad \text{for } 0 \le k < t$$

with

$$G_k^{\lambda|t} := (1-\lambda)\sum_{n=1}^{t-k-1} \lambda^{n-1} G_k^{(n)} + \lambda^{t-k-1} G_k^{(t-k)}$$

online vs offline λ-return algorithm

- performance on a 10-state random walk task for the first 3 episodes ($\lambda = 1, \alpha = 0.2$)



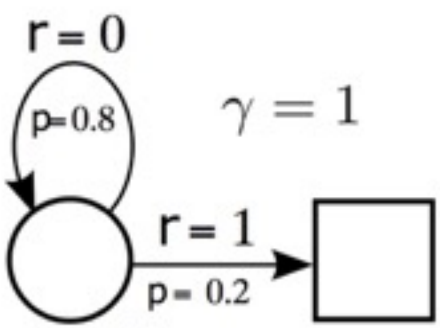
Theorem*

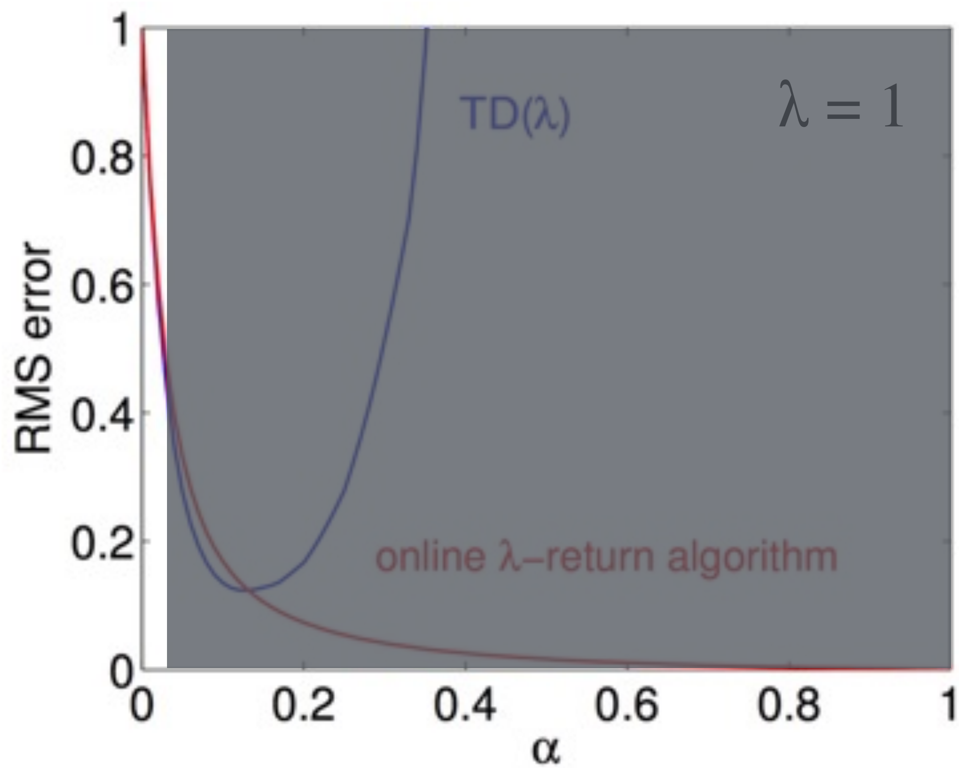
> "For small step-size, the online λ-return algorithm behaves like TD(λ) at all time steps"

*see Theorem 1: *van Seijen, H., Mahmood, A. R., Pilarski, P. M., Machado, M. C., and Sutton, R. S. True online temporal-difference learning. Journal of Machine Learning Research, 17(145):1–40, 2016.*

Sensitivity of TD(λ) to Divergence

RMS error during early learning





Computational Complexity

$$h = 1: \quad \boldsymbol{\theta}_1^1 = \boldsymbol{\theta}_0^1 + \alpha\left[G_0^{\lambda|1} - (\boldsymbol{\theta}_0^1)^\top \boldsymbol{\phi}_0\right]\boldsymbol{\phi}_0$$

$$h = 2: \quad \boldsymbol{\theta}_1^2 = \boldsymbol{\theta}_0^2 + \alpha\left[G_0^{\lambda|2} - (\boldsymbol{\theta}_0^2)^\top \boldsymbol{\phi}_0\right]\boldsymbol{\phi}_0$$

$$\boldsymbol{\theta}_2^2 = \boldsymbol{\theta}_1^2 + \alpha\left[G_1^{\lambda|2} - (\boldsymbol{\theta}_1^2)^\top \boldsymbol{\phi}_1\right]\boldsymbol{\phi}_1$$

$$h = 3: \quad \boldsymbol{\theta}_1^3 = \boldsymbol{\theta}_0^3 + \alpha\left[G_0^{\lambda|3} - (\boldsymbol{\theta}_0^3)^\top \boldsymbol{\phi}_0\right]\boldsymbol{\phi}_0$$

$$\boldsymbol{\theta}_2^3 = \boldsymbol{\theta}_1^3 + \alpha\left[G_1^{\lambda|3} - (\boldsymbol{\theta}_1^3)^\top \boldsymbol{\phi}_1\right]\boldsymbol{\phi}_1$$

$$\boldsymbol{\theta}_3^3 = \boldsymbol{\theta}_2^3 + \alpha\left[G_2^{\lambda|3} - (\boldsymbol{\theta}_2^3)^\top \boldsymbol{\phi}_2\right]\boldsymbol{\phi}_2$$

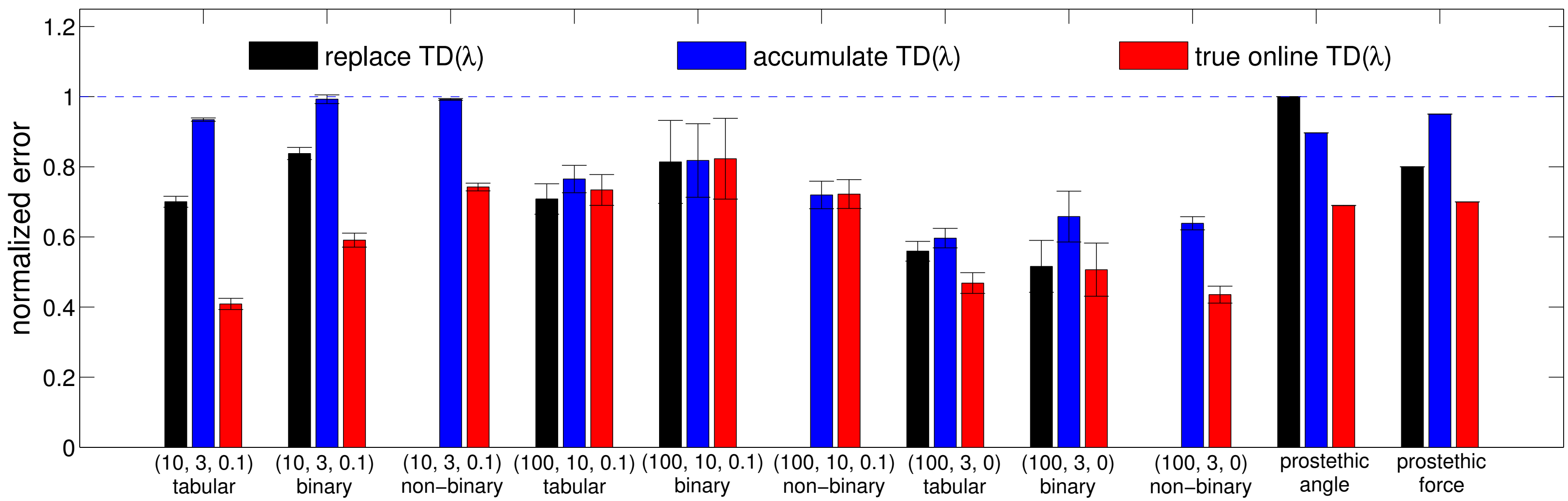
True online TD(λ)

- true online TD(λ) is an efficient implementation of the online λ-return algorithm

dutch trace

$$
\begin{aligned}
\delta_t &= R_{t+1} + \gamma \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_{t+1} - \boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t \\
\mathbf{e}_t &= \gamma\lambda \mathbf{e}_{t-1} + \boldsymbol{\phi}_t - \alpha\gamma\lambda[\mathbf{e}_{t-1}^\top \boldsymbol{\phi}_t]\, \boldsymbol{\phi}_t \\
\boldsymbol{\theta}_{t+1} &= \boldsymbol{\theta}_t + \alpha\delta_t\, \mathbf{e}_t + \alpha[\boldsymbol{\theta}_t^\top \boldsymbol{\phi}_t - \boldsymbol{\theta}_{t-1}^\top \boldsymbol{\phi}_t][\mathbf{e}_t - \boldsymbol{\phi}_t]
\end{aligned}
$$

Empirical Comparison



- in all domains, true online TD(λ) performs at least as good as replace/accumulate TD(λ)

Outline

- part 1: why reinforcement learning?
- part 2: true online temporal-difference learning
- **part 3: effective multi-step learning for non-linear FA**

Computational Cost

- Implementing the online forward view is computationally very expensive.
  - Memory as well as computation time per time step grows over time.

- In the case of linear FA there is an efficient backward view with exact equivalence: true online TD(λ).
  - Computational cost is span-independent and linear in the number of features.

- In the case of non-linear FA such an efficient backward view does not appear to exist.

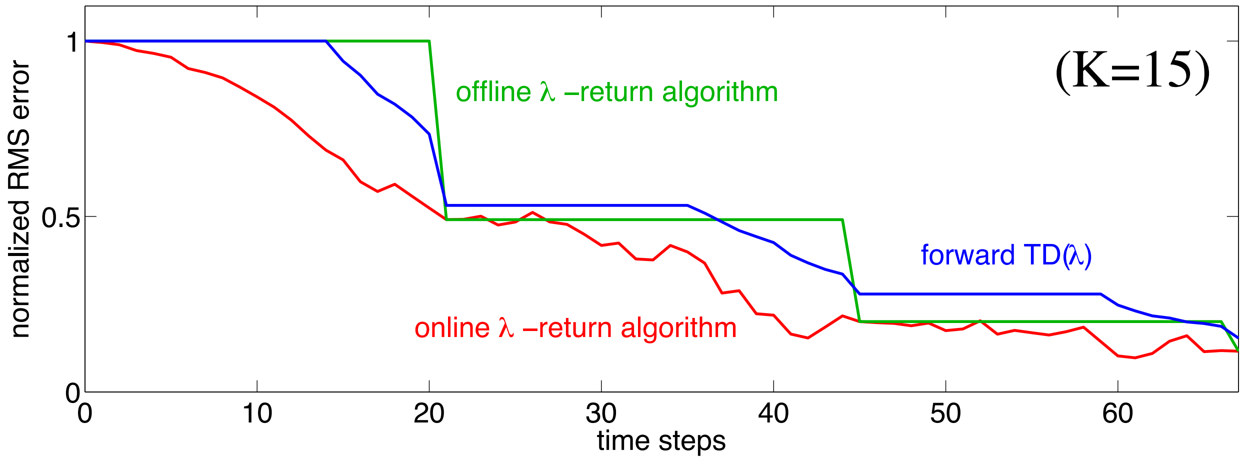
New Research Question

Is it possible to construct a different online forward view, with a performance close to that of the online λ-return algorithm, that can be implemented efficiently?

Answer: Yes

forward TD(λ)

- Uses online λ-return with fixed horizon, K steps ahead: $G_t^{\lambda|t+K}$
- As a consequence, updates occur with a delay of K time steps.
- Computational cost is span-independent and efficient (computational complexity equal to TD(0)).

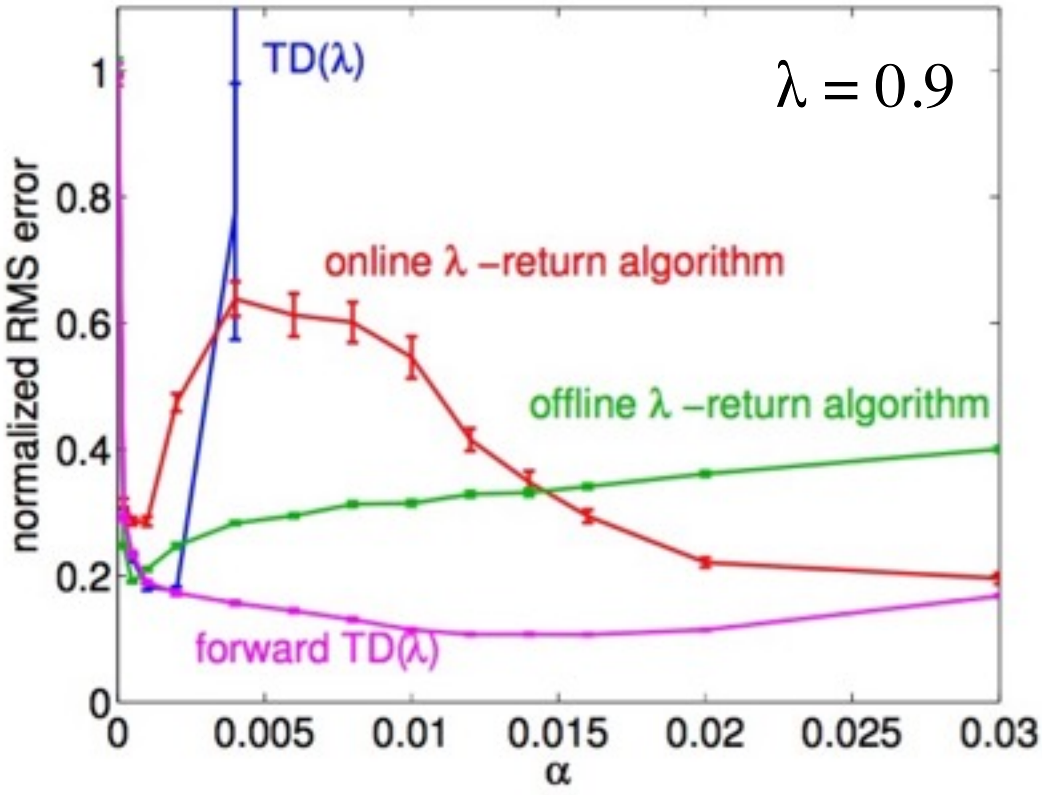


How to set K?

- Setting K involves a trade-off:
  - small K : less delay in updates
  - large K : better approximation of the λ-return
- How well $G_t^{\lambda|t+K}$ approximates G_t^{λ} depends on K, but also on $\gamma\lambda$.
- Whereas the weight of R_{t+1} in G_t^{λ} is 1, the weight of R_{t+n} is only $\gamma\lambda^{n-1}$.
  - Example: $\gamma\lambda = 0.5$ and $n = 20$, then $\gamma\lambda^{n-1}$ is about 10^{-6}.
- Strategy: set K such that $\gamma\lambda^{K-1}$ is just below η, with η some tiny number like 0.01

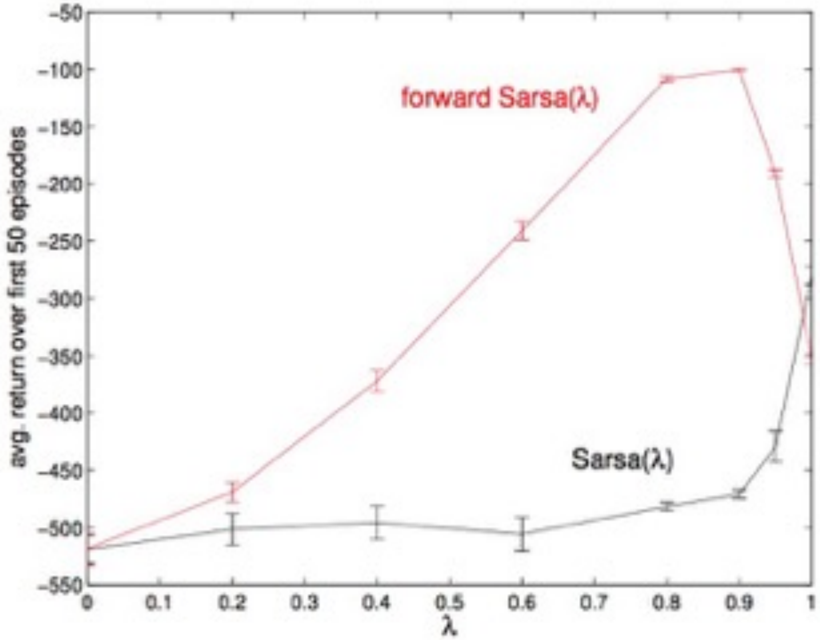
Results on Prediction Task

mountain-car task with non-linear FA

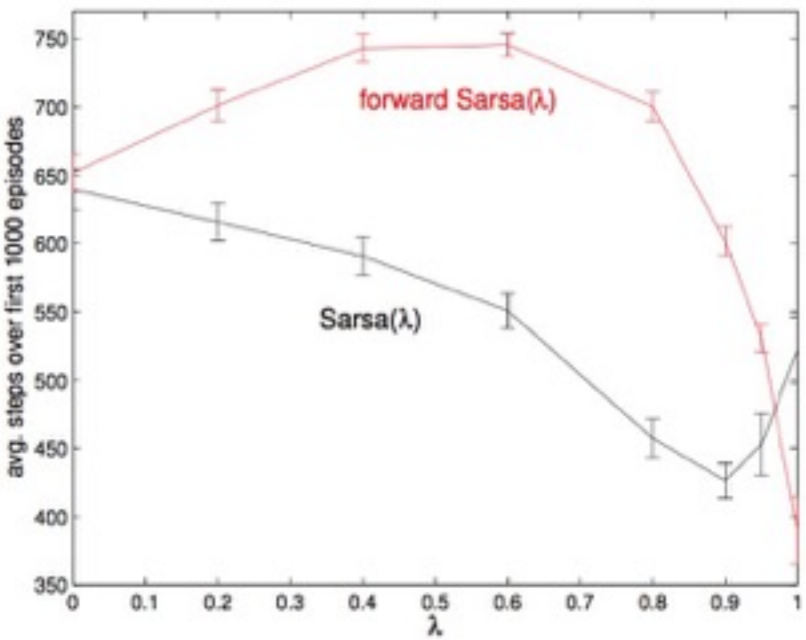


Results on 2 Control Tasks

mountain-car task



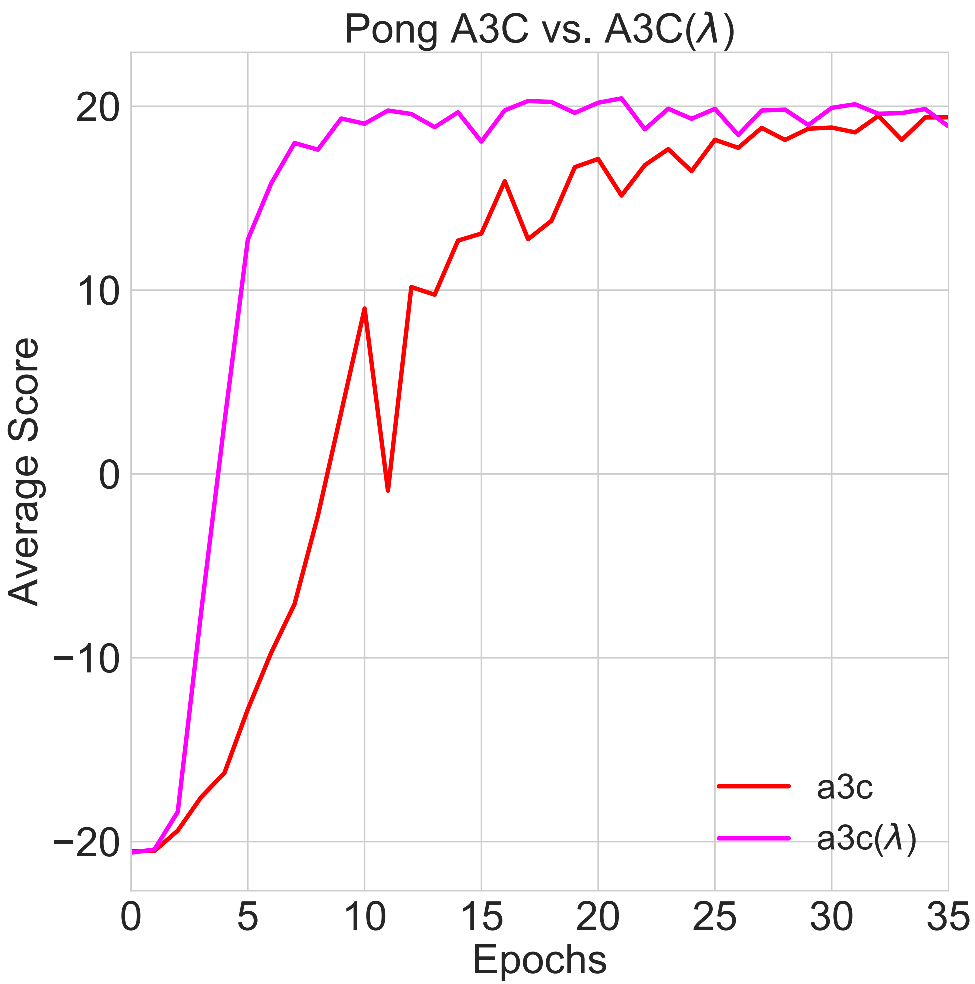
cart-pole task



What about deep RL?

Question: can this technique be applied to DQN?

Results on Atari Pong



Summary

1. The online λ-return algorithm outperforms TD(λ), but is computationally very expensive.
2. For linear FA, an efficient backward view exists with exact equivalence: true online TD(λ).
3. For non-linear FA, such an efficient backward view does not appear to exist.
4. Forward TD(λ) approximates the online λ-return algorithm and can be implemented efficiently for non-linear FA.
5. The price that forward TD(λ) pays is a delay in the updates.
6. Empirically, forward TD(λ) can outperform TD(λ) substantially on domains with non-linear FA.
7. The forward TD(λ) strategy does not work well with experience replay with long histories, but it can be applied to A3C.

Thank you!

References:
1) *van Seijen, H., Mahmood, A. R., Pilarski, P. M., Machado, M. C., and Sutton, R. S. True online temporal-difference learning. Journal of Machine Learning Research, 17(145):1–40, 2016.*
2) *van Seijen, H. Effective multi-step temporal-difference learning for non-linear function approximation. arXiv:1608.05151, 2016.*