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Symbols for nothing:

Different symbolic roles of zero and their gradual emergence in Mesopotamia

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Abstract

Zero plays a number of different roles in our decimal place-value system. To allow for a nuanced discussion of the importance of zero, these roles should be distinguished carefully. We present such a differentiation of symbolic roles of zero and illustrate them by looking at the use of symbols for zero in ancient Mesopotamia. Old and Late Babylonian mathematicians used a place-value system (like ours, but with base sixty instead of ten), but did not use zeros in the way we do now. This shows that our current uses of zero are not a necessary consequence of the adoption of a place-value system and that the lack of a zero does not necessarily render a place-value system unusable.[1]

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1. Introduction

In recent years philosophy of mathematics has begun to pay more attention to mathematical practice, both contemporary and historical. This is evidenced by the recently edited collections by Ferreirós and Gray (2006), Mancosu (2008), and van Kerkhove (2009). With this development, the use of notation has also come to receive more attention. This is not to say that these philosophers adhere to any particular formalist philosophy of mathematics, but rather that they recognize the fact that notation is one of the main tools of working mathematicians. This point has been emphasized by Serfati (2005) and Grosholz (2007). In the present paper we take the emergence of zero as a case study through which to examine some particular aspects of the interplay between mathematical notation and conceptual innovations.

Zero is the number that has received by far the most attention from philosophers and other commentators on the history of mathematics. It is the 'superstar' among the integers, having many popular books written about it with catchy titles such as *The Nothing that Is. A Natural History of Zero* (Kaplan 2000) and *Zero: The Biography of a Dangerous Idea* (Seife 2000). Zero is singled out by both educators and cognitive scientists because of its extraordinary properties. These properties make explicit computations involving zero more difficult for children to learn. They also make accounts of mental representations of it more intricate than those of 1, 2, and 3, its neighbors on the natural number line (see Hughes,1986; and Fayol and Seron, 2005). Another reason why zero has attracted so much attention is that it is often discussed as an integral part of our familiar decimal place-value system, here referred to as the 'indic' system of numerals.[2] In fact, many people are so familiar with the latter that they readily mistake the number ten with the numeral '10'. Moreover, alleged advantages of place-value systems over other systems of numerals have often been attributed to the existence of zero and its particular role.[3]

In this paper we disentangle various notions that have been associated with the symbol for zero. This enables us to discuss the uses and the importance of zero in a more nuanced way than has been common in the literature. In particular, we will take a closer look at the use of zero in a historical precursor of our number system, the sexagesimal place-value system of the Babylonians. Insight into the mathematics of ancient Mesopotamia was greatly advanced by Neugebauer's monumental work (see Neugebauer 1973 [1935–37]). And more recently,much more material and many in-depth scholarly discussions have been published on the subject (see Robson 1999 and 2008; Høyrup

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[2] Traditionally, this has been called the Hindu-Arabic system, but it has neither anything to do with the Hindu religion nor did it originate with the Arabs. The present name was suggested to us by Brendan Gillon to reflect its origins.

[3] Halsted refers to zero as "an indispensable corner-stone of civilization" (1912, 20), while Dantzig calls the invention of zero "the turning-point in the development without which the progress of modern science, industry, or commerce is inconceivable" (1954, 35).

2002a; and Friberg, 2007). Based on these resources we show that historically there was no clear cut distinction between possessing a zero and not possessing no zero; rather there were various intermediate stages. Furthermore, we discuss the misconception that in the absence of a symbol for zero computations in a place-value system are cumbersome and excessively error-prone. According to Høyrup (2002a) and Proust (2000), various computation errors that can be found in ancient computations suggest that computations were not done relying on our customary paper and pencil algorithms, but with some other kind of device. It should be noted that such errors involve zeros much less frequently than one might imagine, which suggests that such devices did not depend on the presence of a symbol for zero in the underlying numeral system.

2. The many sides of zero

In discussions about the importance of zero and its role in mathematics it is important to distinguish different, and independent, aspects of this notion. Failure to do so almost inevitably results in ambiguities and misattributions. For this reason, we aim in the present section to disentangle and briefly discuss several aspects of zero . Our main focus will be on the role of a symbol for zero in a place-value numeral system.

When talking about the history of zero, one must first distinguish the *word* 'zero' and the now common s*hape* of zero, namely '0' (Menninger 1969). In addition, zero can also be understood as denoting the *concept* of absence, void, nothingness, or the cardinality of the empty set, e.g., to signify the result of taking away five apples from five apples. If zero is used in arbitrary computations, like 5+0=5 and 4×(3–3)=0, then it is intended to denote a full-fledged *number*.

It is worth pointing out that all of the above aspects of zero are independent of the particular numeral system that is being used. In other words, there is no logical reason why ancient Romans could not have introduced a symbol denoting the number zero and used it in their calculations.

A further aspect of zero, as we commonly understand it, is intimately connected with its role in a place-value system of numerals, like the decimal system most of us are familiar with. Because of this, the purported advantages of having a zero-numeral and of using a place-value system have been frequently presented as two sides of the same coin. A closer look, however, reveals that matters are not so straightforward and that even within a place-value system, different and independent aspects of zero can be individuated. We shall look in detail at the use of symbols for zero by the ancient Babylonians to illustrate these aspects and to show how they are independent In the discussion that follows,we distinguish five roles that a zero symbol can play in a place value-system.

2.1 Placeholder zeros

Although we assume that most readers are familiar with the decimal place-value system, we give a few notes about it which be useful for a proper understanding of the various aspects of zero discussed below. In general, a place-value system is structured around a specific *base* number b. To record the units from 1 to b–1, b–1 different basic symbols are needed, which are commonly written in the right-most place. In our system, which has a base of ten, these are the digits '1', '2', …, '9'. The place-value that a basic symbol in a numeral represents depends on the place in which it occurs: the value of the symbol written in the second position is to be multiplied by b, those in the third position by b^2, etc., and the value of the whole numeral is obtained by adding the various place-values of the symbols it is composed of. In the example given in Figure 1 we determine the value of '5536' in a base-10 system to be $(5\times10^3)+(5\times10^2)+(3\times10)+6 = 5536$.

Position:	…	4.	3.	2.	1.
Basic symbols:	…	**5**	**5**	**3**	**6**
Place-value:	…	5000	500	30	6
	…	$5\times b^3$	$5\times b^2$	$3\times b$	units

Figure 1: The place-value system (from right to left) with base b=10.

Intermediate zeros.

A slight complication arises in place-value systems, because not all numbers need factors in each of the places to be expressed. But as long as it is made clear that a place is empty no additional symbol needs to be introduced. For example, if numerals are always written on graph paper or within boxes, like

5		3	6

they can be read easily and unambiguously (in the example, as 5036). Similarly, if a counting board or abacus is used to represent numbers, an empty place is simply an empty column on the board. If the numerals are written linearly, an empty place can be marked by a blank space of some fixed width. Alternatively, an additional symbol, say '–', could be introduced to mark the empty places, so that the number in the example would be written as '5–36'. In the indic numeral system the symbol used to accomplish this task is the zero. Following Høyrup (2002a), we shall refer to a symbol that is used to mark empty places *within* a numeral as an *intermediate zero*.

Initial and Final zeros.

A very special case of an intermediate zero occurs if the empty places in a numeral are the right-most ones, as in '5000'. We call such occurrences of the symbol for the empty places *final zeros* (Høyrup 2002a). Similarly, empty places can also occur at the beginning of a numeral, such as in '0.005', and symbols marking empty places at the beginning will be referred to as *initial zeros* (Friberg 2007). In both of these cases the zeros determine the magnitude of the number and they can be omitted if the magnitude is indicated in some other way, e.g., by writing '5K', '5‰', or by saying 'five thousand.' In the case of rational numbers, initial zeros can be avoided altogether by writing the number as a fraction.

Nothing zeros.

An exceptional case of a numerical symbol is the one that signifies the absence of a unit. In the indic system of numerals this role is played again by zero. We shall refer to this particular kind of symbol as a *nothing zero*. As in the cases above, such a symbol can be employed even if it is not considered to stand for a number, but only as a placeholder.

As we shall see later, these aspects of zero are independent of each other and they do not need to be marked by the same symbol at all. The fact that we use the same symbol for all of these tasks obfuscates these distinctions; although, since initial, final, and nothing zeros can be construed as being special cases of intermediate zeros, there are also good reasons for using the same symbol.

2.2 Babylonian sexagesimal system: Separation zeros

The need for another kind of symbol arises specifically in the Babylonian sexagesimal system – a place-value system with base 60 that was developed over 4000 years ago in Mesopotamia. We shall discuss some aspects of the historical development relating to the use of a zero symbol in more detail below.

To represent sexagesimal numerals in this paper we use the standard convention of writing the numerals within a place as decimal place-value numerals and of separating the places by commas. For easier readability we also enclose such transcriptions in single quotation marks. Thus, the value of the numeral transcribed as '2,3' is (2×60)+3=123. In the literature, empty places (intermediate and final zeros) are usually transcribed as '00', regardless of how they are represented in the original tablet. A sexagesimal point is indicated by a semi-colon. Thus, a transcription of '1,00;30' denotes the number 60.5.

Since the Babylonian place-value system has a base of 60, 59 different basic symbols are needed for the units. However, instead of introducing 59 different new symbols for the units, the Babylonians used an additive numeral system with base 10 for these numerals, with small horizontal wedges for the units ('|'), and large vertical wedges for the tens ('〈'). Because in an additive system every symbol represents a fixed numerical value, no intermediate or final zeros are required at all in such a system. Through the combination of the sexagesimal system with an additive system for the units, a peculiar difficulty arises within the Babylonian place-value system. Consider, for example, the numeral '〈〈||,' which unambiguously stands for 22, and which consists of the additive components '〈〈' (20) and '||' (2). If these symbols are used in a place-value system, one needs to be able determine in which places its components belong. They could fill one place and thus stand for 22, but they could also be intended to be split across two places, with '〈〈' in one place and '||' in the other. In this case, the sexagesimal numeral is to be read as '20,2' representing 1202. (Indeed, without a symbol for intermediate zeros it is possible that the numeral could contain any number of empty places and thus be read as ‘20,00,2’ or ‘20,00,00,2’ and so on.) Since the Babylonians usually grouped the tens and units together and wrote them in certain easy identifiable patterns, it is practically impossible to interpret the ‘〈〈||’ as being split up in other ways (e.g., between the two vertical or the two horizontal wedges, resulting in ‘1,12’ or ‘21,1’). To disambiguate between the possible interpretations of such a numeral one could introduce a special symbol to mark the end of a numeral representing 10, 20, 30, 40, and 50. We refer to such a symbol as a *separation zero*. In the decimal place-value system there is no need for such a symbol, because only a single digit is used in each place-value position.

3. Zero in Babylonian computations

To illustrate the uses of the various symbols for zero in Babylonian mathematics we focus on the two historical periods from which many clay tablets have been preserved: Old Babylonia (roughly 2000 to 1600 BCE) and the Seleucid Era (300 BCE to 0). Surviving mathematical problem texts from Old Babylonia mainly deal with problems in line geometry, geometrical algebra and quantity surveying (Robson 1999, 102; also Neugebauer 1969, 44), whereas mathematical problems pertaining to astronomy were more common in the Seleucid period (Neugebauer 1969, 14).

3.1 Differences between Old and Late Babylonians

The Babylonians used different number systems for different purposes (Thureau-Dangin 1939) and claims about zero apply only to the sexagesimal place-value system. According to Thureau-Dangin, the Babylonians used this system as an academic system, while “the system of numeration commonly employed in Babylonia was not sexagesimal but a decimal one” (1939, 117). The sexagesimal system was used only for calculations and different bases were employed for metrological units (1939, 122, in particular footnote 74). According to Friberg, “it is well known that in the Akkadian language number words were decimal. In everyday life in Mesopotamia, decimal numbers were used for counting. Only well educated scribes knew how to count with sexagesimal numbers” (Friberg 2007, 182). Moreover, at least one metrological system used different symbols for 1 and 60: in the talent system, horizontal wedges were used for 1, and vertical wedges for sixty (Friberg 2007, 388). This familiarity with decimal systems might explain the curious use of the base-10 additive system to represent the basic symbols in the Babylonian sexagesimal system.

To appreciate the relevance of the different kinds of symbols for the aspects of zero introduced above, consider the Babylonian sexagesimal system without any symbols for intermediate, final, and separation zeros: How could the numeral ‘〈|’ be interpreted in this system? The most straightforward reading would yield ‘11’, but also the following are possible (among others): ‘10,1’ (=601), ‘11,0’ (=660), and ‘10,00,1’ (=36001). Given the relatively large differences in magnitude of these numbers, however, it is not too difficult to imagine that a skilled reckoner would know from the context of his calculations (e.g., a multiplication in a multiplication table, or the solution to a particular problem) how the numeral was to be interpreted correctly.

If we restrict ourselves only to an initial segment of the natural numbers, intermediate and final zeros occur in the Babylonian system much less frequently than in our decimal system. Since we are dealing here with natural numbers, we omit the consideration of initial zeros. Of the Babylonian numerals that represent the numbers from 1 to 215999 (greatest value of a sexagesimal numeral with 3 positions) only around 3% contain an intermediate or final zero, while these zeros occur in about 43% of the corresponding numerals in the indic decimal system. Two consecutive empty places are extremely rare in practice and we are not aware of any single numeral occurring in a Babylonian tablet that contains two consecutive empty places.

Old Babylonia

During the Old Babylonian period no particular symbol was used as a nothing zero. As an example, consider the tablet TMS 7 (Høyrup 2002a, 181–188). Here 20 is subtracted from 20 and in contrast to the other calculations within the tablet the answer to this problem is omitted. Thus, while the numeric answers to all the other problems within the tablet are stated explicitly, it looks like that the answer to this problem is simply ignored. In similar cases of such subtractions the text claims that the outcome is "missing" (Høyrup 2002a, 184 and 293).

The Old Babylonians also did not use initial or final zeros, and they did not have a sexagesimal point. Thus, for them the values 1, 60, 3600, etc., as well as 1/60, 1/3600, etc., are all represented by the same symbol: '|'.[4] In the decimal place-value system this would be analogous to having all values of 10^n, for any integer n, represented by the same symbol. This can complicate the interpretation of mathematical writings considerably. For example, in the Later Mesopotamia tablet Ash 1924.796 (Robson 2007, 156–159), the scribe writes out the list of squares from one to sixty, but in Robson's translation the list begins and ends by claiming that the square of 1 is 1, even though the context suggests that the second instance of this claim might be better interpreted as claiming that the square of 60 is 3600.

As a consequence of the lack of zero symbols, number magnitude had to be identified on the basis of context. This includes determining whether the number in question is a decimal or a whole number, and, if it is a mixed number, identifying where the decimal begins. Yet this does not appear to have been overly problematic for the Babylonians. Possibly this is because in a sexagesimal system the difference between each place value column is quite large. After all, we sometimes identify numbers based on context even today: if a colleague said that her house cost "four-fifty", it would (probably) be clear that she meant four hundred and fifty-thousand dollars, but if she said that the sandwich she ordered for lunch cost "four-fifty", we would likely understand her

[4] It is worth noting that although the numerals were the same, at least some the number words were not. Summerian and Akkadian both had different words for "one", "sixty" and "thirty-six hundred" (see Melville 2003).

to mean four dollars and fifty cents.

While the Old Babylonians did not have a symbol for intermediate zeros, they did have means for indicating them. An intermediate zero, i.e., an empty place-value column in the middle of a number, would be marked with a space (Neugebauer 1969, 20). Thus, for example, '45,00,2' would be written as '〈〈〈〈||||| ||'.

The Old Babylonians also used a separation zero, and this too was marked by a space. Theoretically this could lead to confusion about how to interpret spaces in numerals. However, it seems that in practice it generally did not. One way such confusion was avoided was by using smaller spaces to mark a separation between two place value columns and larger spaces to mark empty place value columns. An example of such uses of spaces can be seen in the famous tablet Plimpton 322 (Robson 2002). It is interesting that although this convention appears to have worked well for the Babylonians, it posed difficulties to modern translators. Indeed, interpretations of Plimpton 322 remain controversial. The difficulty of reading Old Babylonian numerals is exacerbated by the fact that not all tablets have multiple spaces of different sizes. Without a frame of reference it is difficult to determine what qualifies as a 'big' or 'small' space. To make matters worse, sometimes tablets have seemingly irrelevant spaces (Neugebauer 1969, 20).

Seleucid Era

During the Seleucid Era, more than a thousand years later, there still was no symbol in use for a nothing zero. And as in Old Babylonia, neither initial nor final zeros were used.[5] However, intermediate zeros were marked regularly by a symbol rather than just by a space (Høyrup 2002a*,* 294; Robson 1999). Moreover, in a select number of scribal schools during this era, the symbol for intermediate zero was also used to separate units of different columns, e.g., between '〈|' and '||' in the numeral '11,2' (Neugebauer 1941, 213–215). On this use it was placed between two numbers to indicate that they were in different place-value columns, without indicating an empty place-value column between them, i.e., as a separation zero. In this case, however, '30,4' would again be written using the same symbols as for '34'. Some scribes used a symbol for separation zeros during the Seleucid Era, but this was not adopted widely.[6]

In sum, the main change between Old Babylonia and the Seleucid Era was that intermediate and separation zeros came to be represented by a symbol (see Figure 2). This shows that the notions of initial and final zero are independent from those of

5 However, this claim is contested by Neugebauer (1969, 20), who holds that there are instances of the zero symbol being used as an initial zero.

6 An investigation of why this use was not adopted is beyond the scope of this paper. Otto Neugebauer, the translator who first noticed this use, writes: "From the purely mathematical point of view, this use of the 'zero' sign is doubtless a step backward in the development of a rational number notation, and it is therefore easy to understand that it was not generally accepted in astronomical texts" (Neugebauer 1941, 213-215).

intermediate and separation zero.

Kind of Zero	Old Babylonia	Seleucid Era
Nothing Zero	Not used	Not used
Initial and Final Zeros	Not used	Not used
Intermediate Zero	Represented by a space	Represented by a symbol
Separation Zero	Represented by a space	Represented by a symbol (sometimes)

Figure 2. Comparison of use of zero between Old Babylonia and Seleucid periods.

3.2 Computations

Extant computations

In the following discussion we distinguish between calculations and computations. A *calculation* is a series of operations that yields the solution of a particular problem. For example, finding a particular amount of wheat might involve first dividing the given area of land by the number of people, then multiplying the yield of a unit of land by the previously obtained result. We refer to each single step of such a calculation as a *computation*. Thus, a computation is usually the application of one of the basic arithmetic operations or some more advanced operation, like raising to some power or taking the square root.

If somebody found a contemporary elementary school mathematical notebook they would likely be able to reconstruct the algorithms used for the basic arithmetic operations. Additions would often contain auxiliary marks to record the carries from one column to the next and multiplications would show the carefully arranged intermediate results that have to be added up in order to obtain the final result of the calculation. In general, after a paper and pencil computation we do not end up with just the problem and the final result on the paper, but also with various additional marks and intermediate results, which were used during the calculation. However, such concrete evidence for paper and pencil computations found on Mesopotamian clay tablets is minimal.

Scholars have found that some of the Old Babylonian tables were frequently reused, suggesting that they contained only rough work for calculations. The description by Powell, who discovered such 'scratch pads' dating back to the third millennium BCE is slightly ambiguous as to the content of this work: "Calculations in sexagesimal notation were made on temporary tablets which were then moistened and erased for reuse after the calculation had been transferred to an archival document in standard notation" (Powell 1976). In fact, these "rough work" tablets (Robson 2009) do not contain explicit, written out computations, but only intermediate results of calculations. It thus seems safe to assume that the Babylonians did not use any of the algorithms we use today, but that they made use of some kind of computational device. From an analysis of computational errors, Høyrup (2002b) was able to infer some properties of the devices used during the Old Babylonian period.

Another indication for the use of an external computation device is that apparently no tablets with basic addition facts, like 3+4=7 and 9+8=17 have been preserved, whereas numerous tablets exist for multiplication tables. A possible explanation is that no such addition facts needed to be explicitly memorized, since they were computed on some abacus-like device.

Types of errors

For the discussion of computational errors it is useful to distinguish between several kinds of errors.

First, there is a distinction with respect to the effect a computational error has on a larger calculation in which it features. On the one hand, there are *propagating errors*,[7] which, once they occur, propagate through the remainder of the calculation yielding a false result. This kind of error indicates that the calculation carried on with the results of the erroneous computation. On the other hand, there are *isolated errors*, which occur within a calculation, but do not affect the end result. These errors give us some hints about how the calculation was made. Three possible explanations come to mind for such isolated errors:

a) The calculation was made on some kind of calculating device and only the intermediate results are recorded on the clay tablet. Thus, if a correct intermediate result is on the computational device, but it is copied incorrectly to the tablet, then the erroneous number has no effect on the overall calculation. This is a kind of *copying error*, which is mentioned only rarely in the secondary literature, but from which we can get some indirect hints about the devices used for the computations (see Høyrup 2002b).

b) Under the label ‘copying error’ commentators usually understand a mistake that is made when an entire tablet that contains a finished calculation is copied. Here, too, the calculation continues with what would have been the correct value despite the error, since the correct calculation is what is being copied.

c) A third explanation for isolated errors is that the calculation was in fact carried out in two directions: from the problem to the solution, and from the desired solution to the initial problem. The latter part of such a calculation is necessarily correct, since the result was actually its starting point, so that a computational error in the middle of the calculation would propagate only to the point where the top-down and bottom-up

[7] Most of the terminology of errors is taken from Friberg (2007).

calculations meet.

Our first distinction, between propagating and isolated errors, is based on the effect of the errors on the overall calculation. Errors can further be characterized by their origins. Where mistakes originate will differ for each particular notational system, since particular features of each system might encourage particular errors. For example, if the symbols used are very similar, it will be more likely for them to be mistaken for one another. Or if the columns of an abacus are too close to each other it might be easier for the pebbles to get moved inadvertently from one column to the next. Such mistakes can be broadly classified as *notational errors.*

Finally, we can also classify mistakes within a single computation according to their origins. For this purpose Friberg (2007) has introduced the notion of *telescopic errors* and *reverse telescopic errors.* Here, columns are erroneously added or removed in the course of a computation. The latter are particularly related to the possession or lack thereof of a symbol for zero, as we shall see next.

Babylonian error

Friberg (2007, 32–33) transcribed the tablet VAT 5457, which contains an error in the calculation of the product of '52,44,03,45' and '7,30'. Friberg conjectures that the error arises during the final step of this calculation, when four intermediate results are added incorrectly. During this addition, Friberg suggests that '22,30' and '5,37,30' are mistakenly moved one column to the left, while '6,30' and '5,30' remain properly positioned (see Figure 3 for the correct positioning of the intermediate results).

6	30				
	5	30			
			22	30	
			5	37	30
6	35	30	28	7	30

Figure 3: Intermediate results of '52,44,03,45' times '7,30'.

As a consequence, the "30" from '5,30', the "22" from '22,30' and the "5" from '5,37,30' are all found in the same column. The correct calculation would place the "22" and the "5" one column to the right. Thus, in the erroneous calculation the intermediate results are positioned incorrectly, causing the sum be telescoped inward to remove a place value column.

These so-called reverse telescoping errors are frequently attributed to the lack of initial and final zeros, since such zeros unambiguously show the number of place-value columns for each number (see, for example, Friberg 2007) and this information can be used to line up the columns correctly. Nevertheless, Høyrup (2002b) and Proust (2000) have argued that the Babylonians actually used a device similar to an abacus for their calculations. This would explain why they did not make that many mistakes involving zero. Indeed, the errors that are discussed by Friberg (2007) could arguably also be made on these kinds of devices and are thus not necessarily related to the presence or absence of symbols for intermediate zeros in a numeral system.

4. Concluding remarks

Even in the two main historical periods discussed, the use of the sexagesimal place-value system was far from consistent. For example, different tablets respond to calculations with nothing zero in different ways, giving different results to such problems. Or, during the Seleucid Era, not every tablet uses a symbol to represent separation zeros—some keep using spaces. The tablets that use a symbol for separation zero often use the same symbol as for intermediate zero. However, different tablets use different symbols for this purpose. Thus, not every tablet with a symbol for separation zero uses the same symbol for this purpose: some tablets would use two wedges, and others would use three wedges (Robson 1999, 156–159, 166–168, and 175–176.). In addition, these symbols also had other uses. For example, before two wedges were used for intermediate zero and separation zeros, they were used to mark the ends of sentences (Neugebauer 1941). In addition to this symbol's use as an intermediate zero, it was also used "as a word divider between numbers where [the scribe had] to break up the tabulation because the numbers are so long" (Robson 1999, 166).

We hope to have convinced the reader that, on the one hand, different uses of zero are independent from each other and, on the other hand, that using a place-value system is independent of having a symbol for zero. The independence of the different roles of zero also makes it difficult to pinpoint an exact time when zero was 'invented.' In fact, there isn't necessarily a clear distinction between having a zero and not having a zero. A number system can have a symbol for one role of zero, but not others—or a number system can employ all the roles of zero, but have them represented by different symbols. Furthermore, a careful look at the historical development of numeral systems reveals that number systems lacking a full-fledged zero need not be excessively error prone even if they employ a place-value system.

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