Kalman filtering and friends: Inference in time series models

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slides mostly by Michael Rubinstein

Problem overview

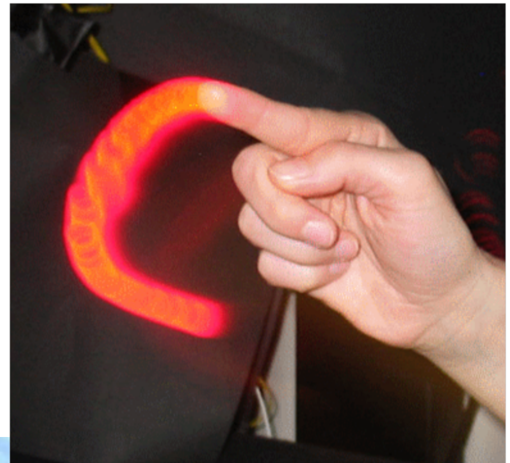
- Goal
  - Estimate most probable state at time k using measurement up to time k’
    - k’<k: **prediction**
    - k’=k: **filtering**
    - k’>k: **smoothing**
- Input
  - (Noisy) Sensor measurements
  - Known or learned system model (see last lecture)

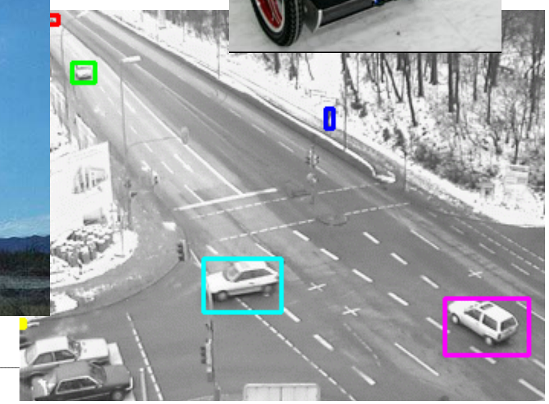
- Many problems require estimation of the state of systems that change over time using noisy measurements on the system

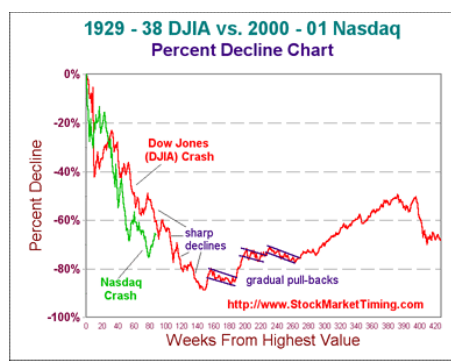
Applications

- Ballistics
- Robotics
  - Robot localization
- Tracking hands/cars/…
- Econometrics
  - Stock prediction
- Navigation

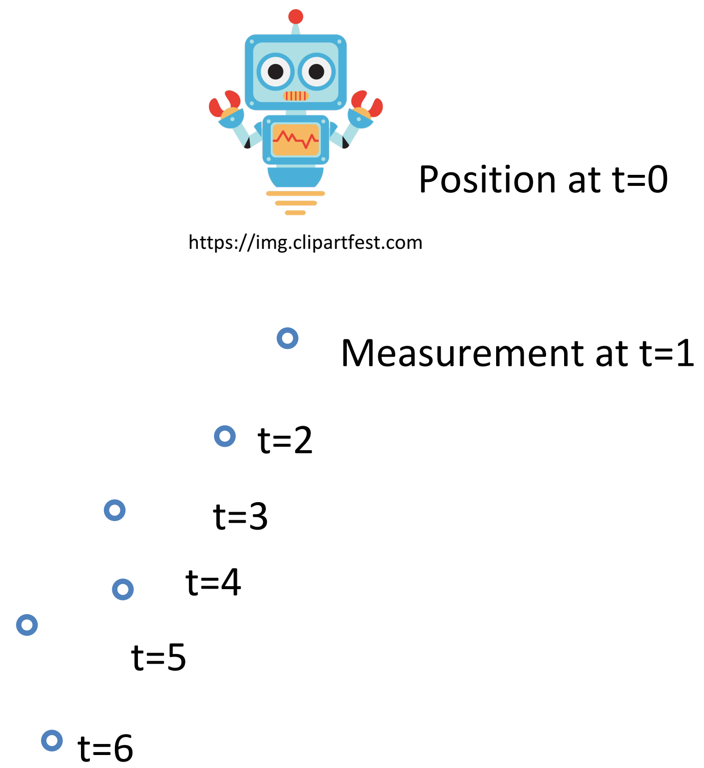
- Many more…







Example: noisy localization



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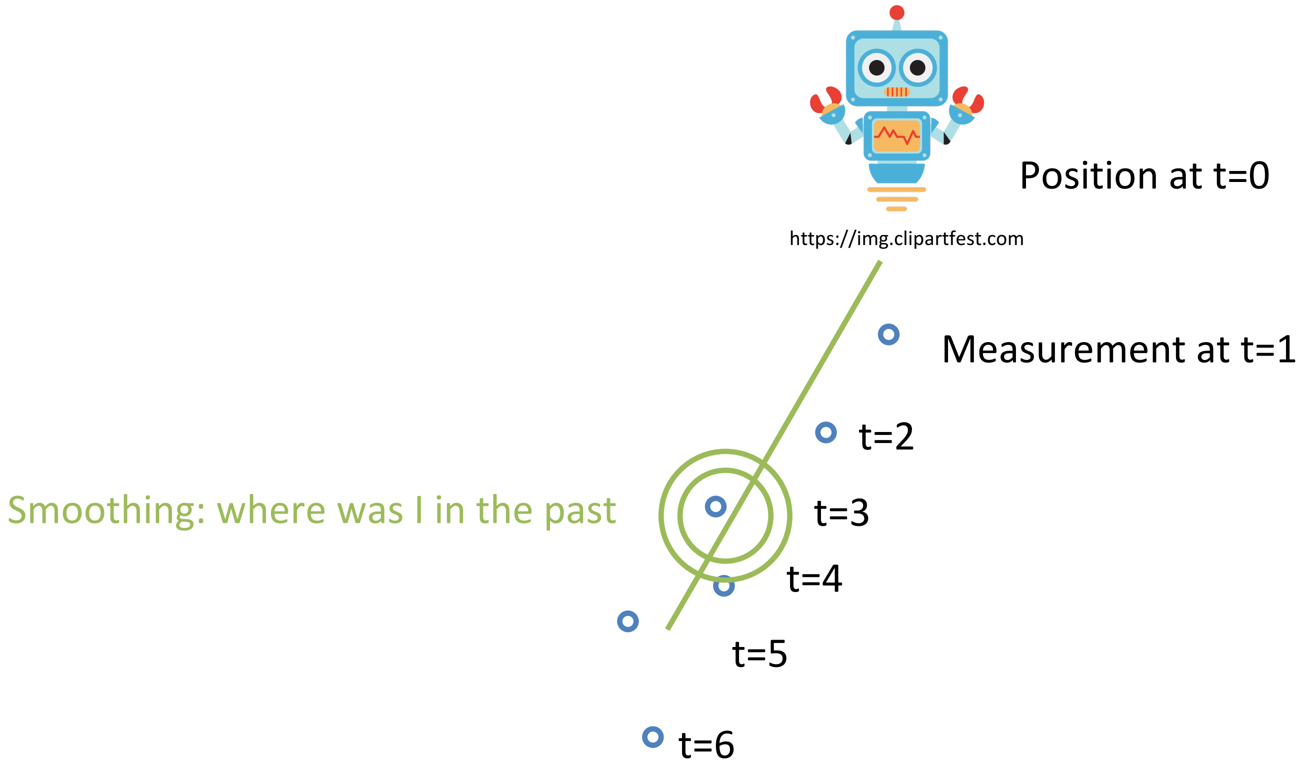

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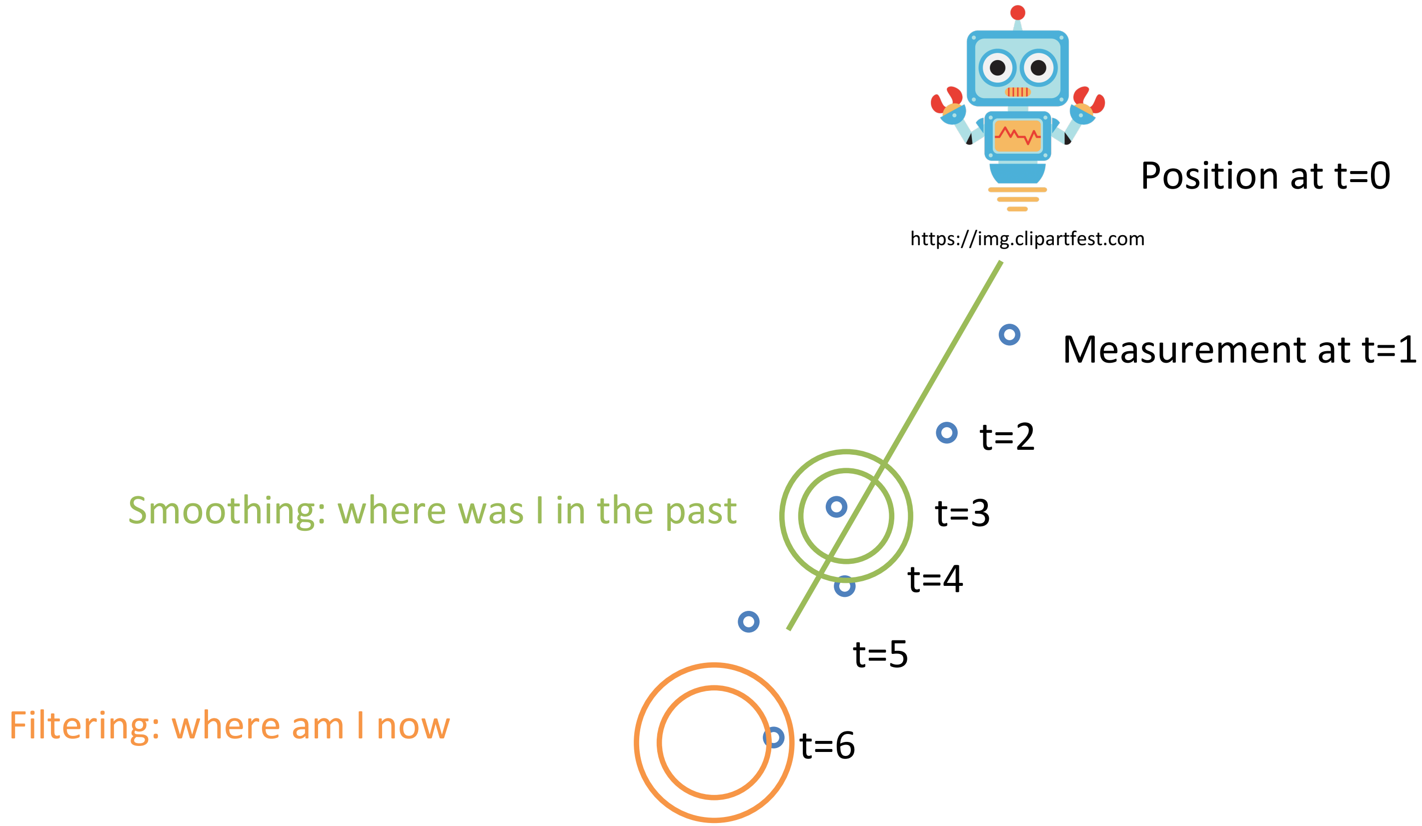

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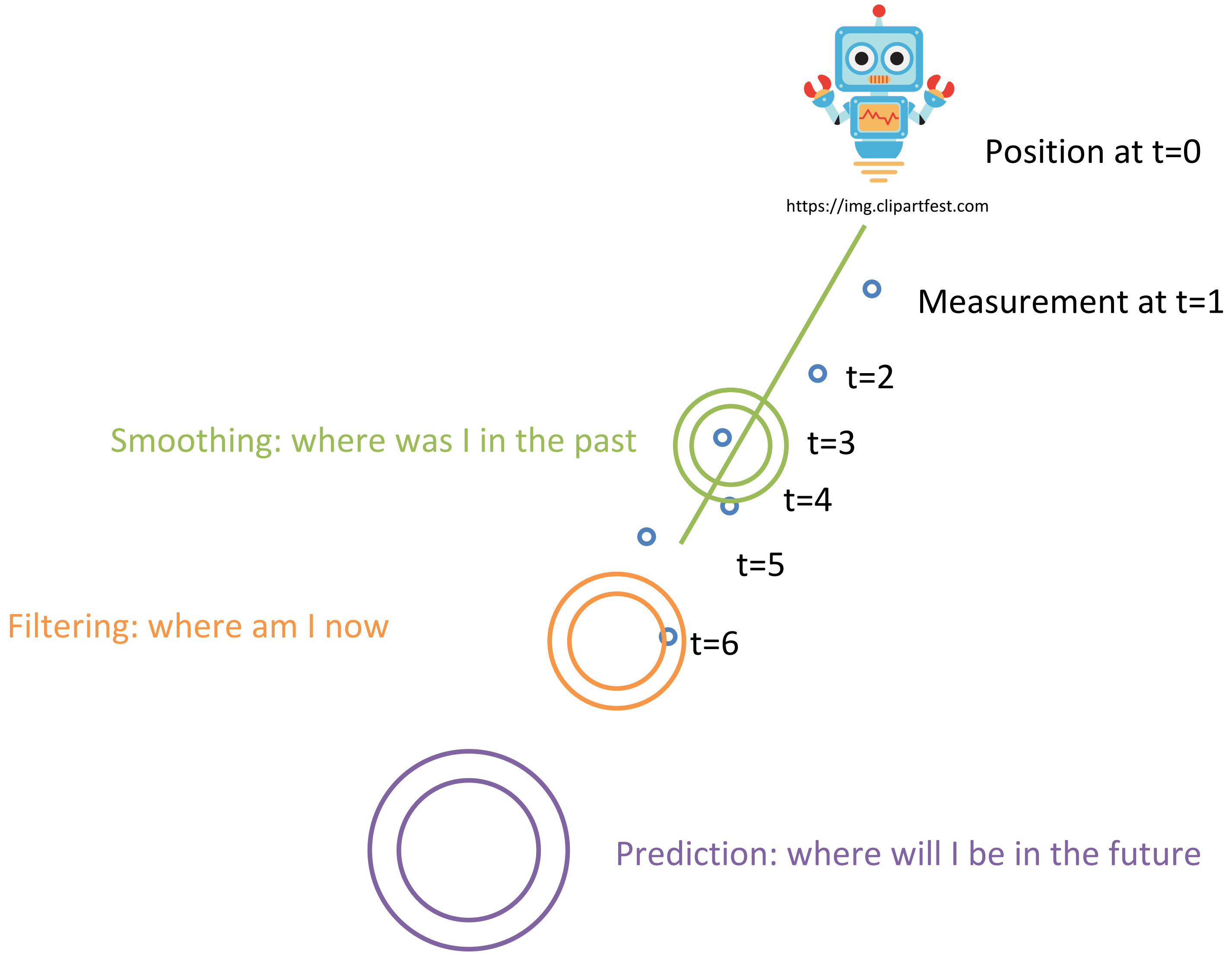

Today's lecture

- Fundamentals
  - Formalizing time series models
  - Recursive filtering
- Two cases with optimal solutions
  - Linear Gaussian models
  - Discrete systems
- Suboptimal solutions

Stochastic Processes

- Stochastic process
  - Collection of random variables indexed by some set
  - Ie. R.V. x_i for every element i in index set
- Time series modeling
  - Sequence of random states/variables
  - Measurements available at discrete times
  - Modeled as stochastic process indexed by $\mathbb{N}$

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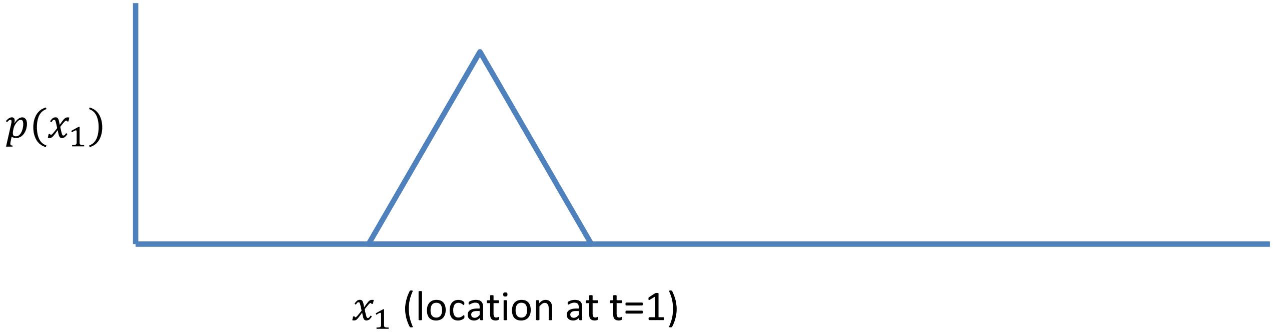

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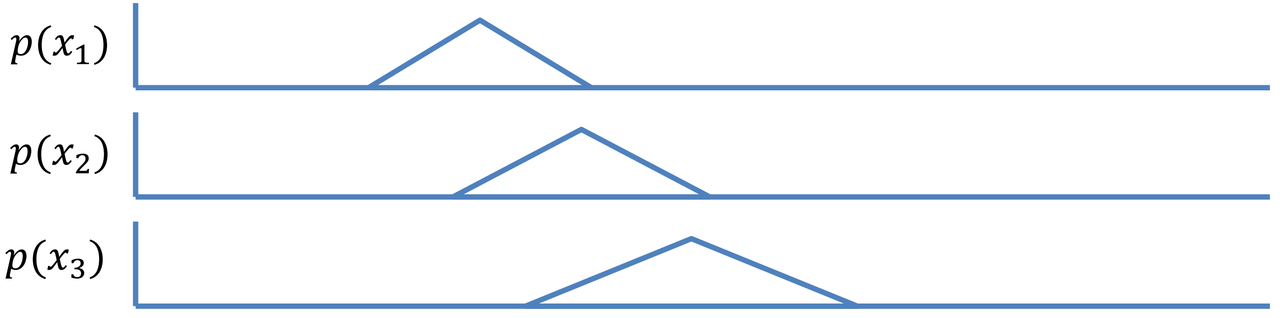
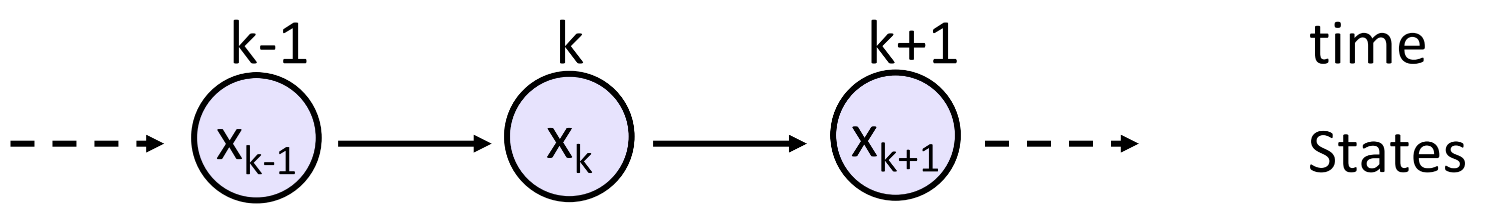


(First-order) Markov process

- The Markov property – the likelihood of a future state depends on present state only

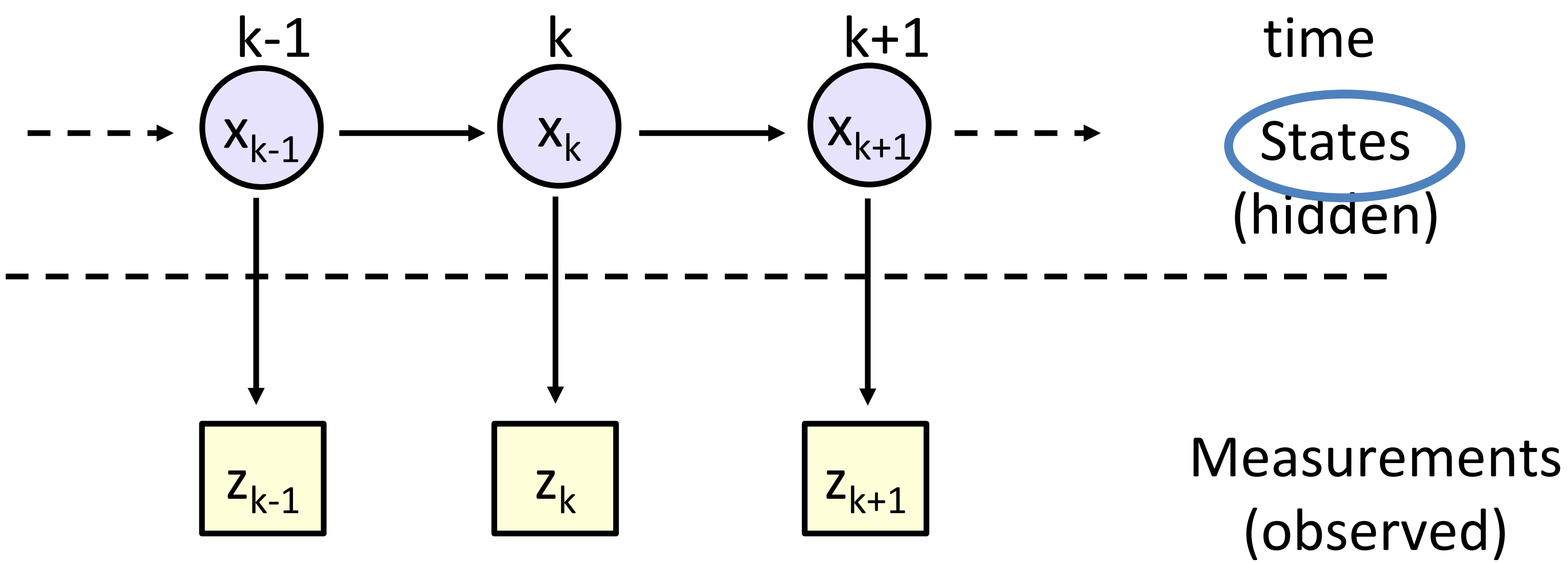
$$\Pr[X(k+h) = y \mid X(s) = x(s), \forall s \le k] = \Pr[X(k+h) = y \mid X(k) = x(k)], \forall h > 0$$

- Markov chain – A stochastic process with Markov property



Hidden Markov Model (HMM)

- the state is not directly visible, but output dependent on the state is visible



State space

- **The state vector** contains all available information to describe the investigated system
  - usually multidimensional: $X(k) \in R^{N_x}$
- **The measurement vector** represents observations related to the state vector $Z(k) \in R^{N_z}$
  - Generally (but not necessarily) of lower dimension than the state vector

State space

- Tracking:

$N_x = 3$
$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$N_x = 4$
$$\begin{bmatrix} x \\ v_x \\ y \\ v_y \end{bmatrix}$$

Econometrics:

- Monetary flow
- Interest rates
- Inflation
- …

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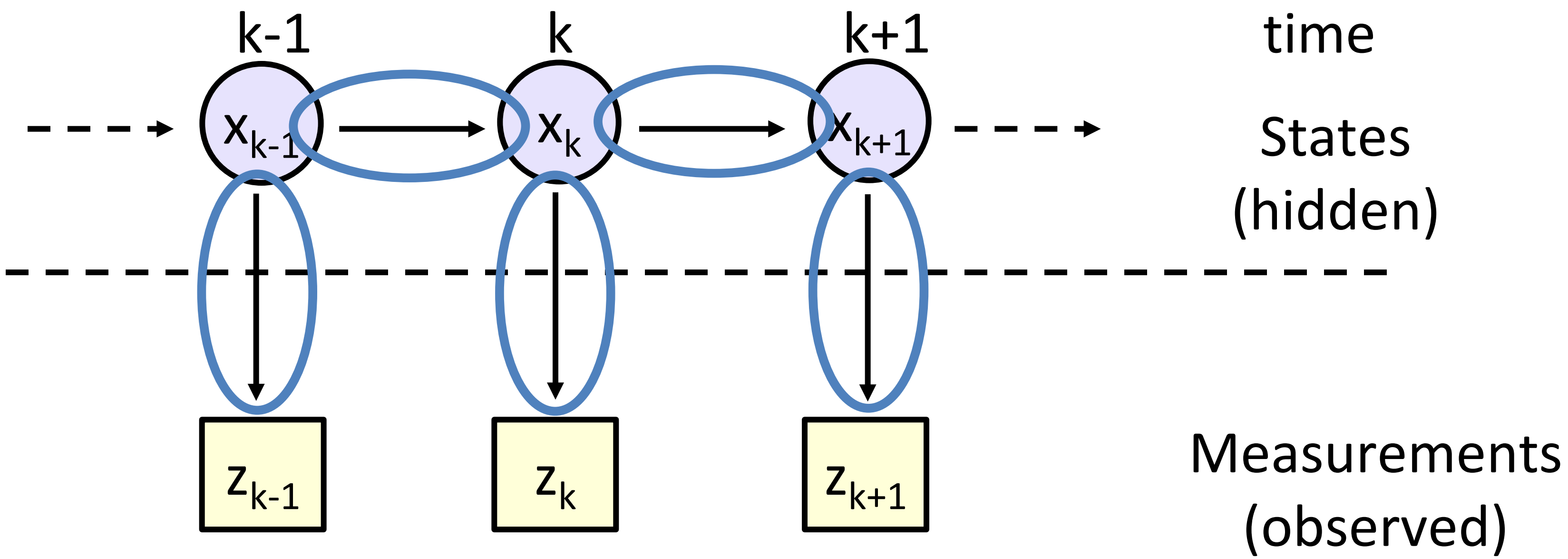
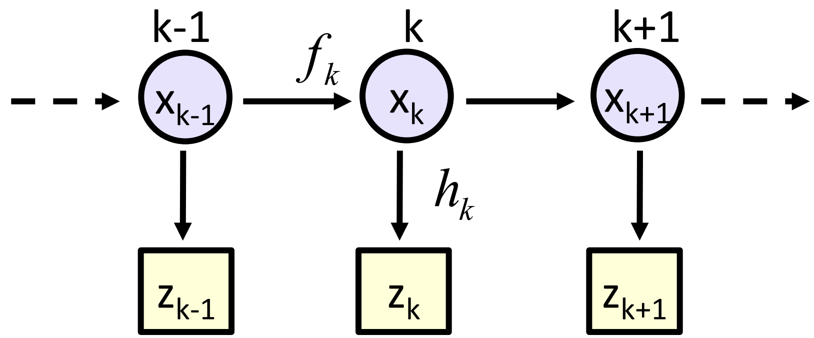


Dynamic System



Stochastic diffusion

State equation: $$x_k = f_k(x_{k-1}, v_k)$$

x_k state vector at time instant k

f_k state transition function, $f_k : R^{N_x} \times R^{N_v} \rightarrow R^{N_x}$

v_k i.i.d process noise

Observation equation: $$z_k = h_k(x_k, w_k)$$

z_k observations at time instant k

h_k observation function, $h_k : R^{N_x} \times R^{N_w} \rightarrow R^{N_z}$

w_k i.i.d measurement noise

A simple dynamic system

- $X = [x, y, v_x, v_y]$ (4-dimensional state space)
- Constant velocity motion:

$$f(X, v) = [x + \Delta t \cdot v_x, y + \Delta t \cdot v_y, v_x, v_y] + v$$

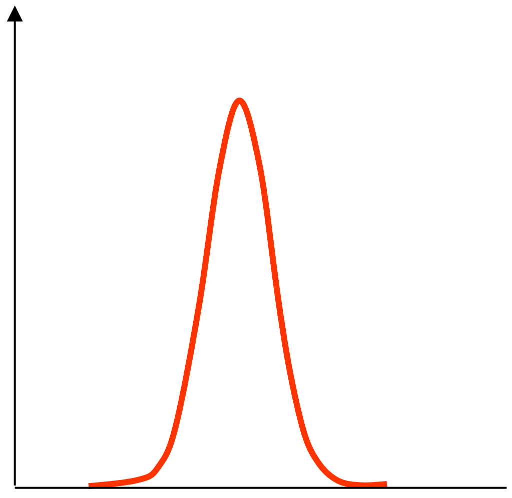
$$v \sim N(0, Q) \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q^2 & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}$$

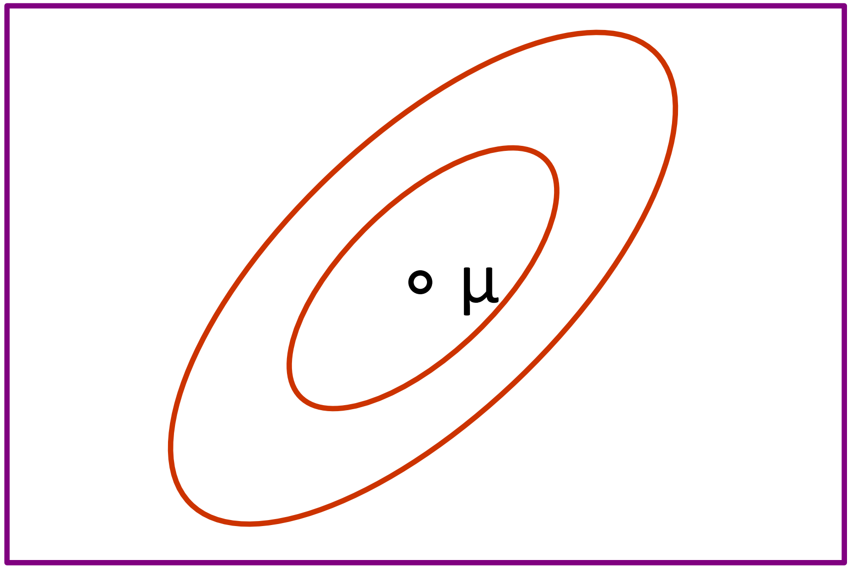
- Only position is observed:

$$z = h(X, w) = [x, y] + w$$

$$w \sim N(0, R) \quad R = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

Gaussian distribution





Yacov Hel-Or

$$p(x) \sim N(\mu, \Sigma) = \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right\}$$

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Recursive filters

- For many problems, estimate is required each time a new measurement arrives

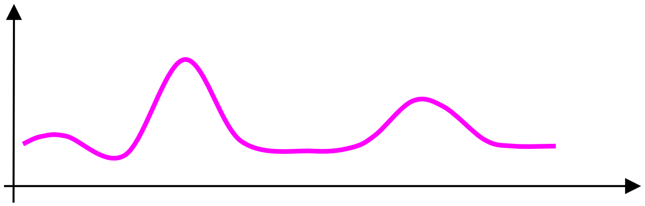
- **Batch** processing
  - Requires *all* available data
- **Sequential** processing
  - New data is processed upon arrival
  - Need not store the complete dataset
  - Need not reprocess all data for each new measurement

  - Assume no out-of-sequence measurements (solutions for this exist as well…)

Bayesian filter


Thomas Bayes

- Construct the posterior probability density function $p(x_k \mid z_{1:k})$ of the state based on all available information



- By knowing the posterior many kinds of estimates for x_k can be derived
  – mean (expectation), mode, median, …
  – Can also give estimation of the accuracy (e.g. covariance)

Recursive Bayes filters

- Given:
  - System models in probabilistic forms

$$x_k = f_k(x_{k-1}, v_k) \leftrightarrow p(x_k \mid x_{k-1})$$

Markovian process

$$z_k = h_k(x_k, w_k) \leftrightarrow p(z_k \mid x_k)$$

Measurements conditionally independent given state

    (known statistics of v_k, w_k)
  - Initial state $p(x_0 \mid z_0) = p(x_0)$ also known as the **prior**
  - Measurements $z_1, \dots, z_k$

Recursive Bayes filters

- Prediction step (a-priori)

$$p(x_{k-1} \mid z_{1:k-1}) \rightarrow p(x_k \mid z_{1:k-1})$$

  - Uses the system model to predict forward
  - Deforms/translates/spreads state pdf due to random noise

- Update step (a-posteriori)

$$p(x_k \mid z_{1:k-1}) \rightarrow p(x_k \mid z_{1:k})$$

  - Update the prediction in light of new data
  - Tightens the state pdf

Prior vs posterior?

- It can seem odd to regard $p(x_k|z_{1:k-1})$ as prior
- Compare

likelihood prior

posterior

$$P(x_k|z_k \qquad) = \frac{p(z_k|x_k \qquad)P(x_k \qquad)}{p(z_k \qquad)}$$

evidence

to

$$P(x_k|z_k, z_{1:k-1}) = \frac{p(z_k|x_k, z_{1:k-1})P(x_k|z_{1:k-1})}{p(z_k|z_{1:k-1})}$$

- In update with z_k, it is a *working prior*

General prediction-update framework

- Assume $p(x_{k-1} \mid z_{1:k-1})$ is given at time k-1
- Prediction:

System model Previous posterior

$$p(x_k \mid z_{1:k-1}) = \int p(x_k \mid x_{k-1}) p(x_{k-1} \mid z_{1:k-1}) dx_{k-1} \quad (1)$$

- Using Chapman-Kolmogorov identity + Markov property

General prediction-update framework

- Update step

$$p(x_k \mid z_{1:k}) = p(x_k \mid z_k, z_{1:k-1})$$

$$p(A \mid B, C) = \frac{p(B \mid A, C)\, p(A \mid C)}{p(B \mid C)}$$

$$= \frac{p(z_k \mid x_k, z_{1:k-1})\, p(x_k \mid z_{1:k-1})}{p(z_k \mid z_{1:k-1})}$$

$$\frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$

Measurement model

Current prior

$$= \frac{p(z_k \mid x_k)\, p(x_k \mid z_{1:k-1})}{p(z_k \mid z_{1:k-1})} \quad (2)$$

Normalization constant

Where

$$p(z_k \mid z_{1:k-1}) = \int p(z_k \mid x_k)\, p(x_k \mid z_{1:k-1})\, dx_k$$

Generating estimates

- Knowledge of $p(x_k \mid z_{1:k})$ enables to compute optimal estimate with respect to any criterion. e.g.
  - Minimum mean-square error (MMSE)

$$\hat{x}_{k|k}^{MMSE} \equiv E\left[x_k \mid z_{1:k}\right] = \int x_k \, p(x_k \mid z_{1:k}) dx_k$$

  - Maximum a-posteriori

$$\hat{x}_{k|k}^{MAP} \equiv \arg\max_{x_k} p(x_k \mid z_k)$$

General prediction-update framework

➔So (1) and (2) give optimal solution for the recursive estimation problem!

- Unfortunately… only conceptual solution
  - integrals are intractable…
  - Cannot represent arbitrary pdfs!

- However, optimal solution *does* exist for several restrictive cases

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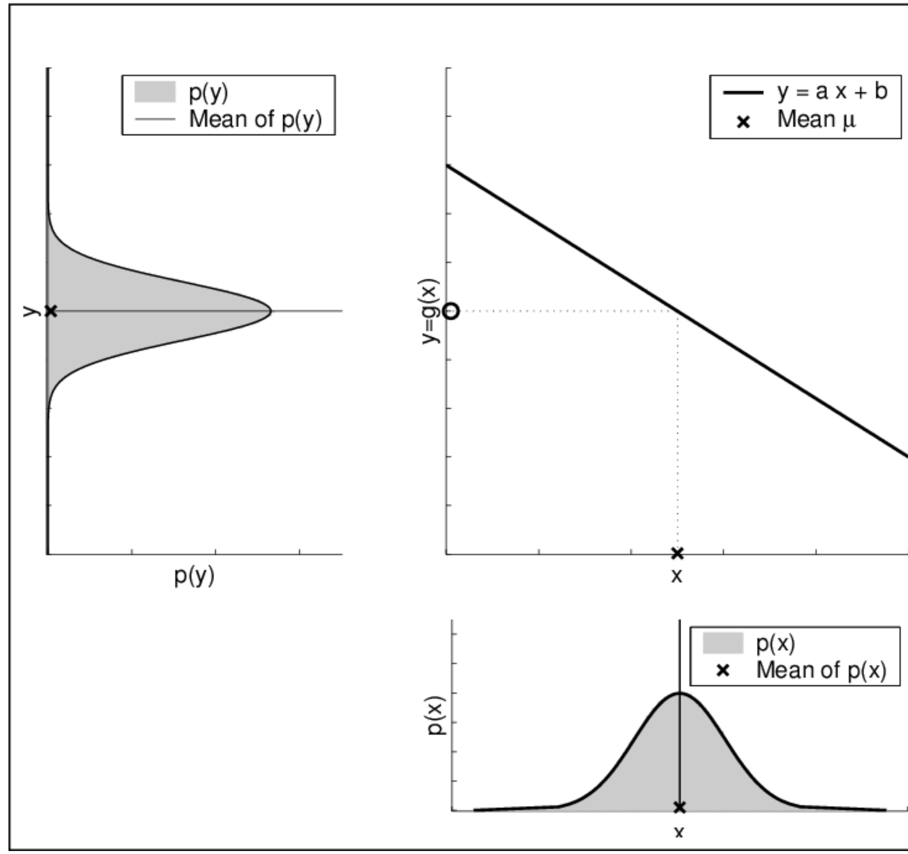
Restrictive case #1

- Posterior at each time step is Gaussian
  - Completely described by mean and covariance
- If $p(x_{k-1} \mid z_{1:k-1})$ is Gaussian it can be shown that $p(x_k \mid z_{1:k})$ is also Gaussian provided that:
  - v_k, w_k are Gaussian
  - f_k, h_k are linear

Restrictive case #1

- Why Linear?

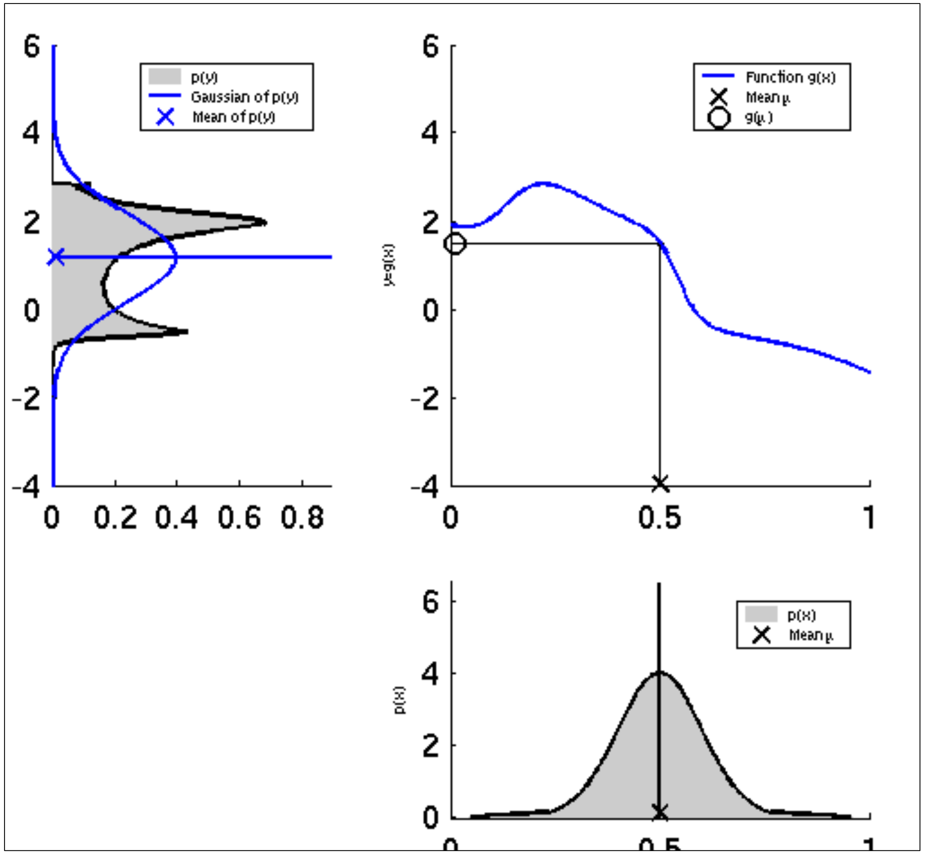


Yacov Hel-Or

$$y = Ax + B \Rightarrow p(y) \sim N\left(A\mu + B, A\Sigma A^T\right)$$

Restrictive case #1

- Why Linear?



Yacov Hel-Or

$$y = g(x) \nRightarrow p(y) \sim N(\ )$$

Restrictive case #1

- Linear system with additive noise

$$x_k = F_k x_{k-1} + v_k$$
$$z_k = H_k x_k + w_k$$
$$v_k \sim N(0, Q_k)$$
$$w_k \sim N(0, R_k)$$

- Simple example again

$$f(X, v) = [x + \Delta t \cdot v_x, y + \Delta t \cdot v_y, v_x, v_y] + v \qquad z = h(X, w) = [x, y] + w$$

$$\begin{pmatrix} x_k \\ y_k \\ v_{x,k} \\ v_{y,k} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{F} \begin{pmatrix} x_{k-1} \\ y_{k-1} \\ v_{x,k-1} \\ v_{y,k-1} \end{pmatrix} + N(0, Q_k)$$

$$\begin{pmatrix} x_{obs} \\ y_{obs} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{H} \begin{pmatrix} x_k \\ y_k \\ v_{x,k} \\ v_{y,k} \end{pmatrix} + N(0, R_k)$$

The Kalman filter

Rudolf E. Kalman

$$p(x_{k-1} \mid z_{1:k-1}) = N(x_{k-1}; \hat{x}_{k-1|k-1}, P_{k-1|k-1})$$
$$p(x_k \mid z_{1:k-1}) = N(x_k; \hat{x}_{k|k-1}, P_{k|k-1})$$
$$p(x_k \mid z_{1:k}) = N(x_k; \hat{x}_{k|k}, P_{k|k})$$

$$N(x; \mu, \Sigma) = |2\pi\Sigma|^{-1/2} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right)$$

- Substituting into (1) and (2) yields the predict and update equations

The Kalman filter

Predict:

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1}$$
$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$$

Update:

$$S_k = H_k P_{k|k-1} H_k^T + R_k$$
$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left(z_k - H_k \hat{x}_{k|k-1} \right)$$
$$P_{k|k} = \left[I - K_k H_k \right] P_{k|k-1}$$

Intuition via 1D example

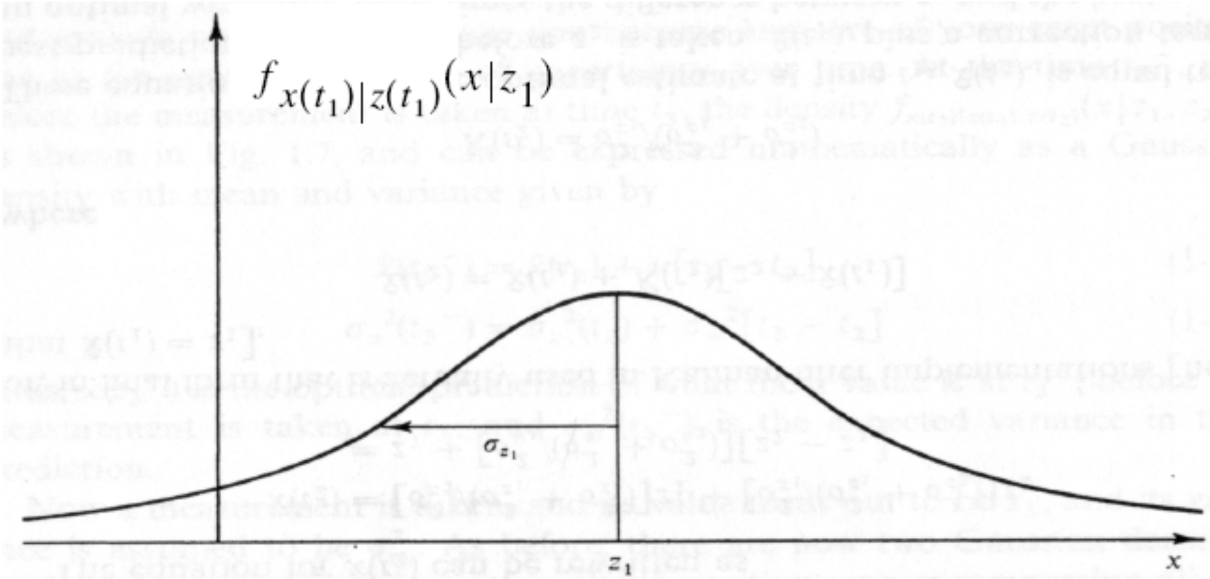
- Lost at sea
  - Night
  - No idea of location
  - For simplicity – let's assume 1D
  - Not moving

* Example and plots by Maybeck, *"Stochastic models, estimation and control, volume 1"*

Example – cont'd

- Time t1: Star Sighting
  - Denote $z(t_1)=z_1$
- Uncertainty (inaccuracies, human error, etc)
  - Denote σ_1 (normal)
- Can establish the conditional probability of $x(t_1)$ given measurement z_1

Example – cont'd

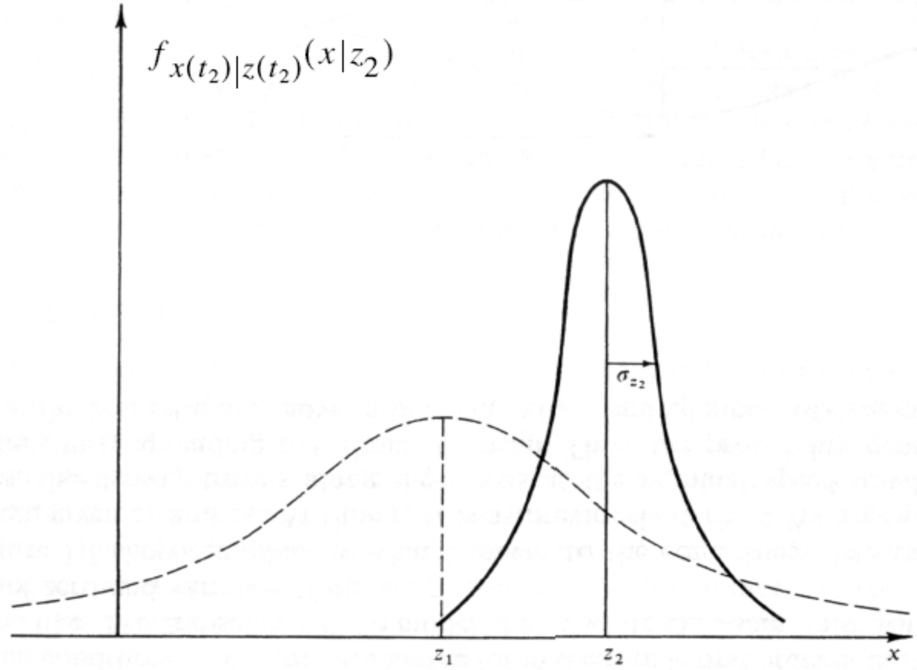


- Probability for any location, based on measurement
- For Gaussian density – 68.3% within ±σ1
- Best estimate of position: Mean/Mode/Median

$$\hat{x}(t_1) = z_1 \qquad \sigma_x^2(t_1) = \sigma_{z_1}^2$$

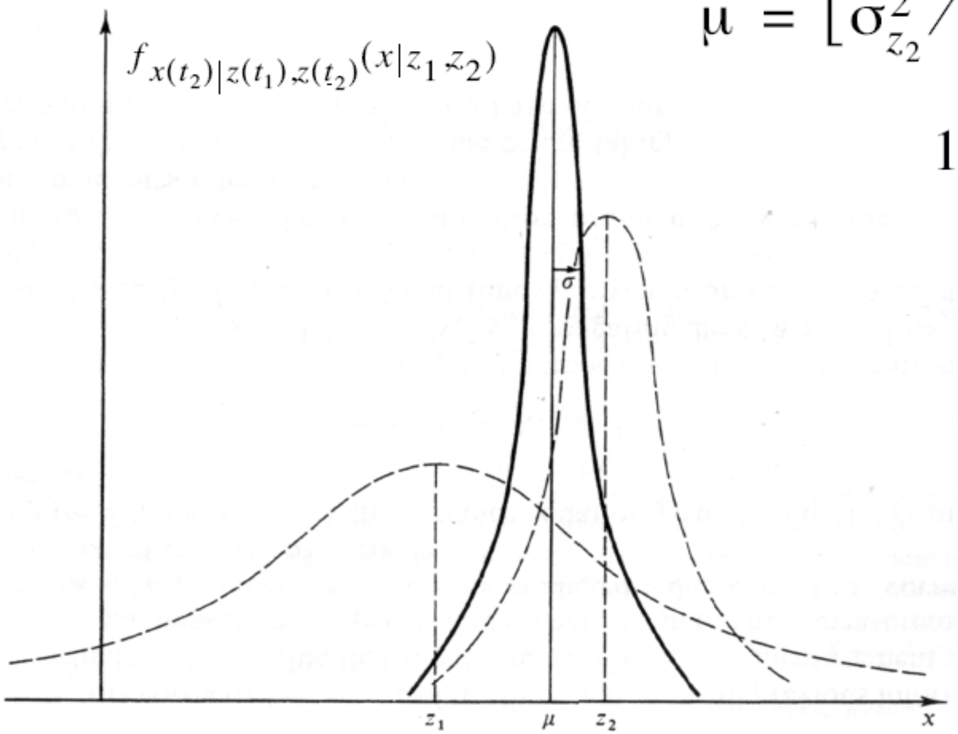
Example – cont'd

- Time t2: friend (more trained)
  - $x(t2)=z2$, $\sigma(t2)=\sigma2$
  - Since she has higher skill: $\sigma2<\sigma1$



Example – cont'd

- f(x(t2)|z1,z2) also Gaussian



$$\mu = [\sigma_{z_2}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_1 + [\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_2$$

$$1/\sigma^2 = (1/\sigma_{z_1}^2) + (1/\sigma_{z_2}^2)$$

Example – cont'd

$$\mu = [\sigma_{z_2}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_1 + [\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_2$$

$$1/\sigma^2 = (1/\sigma_{z_1}^2) + (1/\sigma_{z_2}^2)$$

- σ less than both σ1 and σ2
- σ1= σ2: average
- σ1> σ2: more weight to z2
- Rewrite:

$$\hat{x}(t_2) = [\sigma_{z_2}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_1 + [\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)]z_2$$

$$= z_1 + [\sigma_{z_1}^2/(\sigma_{z_1}^2 + \sigma_{z_2}^2)][z_2 - z_1]$$

Example – cont'd

- The Kalman update rule:

$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)]$$

$$K(t_2) = \sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)$$

Best estimate
Given z2
(a posteriori)

Best Prediction prior to z2
(*a priori*)

Optimal Weighting
(*Kalman Gain*)

Residual

The Kalman filter

Predict:

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1}$$
$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$$

Generally **increases** variance

Update:

$$S_k = H_k P_{k|k-1} H_k^T + R_k$$
$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left(z_k - H_k \hat{x}_{k|k-1} \right)$$
$$P_{k|k} = \left[I - K_k H_k \right] P_{k|k-1}$$

$$K(t_2) = \sigma_{z_1}^2 / (\sigma_{z_1}^2 + \sigma_{z_2}^2)$$
$$\hat{x}(t_2) = \hat{x}(t_1) + K(t_2)[z_2 - \hat{x}(t_1)]$$

Generally **decreases** variance

Kalman gain

$$
\begin{aligned}
S_k &= H_k P_{k|k-1} H_k^T + R_k \\
K_k &= P_{k|k-1} H_k^T S_k^{-1} \\
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + K_k \left(z_k - H_k \hat{x}_{k|k-1} \right) \\
P_{k|k} &= \left[I - K_k H_k \right] P_{k|k-1}
\end{aligned}
$$

- Small measurement error, H invertible:

$$\lim_{\mathbf{R}_k \to 0} K_k = H_k^{-1} \Rightarrow \lim_{\mathbf{R}_k \to 0} \hat{x}_{k|k} = H_k^{-1} z_k$$

- Small prediction error:

$$\lim_{\mathbf{P}_k \to 0} K_k = 0 \Rightarrow \lim_{\mathbf{P}_k \to 0} \hat{x}_{k|k} = \hat{x}_{k|k-1}$$

The Kalman filter

- Pros (compared to e.g. particle filter)
  - Optimal closed-form solution to the tracking problem (under the assumptions)
    - No algorithm can do better in a linear-Gaussian environment!
  - All 'logical' estimations collapse to a unique solution
  - Simple to implement
  - Fast to execute
- Cons
  - If either the system or measurement model is non-linear → the posterior will be non-Gaussian

Smoothing possible with a backward message (cf HMMs, lecture 10)

Restrictive case #2

- The state space (domain) is discrete and finite
- Assume the state space at time k-1 consists of states $x_{k-1}^{i}, i = 1..N_s$
- Let $\Pr(x_{k-1} = x_{k-1}^{i} \mid z_{1:k-1}) = w_{k-1|k-1}^{i}$ be the conditional probability of the state at time k-1, given measurements up to k-1

The Grid-based filter

- The posterior pdf at k-1 can be expressed as sum of delta functions

$$p(x_{k-1} \mid z_{1:k-1}) = \sum_{i=1}^{N_s} w_{k-1|k-1}^{i} \delta(x_{k-1} - x_{k-1}^{i})$$

- Again, substitution into (1) and (2) yields the predict and update equations

Equivalent to belief monitoring in HMMs (Lecture 10)

The Grid-based filter

- Prediction

$$p(x_k \mid z_{1:k-1}) = \int p(x_k \mid x_{k-1}) p(x_{k-1} \mid z_{1:k-1}) dx_{k-1} \qquad (1)$$

$$p(x_k \mid z_{1:k-1}) = \sum_{i=1}^{N_s} \sum_{j=1}^{N_s} p(x_k^i \mid x_{k-1}^j) w_{k-1|k-1}^j \delta(x_{k-1} - x_{k-1}^i)$$

$$= \sum_{i=1}^{N_s} w_{k|k-1}^i \delta(x_{k-1} - x_{k-1}^i)$$

$$w_{k|k-1}^i = \sum_{j=1}^{N_s} w_{k-1|k-1}^j p(x_k^i \mid x_{k-1}^j)$$

- New prior is also weighted sum of delta functions
- New prior weights are reweighting of old posterior weights using state transition probabilities

The Grid-based filter

- Update

$$p(x_k \mid z_{1:k}) = \frac{p(z_k \mid x_k)\, p(x_k \mid z_{1:k-1})}{p(z_k \mid z_{1:k-1})} \qquad (2)$$

$$p(x_k \mid z_{1:k}) = \sum_{i=1}^{N_s} w_{k|k}^{i} \delta\,(x_{k-1} - x_{k-1}^{i})$$

$$w_{k|k}^{i} = \frac{w_{k|k-1}^{i}\, p(z_k \mid x_k^{i})}{\sum_{j=1}^{N_s} w_{k|k-1}^{j}\, p(z_k \mid x_k^{j})}$$

- Posterior weights are reweighting of prior weights using likelihoods (+ normalization)

The Grid-based filter

- Pros:
  - $p(x_k \mid x_{k-1}), p(z_k \mid x_k)$ assumed known, but no constraint on their (discrete) shapes
  - Easy extension to varying number of states
  - Optimal solution for the discrete-finite environment!
- Cons:
  - Curse of dimensionality
    - Inefficient if the state space is large
  - Statically considers *all* possible hypotheses

Smoothing possible with a backward message (cf HMMs, lecture 10)

Today's lecture

- Fundamentals
  - Formalizing time series models
  - Recursive filtering
- Two cases with optimal solutions
  - Linear Gaussian models
  - Discrete systems
- Suboptimal solutions

Suboptimal solutions

- In many cases these assumptions do not hold
  - Practical environments are nonlinear, non-Gaussian, continuous

➔Approximations are necessary…

  - Extended Kalman filter (EKF)
  - Approximate grid-based methods
  - Multiple-model estimators
  - Unscented Kalman filter (UKF)
  - Particle filters (PF)
  - …

Analytic approximations

Numerical methods

Gaussian-sum filters

Sampling approaches

The extended Kalman filter

- The idea: local linearization of the dynamic system might be sufficient description of the nonlinearity
- The model: nonlinear system with additive noise

$$
\begin{aligned}
x_k &= F_k x_{k-1} + v_k & x_k &= f_k(x_{k-1}) + v_k \\
z_k &= H x_k + w_k & z_k &= h_k(x_k) + w_k \\
v_k &\sim N(0, Q_k) & v_k &\sim N(0, Q_k) \\
w_k &\sim N(0, R_k) & w_k &\sim N(0, R_k)
\end{aligned}
$$

The extended Kalman filter

- *f, h* are approximated using a first-order Taylor series expansion (eval at state estimations)

Predict:

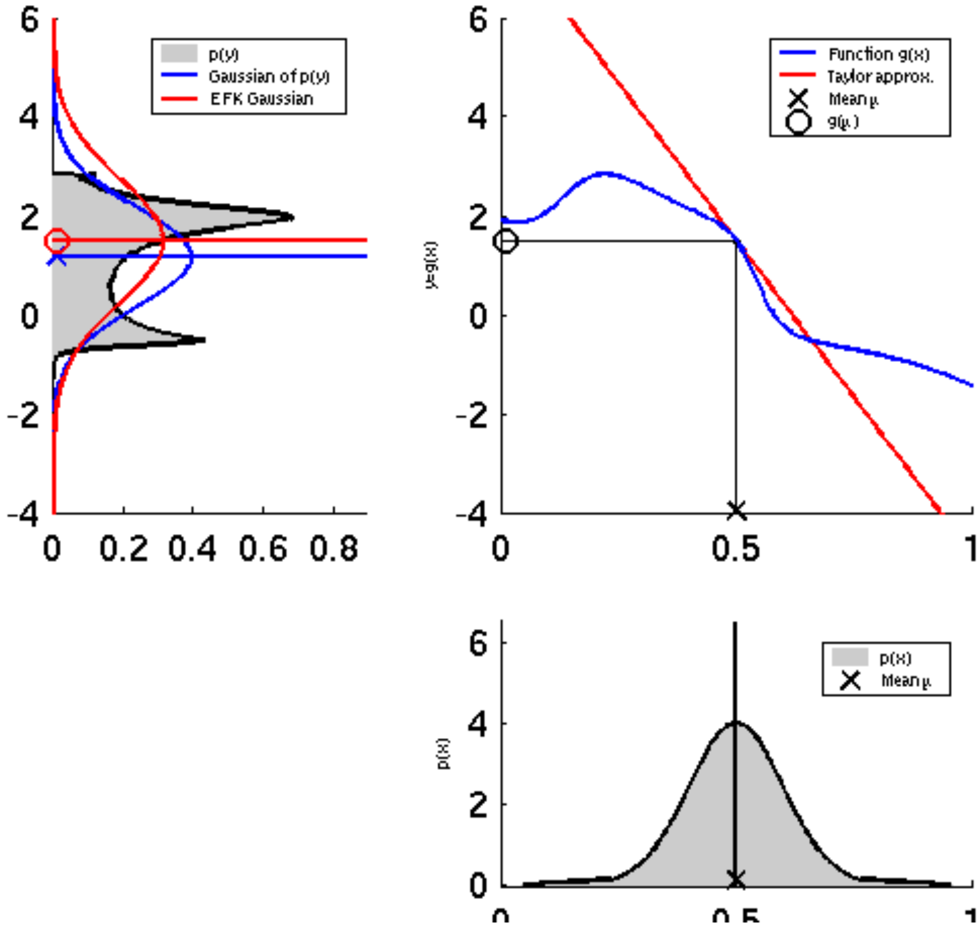
$$\hat{x}_{k|k-1} = f_k(\hat{x}_{k-1|k-1})$$
$$P_{k|k-1} = \hat{F}_k P_{k-1|k-1} \hat{F}_k^T + Q_k$$

Update:

$$S_k = \hat{H}_k P_{k|k-1} \hat{H}_k^T + R_k$$
$$K_k = P_{k|k-1} \hat{H}_k^T S_k^{-1}$$
$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \left(z_k - h_k(\hat{x}_{k|k-1}) \right)$$
$$P_{k|k} = \left[I - K_k H_k \right] P_{k|k-1}$$

$$\hat{F}_k[i,j] = \left. \frac{\partial f_k[i]}{\partial x_k[j]} \right|_{x_k = \hat{x}_{k-1|k-1}}$$
$$\hat{H}_k[i,j] = \left. \frac{\partial h_k[i]}{\partial x_k[j]} \right|_{x_k = \hat{x}_{k|k-1}}$$

The extended Kalman filter

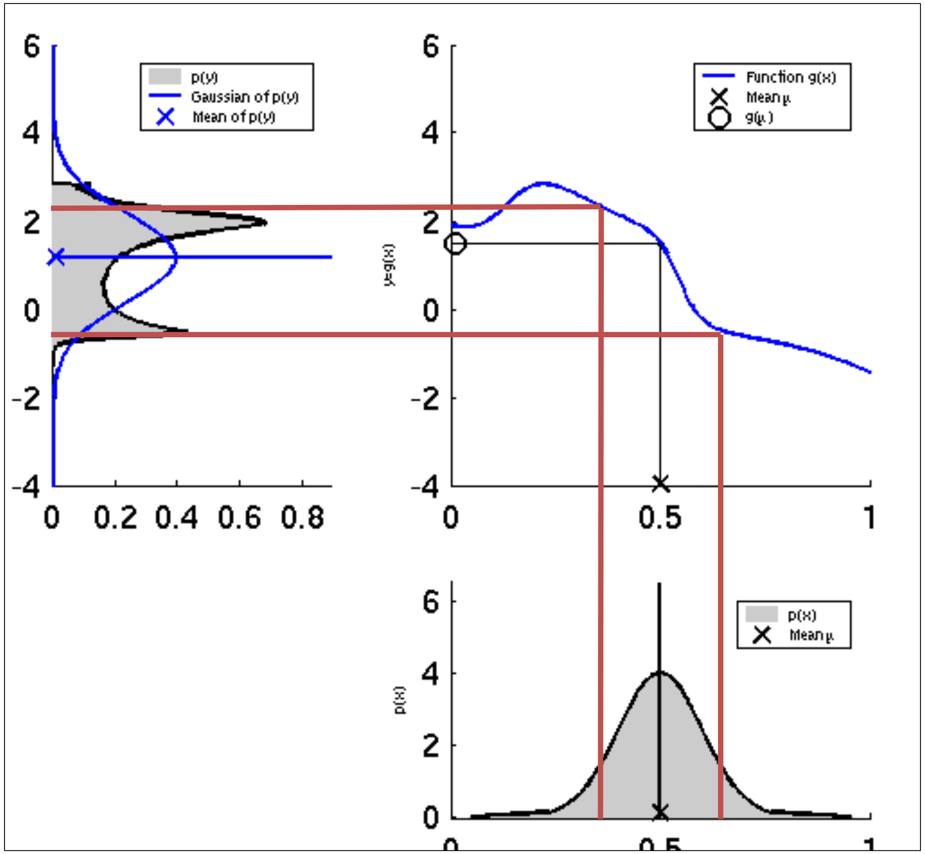


The extended Kalman filter

- Pros
  - Good approximation when models are near-linear
  - Efficient to calculate

  (de facto method for navigation systems and GPS)
- Cons
  - Only approximation (optimality not proven)
  - Still a single Gaussian approximations
    - Nonlinearity → non-Gaussianity (e.g. bimodal)
  - If we have multimodal hypothesis, and choose incorrectly – can be difficult to recover
  - Inapplicable when *f,h* discontinuous

The unscented Kalman filter



Yacov Hel-Or

- Can give more accurately approximates posterior

Challenges

- Detection specific
  - Full/partial occlusions
  - False positives/false negatives
  - Entering/leaving the scene
- Efficiency
- Multiple models and switching dynamics
- Multiple targets
- …

Conclusion

- Inference in time series models:
  - Past: smoothing
  - Present: filtering
  - Future: prediction
- Recursive Bayes filter optimal
- Computable in two cases
  - Linear Gaussian systems: Kalman filter
  - Discrete systems: Grid filter
- Approximate solutions for other systems