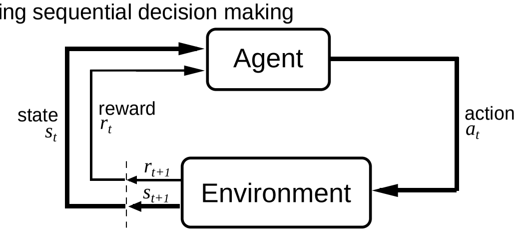
Lecture 15: Markov Decision Processes (MDPs)

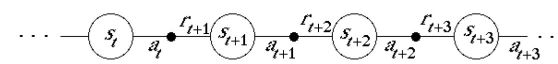
- Definition of MDPs
- Policies and value functions
- Bellman equations
- Dynamic programming methods
- Monte Carlo methods

Sequential decision making

- Utility theory provides a foundation for one-shot decision making. But typically agents have *repeated* interaction with the environment over time.
- Greedy agents, which try to maximize the immediate utility, are not necessarily optimal. A long-term view is needed
- Markov Decision Processes (MDPs) provide a framework for modeling sequential decision making



Markov Decision Processes (MDPs)



Assume:

- Finite set of states S (we will lift this later)
- Finite set of actions $A(s)$ available in each state s
- γ = discount factor for later rewards (between 0 and 1, usually close to 1)
- *Markov assumption:* s_{t+1} and r_{t+1} depend only on s_t, a_t and not on anything that happened before t
- Similar to a Markov chain, but has actions and rewards

Models for MDPs

- r_s^a = expected value of the immediate reward if the agent is in s and does action a

$$r_s^a = E\{r_{t+1} \mid s_t = s, a_t = a\}$$

- $p_{ss'}^a$ = probability of going from s to s' when doing action a

$$p_{ss'}^a = E\big\{s_{t+1} = s' \mid s_t = s, a_t = a\big\}$$

These form the *model* of the environment

Policies

A *policy* is a way of behaving (choosing actions:

$$\pi : S \times A \to [0,1], \;\; \pi(s,a) = P\{a_T = a | s_t = s\}$$

Deterministic policy: $\pi : S \to A$.

- One a policy is fixed, the MDP becomes a *Markov chain with rewards*
- Every policy induces a different Markov chain
- We want to find a policy that receives a large cumulated reward

Returns

The **return** R_t received after time t along a system trajectory is a function of all rewards

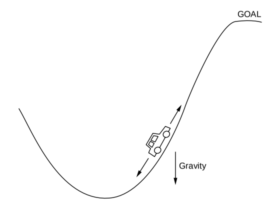
- Episodic tasks (e.g. games, trips through a maze etc)

$$R_t = r_{t+1} + r_{t+2} + \cdots + r_T$$

  where T is the time when a terminal state is reached
- Continuing tasks:
  - Discounted reward:
    $R_t = r_{t+1} + \gamma r_{t+2} + \ldots = \sum_{k=1}^{\infty} \gamma^{t+k-1} r_{t+k}$
  - Average reward: $R_t = \lim_{T \to \infty} \frac{1}{T} \left(\sum_{k=1}^{T} r_{t+k} \right)$

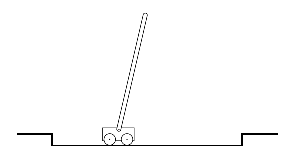
We want to find a policy that maximizes the **expected return**, $\forall t$

Example: Mountain-Car



- States: position and velocity
- Actions: accelerate forward, accelerate backward, coast
- Rewards:
  - reward = -1 for every time step, until car reaches the top
  - reward = 1 at the top, 0 otherwise $\gamma < 1$
- Return is maximized by minimizing the number of steps to the top of the hill

Example: Pole Balancing



Avoid failure: pole falling beyond a given angle, or cart hitting the end of the track

- Episodic task formulation: reward = +1 for each step before failure
  ⇒ return = number of steps before failure
- Continuing task formulation: reward = -1 upon failure, 0 otherwise, $\gamma < 1$
  ⇒ return = $-\gamma^k$ if there are k steps before failure

Value Functions

The **value of a state** is the expected return starting from that state, when following the policy:

$$V^\pi(s) = E_\pi\{R_t \mid s_t = s\} = E_\pi\left\{\sum_{k=1}^{\infty} \gamma^{k-1} r_{t+k} \mid s_t = s\right\}$$

The value of taking action a in state s and following policy π afterwards is the expected return when starting in that state, taking the action and following π afterwards:

$$Q^\pi(s,a) = E_\pi\{R_t \mid s_t = s, a_t = a\} = E_\pi\left\{\sum_{k=1}^{\infty} \gamma^{k-1} r_{t+k} \mid s_t = s, a_t = a\right\}$$

Bellman Equation for Policy π

$$\begin{aligned} R_t &= r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots \\ &= r_{t+1} + \gamma\left(r_{t+2} + \gamma r_{t+3} + \cdots\right) \\ &= r_{t+1} + \gamma R_{t+1} \end{aligned}$$

Based on this observation, V^π becomes:

$$V^\pi(s) = E_\pi\{R_t \mid s_t = s\} = E_\pi\{r_{t+1} + \gamma V^\pi(s_{t+1}) \mid s_t = s\}$$

Without the expectation:

$$V^\pi(s) = \sum_a \pi(s,a)\left(r_s^a + \gamma \sum_{s'} p_{ss'}^a V^\pi(s')\right)$$

This is a system of *linear equations* whose unique solution is V^π.

Iterative Policy Evaluation

Main idea: turn Bellman equation into an update rule.

1. Start with some initial guess V_0
2. During every iteration k, perform a **full backup of the value function**:

$$V_{k+1}(s) \leftarrow \sum_a \pi(s,a)\left(r_s^a + \gamma \sum_{s'} p_{ss'}^a V_k(s')\right)$$

3. Stop when the maximum change between two iterations is smaller than a desired threshold (the values stop changing)

Key idea: bootstrapping!

The value of one state is updated based on the values of the other states

Optimal Value Functions

- Policies can be partially ordered: $\pi \geq \pi'$ iff $V^\pi(s) \geq V^{\pi'}(s) \forall s$
- In an MDP there always exists at least one policy better than all others. This is called the **optimal policy**, π^*.
- The **optimal state-value function** is the value function shared by all optimal policies:

$$V^*(s) = \max_\pi V^\pi(s), \forall s \in S$$

- Similarly, we can define the **optimal action-value function**:

$$Q^*(s,a) = \max_\pi Q^\pi(s,a), \forall s \in S, \forall a \in A$$

This is the expected value for taking action a in state s and following an optimal policy afterwards

Bellman Optimality Equation for V^*

The value of a state under the optimal policy must be equal to the expected return for the best action in the state:

$$\begin{aligned} V^*(s) &= \max_a Q^*(s,a) \\ &= \max_a E\{r_{t+1} + \gamma V^*(s_{t+1}) \mid s_t = s, a_t = a\} \\ &= \max_a (r_s^a + \gamma \sum_{s'} p_{ss'}^a V^*(s')) \end{aligned}$$

V^* is the **unique solution** of this system of non-linear equations

Bellman Optimality Equation for Q^*

$$\begin{aligned} Q^*(s,a) &= E\Big\{r_{t+1} + \gamma \max_{a'} Q^*(s_{t+1}, a') \mid s_t = s, a_t = a\Big\} \\ &= r_s^a + \gamma \sum_{s'} p_{ss'}^a \max_{a'} Q^*(s', a') \end{aligned}$$

Q^* is the **unique solution** of this system of non-linear equations.

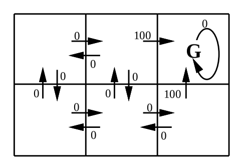
Why Optimal Value Functions are Useful

Any policy that is greedy with respect to V^* is an optimal policy!

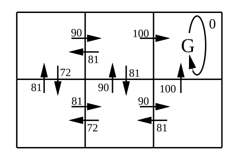
- If we know V^* and the model of the environment, one step of look-ahead will tell us what the optimal action is
- If we know Q^*, look-ahead is not even needed!

$$\pi^*(s) = \arg\max_a Q^*(s,a), \forall s$$

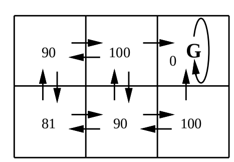
Illustration: A Deterministic Gridworld



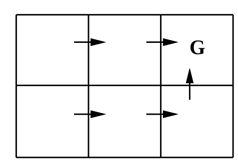
$r(s,a)$ (immediate reward) values



$Q^*(s,a)$ values ($\gamma = 0.9$)



$V^*(s)$ values



One optimal policy

Policy Improvement

Suppose we have computed V^{π} for some deterministic policy π

When is it better to do an action $a \neq \pi(s)$?

$$Q^{\pi}(s,a) > V^{\pi}(s)$$

If we make the change at all states, we get a policy π' which is **greedy** with respect to Q^{π}:

$$\pi'(s) = \arg\max_a Q^{\pi}(s,a) = \arg\max_a r_s^a + \gamma \sum_{s'} p_{ss'}^a V^{\pi}(s')$$

Then $V^{\pi'}(s) \geq V^{\pi}(s), \forall s$

Policy Improvement (continued)

What if at some point $V^{\pi'} = V^{\pi}$?

Then we have:

$$V^{\pi}(s) = \max_a r_s^a + \gamma \sum_{s'} p_{ss'}^a V^{\pi}(s')$$

But this is the Bellman optimality equation!

So if the value does not change at some point, both π and π' are optimal.

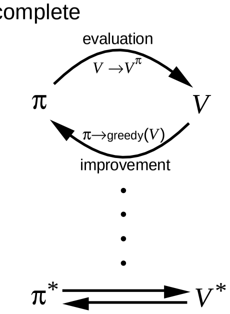
Policy Iteration

1. Start with an initial policy π_0
2. Repeat:
   (a) Compute V^{π_i} using policy evaluation
   (b) Compute a new policy π_{i+1} that is greedy with respect to V^{π_i}

   until $V^{\pi_i} = V^{\pi_{i+1}}$

Generalized Policy Iteration

Any combination of policy evaluation and policy improvement steps, even if they are not complete



Value Iteration

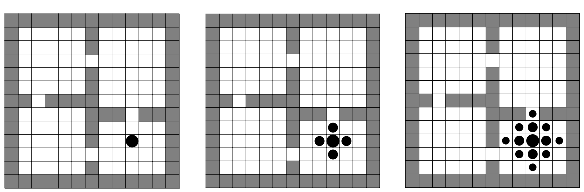
Main idea: Turn the Bellman optimality equation into an update rule (same as done in policy evaluation):

1. Start with an arbitrary initial approximation V_0
2. $V_{k+1}(s) \leftarrow \max_a r_s^a + \gamma \sum_{s'} p_{ss'}^a V_k(s), \forall s$

Illustration: Rooms Example

Four actions, fail 30% of the time

No rewards until the goal is reached, $\gamma = 0.9$.



What if the model is unknown?

- Observe transitions in the environment, learn an *approximate model* $\hat{r}_s^a, \hat{p}_{ss'}^a$

  We already discussed methods for approximating probabilities
- Pretend the approximate model is correct and use it for any dynamic programming method
- This approach is called **model-based reinforcement learning**
- Many believers, especially in the robotics community

Asynchronous dynamic programming

- All the methods described so far require sweeps over the entire state space
- A more efficient idea: repeatedly pick states at random, and apply a backup, until some convergence criterion is met
- How should states be selected?

  Based on the agent's experience! I.e. along trajectories.
- Still needs lots of computation, but does not get locked into very long sweeps

Efficiency of DP

- Good news: finding an optimal policy is polynomial in the number of states
- Bad news: finding an optimal policy is polynomial in the number of states!

  Number of states is often astronomical; typically number of states is exponential in the number of state variables
- In practice, classical DP can be applied to problems with a few millions states
- Asynchronous DP can be applied even to larger problems, and is appropriate for parallel computation
- But it is surprisingly easy to find problems for which DP methods are not feasible

Monte Carlo Methods

- Suppose we have an episodic task (trials terminate at some point)
- The agent behave according to some policy π for a while, generating several trajectories. How can we compute V^π?
- Compute $V^\pi(s)$ by **averaging the observed returns** after s on the trajectories in which s was visited.
- Two main approaches:
  - Every-visit: average returns for every time a state is visited in a trial
  - First-visit: average returns only for the first time a state is visited in a trial

Implementation of Monte Carlo Policy Evaluation

$$\begin{aligned} V_{n+1} &= \frac{1}{n+1}\sum_{i=1}^{n+1} R_i = \frac{1}{n+1}(\sum_{i=1}^{n} R_i + R_{n+1}) \\ &= \frac{n}{n+1}\frac{1}{n}\sum_{i=1}^{n} R_i + \frac{1}{n+1}R_{n+1} \\ &= \frac{n}{n+1}V_n + \frac{1}{n+1}R_{n+1} \end{aligned}$$

If we do not want to keep counts of how many times states have been visited, we can use a *learning rate* version:

$$V(s_t) \leftarrow V(s_t) + \alpha(R_t - V(s_t))$$

Monte Carlo Estimation of Q values

- We use the same idea: $Q^\pi(s,a)$ is the average of the returns obtained by starting in state s, doing action a and then following π
- Like the state-value version, it converges asymptotically *if every state-action pair is visited*
- But π might not choose every action in every state!
- *Exploring starts:* Every state-action pair has a non-zero probability of being the starting pair

Dynamic Programming vs. Monte Carlo

	DP	MC
Need model	yes	no (+)
Bootstrapping	yes (+)	no
Improve directly with interaction	no	yes (+)
Focus on visited states	no	yes (+)

Can we combine the advantages of both methods?