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Forthcoming in: *History and Philosophy of Logic*.

On Frege's Begriffsschrift notation for propositional logic: Design principles and trade-offs

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April 6, 2017

Abstract

Well over a century after its introduction, Frege's two-dimensional Begriffsschrift notation is still considered mainly a curiosity that stands out more for its clumsiness than anything else. This paper focuses mainly on the propositional fragment of the Begriffsschrift, because it embodies the characteristic features that distinguish it from other expressively equivalent notations. In the first part, I argue for the perspicuity and readability of the Begriffsschrift by discussing several idiosyncrasies of the notation, which allow an easy conversion of logically equivalent formulas, and presenting the notation's close connection to syntax trees. In the second part, Frege's considerations regarding the design principles underlying the Begriffsschrift are presented. Frege was quite explicit about these in his replies to early criticisms and unfavorable comparisons with Boole's notation for propositional logic. This discussion reveals that the Begriffsschrift is in fact a well thought-out and carefully crafted notation that intentionally exploits the possibilities afforded by the two-dimensional medium of writing like none other.

1 Introduction

The notation that Frege introduced in his *Begriffsschrift. Eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (A formula language of pure thought, modeled upon that of arithmetic) (Frege 1879b) is certainly one of the most curious systems of logic.[1] It easily stands out because of its two-dimensional design consisting mainly of connected horizontal and vertical strokes. Frege's innovations and contributions to both logic and philosophy, in particular the development of a formal system for quantificational logic and the analysis of concepts in terms of functions, are nowadays widely acknowledged (though this was not the case during his lifetime). Indeed, one commonly finds Frege being hailed as 'the father of modern logic' and the 'founder of analytic philosophy'. However, although some of his contemporaries like Peano and Russell became versed enough in Frege's notation to be able to translate it into their

[1] On the background of Frege's Begriffsschrift, see Kreiser (2001), in particular Ch. 3, pp. 135–275, and Sluga (1980, Ch. II).

own,[2] I am not aware of any author other than Frege who used this notation in any published work on logic.[3]

The most literal translation of 'Begriffsschrift' is 'concept script' and it has also been translated as 'conceptual notation' (Frege 1972), but it has become common to use the original German term to refer to Frege's notation. I shall follow this custom and write it in italics when referring to the title of Frege's book and without italics when referring to his notation. In accordance with the practice at the time, Frege himself used the term 'Begriffsschrift' not only for his own notation, but more generally in the sense of ideography or pasigraphy, e. g., when speaking of 'the justification of a Begriffsschrift' (Frege 1882b) or when distinguishing 'Peano's Begriffsschrift' from his own (Frege 1896).[4]

Let me briefly mention some of Frege's other works in order to provide some context for the development of his notation. After the groundbreaking publication of *Begriffsschrift*, Frege turned towards less technical expositions of his ideas of developing arithmetic on the basis of logic, most notably in his *Grundlagen der Arithmetik* (1884), towards more general reflections on philosophy of language, in particular 'Function und Begriff' (1891), 'Über Sinn und Bedeutung' (1892a), and 'Über Begriff und Gegenstand' (1892b), and he also started working on his *magnum opus*, the *Grundgesetze der Arithmetik*. The latter consists of the formal development of his logicist programme, i. e., to show that arithmetic and analysis could be completely reduced to logic. Frege himself acknowledges that 'a deep-reaching development of [his] logical views' (Frege 2013, x) took place during this time, but that the external appearance of his system hardly changed.[5] The first volume of the *Grundgesetze*, in which Frege used a modified version of his formula language, appeared in 1893. Shortly before the publication of the second volume (1903), Frege was informed in a letter from Russell about an inconsistency in his system (due to his Basic Law V), which eventually brought his efforts of providing a logicist foundation for mathematics to an end.

Despite the insurmountable difficulties that Frege saw for his logicist programme, Frege continued to give university lectures on the Begriffsschrift in Jena until 1917, a year before his retirement (Kreiser 2001, 280–284). Notes from the lectures held in the Winter semester 1910/11 and the Summer semester 1913, taken by Rudolf Carnap, have been preserved and were recently published (Frege 1996, Reck and Awodey 2004).[6]

The remainder of this paper consists of two main parts. In the first, systematic part, a brief introduction to the propositional fragment of Frege's Begriffsschrift is given with the aim of highlighting some idiosyncrasies of the notation and arguing that,

[2]See, e. g., (Dudman 1971, 30) and (Frege 1980, 148).

[3]But, see the reviews of Frege's books (Vilkko 1998), the published parts of Frege's correspondence (Frege 1980), and recent philosophical reflections (Macbeth 2014).

[4]See Barnes (2002) for various uses of the term 'Begriffsschrift'.

[5]See Cook (2013) for an excellent introduction to Frege's notation in *Grundgesetze*; for discussions of the differences between the 1879 and later versions, see Simons (1996), Thiel (2005), and the Introduction to Frege (2013).

[6]For some background on the audience of Frege's lectures, see Schlotter (2012).

despite popular opinion, the notation is quite advantageous in terms of perspicuity and readability. In the second, more historical part of the paper some of the debates regarding the advantages and disadvantages of the Begriffsschrift notation between Frege and his contemporaries, in particular Schröder, are examined. The main criteria that were used by the historical protagonists to assess notations for propositional logic will emerge in this discussion. Since these criteria are often incompatible with each other, the design of a particular notation is always guided by specific aims and involves various trade-offs. Because many historical arguments focused on those aspects of the Begriffsschrift that pertain to the propositional part of the notation, our restriction to propositional logic, while excluding some of Frege's seminal innovations like quantifiers and bound variables, allows us to focus on some contentious aspects of the notation without being distracted by other considerations.

2 The Begriffsschrift notation

2.1 Frege's conceptions of the Begriffsschrift

Frege understood his notation in ways that are substantially different from the modern truth-functional account of logic and, moreover, he changed his views between the publication of *Begriffsschrift* (1879b) and *Grundgesetze* (1893, 1903). One of the main differences in Frege's conceptions of logic concerns the meaning of the *content* or *horizontal* stroke. In 1879, a horizontal stroke written in front of an expression indicates that the expression forms a judgeable content; it 'binds the symbols that follow it into a whole' (Frege 1879b, 2, original in italics; quoted from Beaney 1997, 53). Such a judgeable content can either be denied or affirmed. That such a content is indeed affirmed, or held to be true, is a *judgment*, which is indicated by the *judgment* stroke, a small vertical stroke at the left end of the content stroke. Thus, '$\vdash\!\!-\, 3 \times 7 = 21$' expresses that the proposition '$3 \times 7 = 21$' is affirmed, while '$-\!\!-\, 3 \times 7 = 21$' merely expresses that '$3 \times 7 = 21$' is a judgeable content, without indicating whether it is affirmed or denied. This distinction between judgeable content and judgment is not expressed in the current standard notation for classical propositional logic.

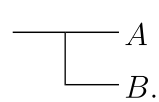
In Frege's 1879 conception of logic, attaching negation or conditional strokes to a content stroke are understood as modifying the content of the expression. However, by 1893 Frege had introduced a distinction between the truth value and the thought of a judgeable content, and, as a consequence, the horizontal stroke was then understood by Frege as signifying a function from a name to a truth value. Thus, '$-\!\!-\, 3 \times 7 = 21$' would refer to the True according to this view. Because Frege allows *all* names that refer to an object to be part of an expression, the horizontal must assign a truth value also to names such as '2'; for Frege, expressions that are not true are false, such that, for example, the expression '$-\!\!-\, 2$' denotes the False. This differs from the modern conception of propositional logic, where only *propositions*, i. e., statements that can be true or false, are considered as basic non-logical constituents of expressions.

In the following discussion we do not take into account these different conceptions that Frege had of the Begriffsschrift. Instead, we treat it as if it were a notation for modern, truth-functional logic. Thus, the variables are understood as propositional variables (i. e., they stand for propositions, not for judgeable contents or names) and we also do not consider the judgment stroke. The reason for this is that the topic of the present paper is not Frege's specific views on logic, about which a large body of scholarly literature exits by now, but particular notational aspects of the Begriffsschrift. For this, not all nuances of Frege's views, in particular the more philosophical ones, are relevant.[7] Indeed, such considerations can even distract from or stand in the way of analysing the cognitive and pragmatic aspects of the Begriffsschrift *as a notation*. We also do not consider those parts of the Begriffsschrift that involve quantifiers, bound variables, definitions, etc.; while these are, of course, important aspects of Frege's original and innovative contributions in their own right, they are not part of the propositional fragment of the Begriffsschrift. By separating the discussion from Frege's logical innovations it becomes easier to keep apart what is conceptual and what is purely notational.

2.2 Formulas in the Begriffsschrift

Let me now introduce the basic elements of Begriffsschrift formulas that represent the propositional connectives of implication and negation. After a brief discussion of two different readings of the Begriffsschrift, I shall turn to Frege's rules of 'interchange' and 'transposition' to obtain logically equivalent formulas and to the representations of conjunction and disjunction. The latter can be understood as complex symbols that are formed from the primitive elements.

Three different kinds of strokes suffice to represent formulas of propositional logic in Frege's Begriffsschrift notation: the *content* stroke (also called *horizontal*), the *conditional* (or *vertical*), and the *negation* stroke.[8] A negation stroke is a short vertical line, that, if attached to a horizontal stroke, changes the truth value from True to False, and from False to True. (In Frege's 1879 way of speaking, the negation stroke negates the content.) Thus, '—┬— $3 \times 7 = 21$' expresses that $3 \times 7 \neq 21$. The only primitive element that connects two propositions A and B is the vertical conditional stroke, as in:

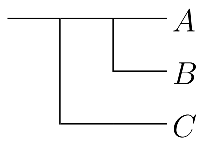


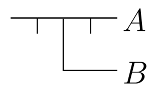
Frege introduces the conditional stroke by stating that to assert the above formula means that 'A is denied and B is affirmed' does not hold. Understood truth-functionally, it thus corresponds to material implication, '$B \to A$' in modern notation.

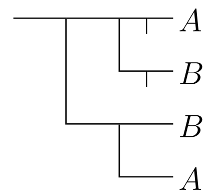
[7]Macbeth (2005) argues for a close connection between Frege's notation and his philosophy of logic, but we leave that aside here, too.

[8]The judgment stroke, mentioned above, will not be used in our discussion.

By combining these elements, complex formulas can be obtained. For example:

(1) 

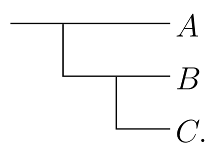
(2) 

(3) 

According to Frege's conventions for reading Begriffsschrift formulas, these formulas are true and false just in case the following are:[9]

$$(1') \quad C \to (B \to A) \qquad (2') \quad \neg(B \to \neg A) \qquad (3') \quad (A \to B) \to (\neg B \to \neg A).$$

When comparing (1) with (1′) we notice that the Begriffsschrift formula is intended to be read as if implication was right-associative, which is indicated by the parentheses in the modern representation. Thus, the order in which conditional strokes appear on the upper horizontal stroke determines the general structure of a formula. A negation stroke applies to the entire subformula that follows to the right of it. The left-most connective on the upper horizontal is therefore the main connective in a Begriffsschrift formula. Consequently, the formula '$(C \to B) \to A$' is rendered in the Begriffsschrift as



(At this point the reader is invited to practice translating some formulas from the modern representation to the Begriffsschrift, and vice versa.)

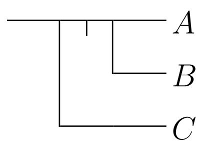
Secondary reading of sequences of vertical strokes. While I have interpreted the conditional stroke as material implication ('$\to$') in the above reading of (1), an alternative reading, suggested by Frege himself, is also possible. For this, it is useful to distinguish the *upper term* 'A' from the *lower terms* 'B' and 'C',[10] such that each vertical stroke on the upper horizontal determines a lower term. For example, in Formula (3) the lower terms are, in modern notation, '$A \to B$' and '$\neg B$', while the upper term is '$\neg A$'. In the alternative reading the lower terms are connected by conjunctions. For the Begriffsschrift formula (1), this yields the formula '$(C \wedge B) \to A$', which is logically equivalent to (1′). This reading is particularly advantageous in the case of multiple conditional strokes, because of the associativity of conjunction, which allows for the omission of parentheses in the modern formulation. Compare, for example,

[9] The numbering conventions for formulas are as follows: the label '(1′)', with a prime, denotes a formula in modern notation that corresponds to the Begriffsschrift formula (1), while subscripts, like '(1_a)' and '$(1'_a)$', indicate formulas that are logically equivalent to (1) and (1′), respectively.

[10] We follow here the terminology of Reck and Awodey (2004, 52). The English translation of *Grundgesetze* uses 'supercomponent' and 'subcomponent' (Frege 2013, 22).

$$(A_1 \to (A_2 \to \cdots (A_{n-1} \to (A_n \to C)) \cdots)) \text{ with } (A_1 \wedge A_2 \wedge \ldots A_{n-1} \wedge A_n) \to C.$$

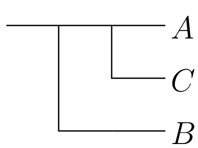
In fact, the latter is also closer to the way in which mathematical theorems are often expressed: C holds under the conditions $A_1, \ldots, A_n$. This alternative reading of the vertical strokes has led some commentators to regard this particular ambiguity as a characteristic feature of Frege's notation.[11] However, the two readings are not completely on par: the first reading is universally applicable, while the second one has exceptions. Namely, if there is a negation stroke on an internal segment of the upper horizontal stroke between two conditional strokes, then the second reading fails. For example, the formula



cannot be read as A being implied by a combination of conjunctions and negations of B and C. It therefore seems appropriate to consider the reading of the conditional stroke as implication the *primary* reading of a Begriffsschrift formula and the alternative reading as the *secondary* one. This view is also supported by Frege's own terminology of calling the vertical stroke the 'conditional stroke' ('*Bedingungsstrich*', Frege 1879b, 6).

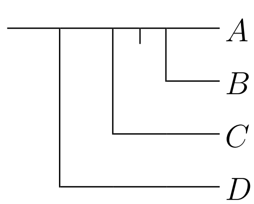
Rules of interchange and transposition. The above observations suggest some simple rules for obtaining logically equivalent formulas, which are discussed by Frege himself. For example, we have seen that $(C \wedge B) \to A$ is logically equivalent to Formula ($1'$), and by commutativity of conjunction also to $(B \wedge C) \to A$, which again is logically equivalent to $B \to (C \to A)$. In the Begriffsschrift notation, the latter is represented by

(1_a)



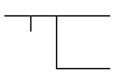
By comparing formulas (1) and (1_a), and considering the general principles underlying this transformation, Frege was led to the principle that in the Begriffsschrift notation 'The lower terms are interchangeable' (Reck and Awodey 2004, 52).[12]

However, analogously to the exception mentioned above regarding the secondary reading of Begriffsschrift formulas, also in the case of the formula



[11]In particular, Macbeth (2005); see also (Thiel 2005, 15–16) and (Moktefi and Shin 2012, 657–661).

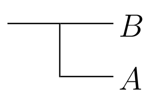
[12]See also (Frege 1972, 147) and (Frege 2013, 52).

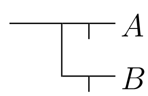
the terms B and C cannot be interchanged while retaining logical equivalence, because of the negation stroke between the conditional strokes.[13] Frege himself was well aware of cases like this and he remarks in his lectures that here B cannot be regarded as a lower term (Reck and Awodey 2004, 53). Thus, we recognize Frege's implicit understanding that 'lower terms' are only those subformulas that do not have a negation stroke on any internal upper horizontal stroke on the left of their connecting vertical stroke. In the case of the above example this means that only C and D are lower terms, while 

is the upper term.

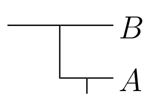
With this refined understanding of 'lower terms' we can now also formulate the restriction on the above mentioned secondary reading of Begriffsschrift formulas as follows: If $A_1, \ldots, A_n$ are the lower terms and C the upper term of a formula, then we can read the formula as $(A_1 \wedge \ldots \wedge A_n) \rightarrow C$.

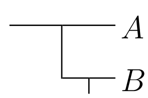
Frege also noticed that the formulas

(4_a) 

and (4_b) 

as well as

(5_a) 

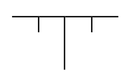
and (5_b) 

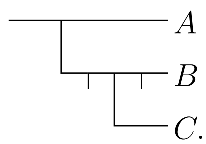
are logically equivalent and he called the modification of one into the other a *transposition*. As a general rule, it reads: 'In a transposition the upper term negated takes the place of the lower term, and the lower term negated, the place of the upper term' (Reck and Awodey 2004, 56). Note that this rule assumes that two consecutive negation strokes can be eliminated. (More on double negations later.) Together with the interchangeability of lower terms observed above, transposition can also be applied in the case of more than one lower terms to obtain logically equivalent formulas.

Complex symbols for conjunction and disjunction. It is an interesting feature of Frege's Begriffsschrift that certain combinations of primitive symbols (i. e., of long and short vertical strokes) can be interpreted as forming individual complex symbols that stand for other logical connectives. Because of the graphical nature of Begriffsschrift and the fact that all logical symbols are grouped together on the left-hand side of an expression, these complex symbols have their own perceptual features that make them especially conspicuous and easy to identify at a glance. Such complex symbols are also referred to as 'chunks', i. e., meaningful units formed from collections of simpler elements, in the literature on the psychology of expert reasoning.[14] Frege noted that certain sequences of strokes can be interpreted as forming complex symbols

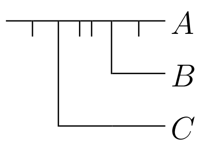
[13]This restriction is not explicitly discussed in the presentations of the Begriffsschrift by Macbeth (2005) and Cook (2013, A-8).

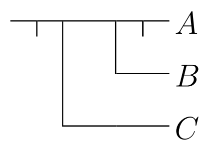
[14]See, for example, Miller (1956) and Chase and Simon (1973).

for conjunction and disjunction, which allows for a direct reading of these connectives in Begriffsschrift. For example, it was clear to Frege that the truth conditions for Formula (2) are exactly those of the natural language connective 'and'.[15] In other words, a sequence of negation-conditional-negation strokes (i. e., ) in a Begriffsschrift formula can also be interpreted as a complex symbol standing for logical conjunction ($\wedge$). So, while we've seen that Formula (1) can be read as $(C \wedge B) \rightarrow A$, the latter can also be represented directly in Begriffsschrift notation as

(1_b) 

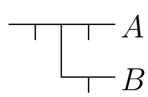
The representation of multiple conjunctions, for example in the formula '$C \wedge B \wedge A$' (or, more exactly, '$C \wedge (B \wedge A)$'), can be simplified in the Begriffsschrift notation by exploiting the fact that two consecutive negation strokes cancel each other out (the negation of the negation of A is A itself), as Frege observed (Frege 1893, 23):



is logically equivalent to


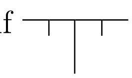
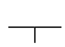
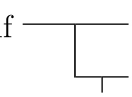
Because the disjunction '$A \vee B$' is logically equivalent to '$\neg A \rightarrow B$', it can be represented by Formula (5_a). Thus, analogously to the case of conjunction, we can consider the combination of a vertical stroke with a negation stroke on its lower leg as a single complex symbol that stands for disjunction.

It follows from the above considerations, that in the formula

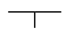
(6) 

the vertical stroke can be interpreted individually or as being a constituent of two different complex symbols. These three different interpretations of the vertical stroke yield three different representations in modern notation:

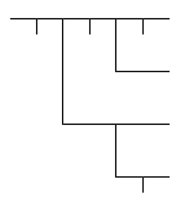
($6'_a$) $\neg(\neg B \rightarrow \neg A)$ ($6'_b$) $\neg B \wedge A$ ($6'_c$) $\neg(B \vee \neg A)$

Formula ($6'_a$) results when the vertical stroke of (6) is read as a conditional; if  is read as the conjunction of  B and A, then we get ($6'_b$); finally, if  is read

[15]See (Frege 1879b, § 7), (Frege 1880/81, 12), and the discussion of 'and', 'neither—nor', and 'or' in (Frege 1893, § 12).

as the disjunction of B and  A, which is itself negated by the left-most negation stroke on the upper horizontal, then we obtain ($6'_c$). These interpretations clearly exhibit how conjunction and disjunction are related to implication and negation. If separate symbols are used for each connective, these relations between the connectives are not made explicit by the notation itself.[16]

Another illustrative example for the various readings of complex symbols is the formula that Frege proposed for exclusive disjunction (Frege 1879b, 12):

(7) 

Using only implications and negations, this formula can be straightforwardly rendered as

$$\neg((\neg B \to A) \to \neg(B \to \neg A)).$$

In addition, interpreting the first negation–implication–negation sequence as a conjunction yields

$$(\neg B \to A) \wedge (B \to \neg A),$$

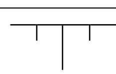
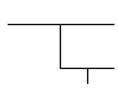
which is Frege's own reading. Alternatively, interpreting the second negation–implication–negation sequence as a conjunction, we get

$$\neg((\neg B \to A) \to (B \wedge A)).$$

Finally, we could also interpret the lowest branch (including the negation stroke) as the symbol for disjunction, $B \vee A$. Thus, in all three of the above formulas, we can replace $(\neg B \to A)$ by $B \vee A$, yielding, for example, together with the previous formula:

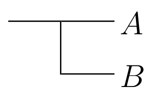
$$\neg((B \vee A) \to (B \wedge A))$$

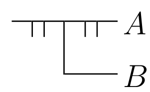
Double negations. As already mentioned above, adding two negation strokes to a content stroke in a formula does not change the truth value of the formula.[17] This yields a further way in which we can systematically obtain logically equivalent formulas.

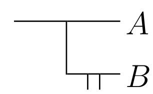
[16] The interpretations of  and  as conjunction and disjunction, are not to be confused with the primary and secondary readings of the Begriffsschrift discussed above; here, local combinations of conditional and negation strokes are interpreted as a unit (a complex symbol) representing a particular connective. For the secondary reading, whether a vertical stroke stands for an implication or a conjunction depends on the position of the stroke within the formula. This issue is taken up again in Section 2.3.

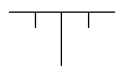
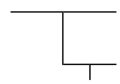
[17] In his letter to Anton Marty (August 29, 1882), Frege transforms a formula by adding two negation strokes (Frege 1980, 101–102); see also (Frege 2013, 23).

While these formulas might at first just look more complicated because they consist of a greater number of strokes, this modification can in some cases lead to expressions that can be easily interpreted in terms of conjunctions and disjunctions. For example, the addition of pairs of negation strokes to the horizontal strokes in Formula (8) yield the following logically equivalent Begriffsschrift formulas (8_a) and (8_b):

(8) 

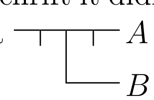
(8_a) 

(8_b) 

These syntactical modifications now allow for different direct readings of the formulas. Recall, that  can be read as '$\wedge$', and that  can be read as '$\vee$'. Thus, considering these complex symbols, we can translate the above formulas into modern notation as:

$$(8') \quad B \to A \qquad\qquad (8'_a) \quad \neg(B \wedge \neg A) \qquad\qquad (8'_b) \quad \neg B \vee A$$

To summarize, as illustrated by the formulas ($8'_a$) and ($8'_b$), each implication can be transformed into a conjunction by adding a pair of negation strokes to the left and to the right of the conditional stroke, and into a disjunction by adding a pair of negation strokes to the horizontal that begins at the bottom of the conditional stroke.

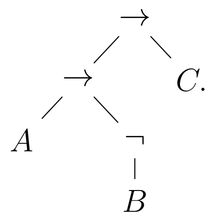
Discussion. The above considerations show that, despite Frege's conscious decision to use only two primitive notions, namely implication and negation, represented by the vertical stroke and the short vertical stroke respectively, the system can easily express conjunctions and disjunctions as complex symbols. In fact, because these connectives are not introduced by new primitive symbols, but by combinations of the primitive symbols, the same Begriffsschrift formula can be given different interpretations. The graphical nature of the notation makes this particularly suggestive. When I introduced students to the Begriffsschrift it didn't take them very long to naturally switch between interpreting the formula 

as $\neg(B \to \neg A)$ and $B \wedge A$, but this connection is not so obvious when the modern notation is used.

2.3 On the structure of Begriffsschrift formulas

2.3.1 The Begriffsschrift and syntax trees

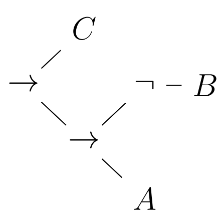
In order to parse and understand a propositional formula, presented in any notation, the reader has to be able to perform two main tasks: to identify the main connective of the formula and to determine the subformulas. The understanding of the subformulas is then achieved by applying the same reasoning recursively. A particularly transparent way of representing logical formulas is in terms of *syntax trees*, because here the main connective of a formula is the top node and the subformulas are simply the subtrees of a node. For example, the formula $(A \to \neg B) \to C$ is represented by the following syntax tree:



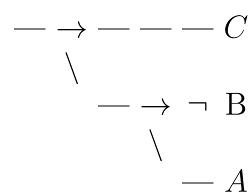
To determine the main connective of this formula written in the modern notation, i. e., '$(A \to \neg B) \to C$', one has to parse the entire formula while at the same time keeping track of the depth of parentheses. For short formulas like this one, which can be parsed at a glance, this process does not appear to be particularly cumbersome, but the difficulties become clear if expressions that span over multiple lines are to be read. Finding the main connective in a syntax tree is simply a matter of looking at the top node, regardless of the complexity of the entire formula. Similarly, the subformulas can be easily individuated, because they are simply the subtrees to the left and to the right of the top node; no keeping track of parentheses is necessary.

With the previous considerations I hope to have been able to convince the reader about the perspicuous way in which syntax trees represent propositional formulas. If this is the case, the reader might be surprised to learn that exactly the same features that make syntax trees a transparent representation of formulas are also present in the Begriffsschrift. In fact, by turning a syntax tree 90 degrees in an anti-clockwise direction and shifting the root node to the top, it is converted into the corresponding Begriffsschrift formula! For example, the above syntax tree corresponds to the following:

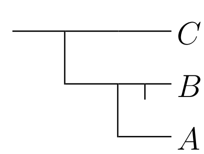
Syntax tree, rotated:



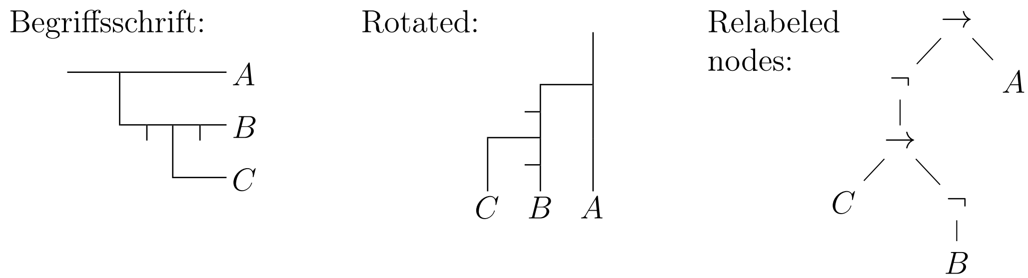
Root shifted to the top:



Begriffsschrift:

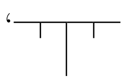
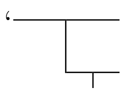


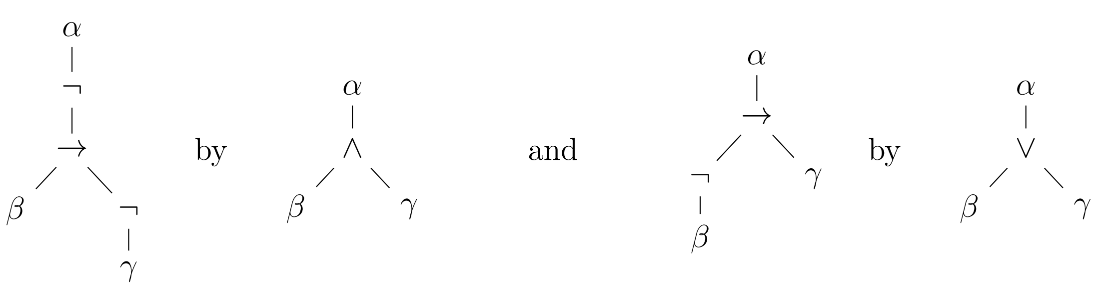
Conversely, to obtain a syntax tree from a Begriffsschrift formula, all we need to do is to rotate the formula clockwise 90 degrees and then relabel the nodes, such that a conditional stroke becomes a $\to$-node and a negation stroke becomes a $\neg$-node:



Thus, if we agree with the above, that syntax trees offer a clear and concise representation of formulas of propositional logic, and notice the very close connection between the tree representation and Frege's Begriffsschrift, we must conclude that the Begriffsschrift is also a representation that displays the structural connections within a formula in a perspicuous way. Any further considerations about syntax trees carry over almost directly to the Begriffsschrift, and vice versa.

2.3.2 On complex symbols and alternative readings

With the help of syntax trees we can now shed some more light on our earlier discussions of interpreting collections of primitive symbols as individual complex symbols and of the secondary reading of Begriffsschrift. In terms of syntax trees, the interpretation of the negation-implication-negation sequence '' as a complex symbol that stands for conjunction and of the implication-negation sequence '' as a symbol for disjunction is tantamount to replacing



respectively, in the syntax tree of a formula (where β and γ are subformulas and α is a possibly empty, additional part of the syntax tree). The fact that in Begriffsschrift all logical symbols are gathered on the left-hand side of an expression, just as they appear above the propositional variables in a syntax tree, allows for these replacements, which are *local*, i. e., characterized by affecting only a small contiguous part of the tree, and *context-free*, i. e., independent of the structure of the tree and of the particular values of α, β, γ. These features guarantee that the primitive symbols form a perceptual unit that corresponds to the semantic unit (which is in this case a non-primitive logical connective), which justifies the chunking of complex symbols in Begriffsschrift.[18]

Let us compare the situation in a modern linear notation, restricted to symbols for implication and negation. To make replacements that are analogous to the ones just mentioned, say for the case of conjunction, one would have to interpret the subformula '$\neg(\beta \to \neg\gamma)$' as '$(\beta \wedge \gamma)$'. This, however, is not just a matter of replacing a sequence of adjacent symbols by another symbol, but requires the identification of the pattern

$$\ldots\ \neg\ (\ \ldots \to \neg\ \ldots)\ \ldots$$

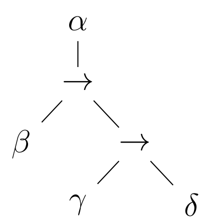
within the formula and the processing of the parts denoted by the ellipses. Here the two inner ellipses can stand for arbitrarily complex subformulas β and γ and the two outer ones make up the remaining parts of the formula (corresponding to α in the above syntax trees). Thus, the symbols to be replaced (or chunked as a complex symbol) do not form a perceptual unit, but can in fact be very distant from each other in the formula. In other words, one has to take into consideration the global structure of the formula to identify the pattern. As a practical consequence, noticing the pattern might not be as immediately obvious as it is always in the case of Begriffsschrift or syntax trees. This comparison between Begriffsschrift and the modern notation shows that the interpretation of collections of primitive symbols as complex symbols is not just a matter of having only implication and negation as primitive symbols, but also depends on structural features of the notation, namely the way in which the logical symbols are arranged in the formulas.

We have seen that in Begriffsschrift it is the two-dimensionality of the notation that allows for the gathering of logical symbols into complex symbols. Because of this, one might be tempted to think that two-dimensionality is a necessary condition for this chunking of complex symbols in notations for propositional logic. However, this is not the case, as the following example using Polish notation restricted to symbols for implication ('C') and negation ('N') shows. Here, '$\neg\,\beta \to \gamma$' is represented as '$CN\beta\gamma$'. This grouping of the connectives is local and non-contextual, so that we can consider 'CN' to be a complex symbol that stands for disjunction, illustrating how complex symbols can be introduced also in linear notations for propositional logic. Note, however, that in this notation the locality condition is violated in the case of '$\neg(\beta \to \neg\,\gamma)$', which is represented as '$NC\beta N\gamma$': the primitive logical symbols are not adjacent to each other, but separated by the arbitrarily complex subformula β. Thus, we cannot introduce a complex symbol for conjunction in this notation. This discussion has shown that whether complex symbols can be introduced in a notation or not, depends on very particular structural features of the notation.

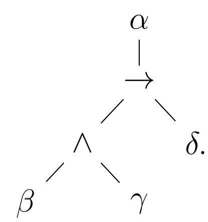
Considering the syntax tree representation of formulas also reveals how the introduction of complex symbols for conjunction and disjunction in Begriffsschrift is structurally different from the secondary reading of sequences of conditional strokes discussed in Section 2.2. Take, for example, the secondary reading of Formula (1) as

[18] On the notion of chunking, see the references in Footnote 14.

$(C \wedge B) \rightarrow A$. This can be illustrated as replacing one syntax tree by another as follows:



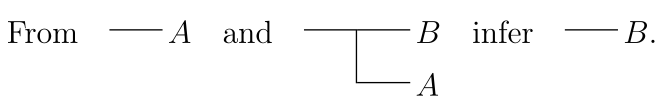
by



In contrast to the earlier replacements used to illustrate the introduction of complex symbols, we notice here that a collection of logical connectives is not simply replaced by a single one, but that this change also affects the structure of the connections. Although this change is local, in that it affects only an adjacent sequence of logical symbols, it does not involve the replacement of a group of symbols by a single one, and whether an implication symbol can be replaced by a conjunction symbol depends on the context, namely the presence of another implication symbol. The different structure of the syntax trees is reflected by the change in parentheses in the representations of the primary and the secondary readings in the modern notation. Thus, the distinction made above between the secondary reading of sequences of vertical strokes and the interpretations of complex symbols for conjunction and disjunction is based on underlying structural differences, which are displayed in a clear way in the representation of formulas in terms of syntax trees.

2.4 The Begriffsschrift calculus

To complete the presentation of the Begriffsschrift notation we now take a brief look at inferences in Frege's system for propositional logic. To move beyond the transformation of a formula into a logically equivalent one to the deduction of a conclusion from given premises, where the content of the conclusion differs from that expressed in each of the premises, Frege chose to adopt a single rule of inference, namely *modus ponens*:



So, to make an inference involving a conditional formula one must construct a formula that corresponds to the lower term, which is then detached from the conditional formula by one application of the *modus ponens* rule.[19] Using this inference rule (and an implicit substitution rule that allows for arbitrary formulas to be substituted for the propositional variables), Frege's rule of transposition can be expressed as an axiom, namely as Formula (3), above. Indeed, as Frege argued, any additional rule of inference that one would want for his calculus can be reformulated in a similar way as an axiom, so that modus ponens is sufficient to cover all possible inferences. He also noted that

[19] In fact, this rule of inference is sometimes referred to as 'rule of detachment' (Tarski 1994).

this might not be the most practical way of setting up a calculus, but practicality was not his main concern.

The logical calculus presented in *Begriffsschrift* is based upon nine axioms, which Frege refers to as propositions that form the 'Kern' (kernel or core) of his presentation (Frege 1879b, 26).[20] Three of these involve only implication, three contain also the negation symbol, two are for the identity symbol, and one is for the universal quantifier. Thus, for the calculus of propositional logic (without identity), only six of these axioms are needed (namely, propositions 1, 2, 8, 28, 31, and 41 in Frege 1879b). They are presented here in modern notation:

a) $A \to (B \to A)$

b) $\big(C \to (B \to A)\big) \to \big((C \to B) \to (C \to A)\big)$

c) $\big(D \to (B \to A)\big) \to \big(B \to (D \to A)\big)$

d) $(B \to A) \to (\neg A \to \neg B)$

e) $\neg\neg A \to A$

f) $A \to \neg\neg A$

Although Frege did not have the modern notions of semantics and completeness at his disposal, he was fairly confident that his system would be strong enough to allow for the derivation of all logical truths (Frege 1879b, 25–26). Indeed, in 1934 Łukasiewicz showed that Frege's system, using modus ponens and substitution, is complete. He also showed that the third axiom can be deduced from the first two and that the remaining axioms are independent.[21]

3 On Frege's design decisions for the Begriffsschrift

We turn now to the historical part of the present investigation. After its publication, Frege's *Begriffsschrift* was reviewed several times. The most critical was by Ernst Schröder (1881), who had just published a book on the algebra of logic (Schröder 1877) based upon the work of George Boole (Boole 1854). Frege submitted for publication two replies to Schröder, 'Booles rechnende Logik und die Begriffsschrift' (1880/81) and 'Booles logische Formelsprache und meine Begriffsschrift' (1882a), both of which, however, were rejected by the journals and were published only posthumously. Feeling misunderstood by this reaction to his work, Frege published two short pieces in which he explained the purpose of and justification for his Begriffsschrift (Frege 1882b, 1882/83). Also the publication of the *Grundgesetze* (1891) prompted a debate between Frege and a reviewer, this time Giuseppe Peano (Frege 1896, Dudman 1971), and Frege returned to this discussion in the second volume of *Grundgesetze* (Frege 1903, §58).

Thanks to these discussions between Frege and his critics, we get a more detailed picture of the arguments that were put forward against the Begriffsschrift notation and the reasoning that Frege himself employed to justify its design. As we shall see, one of the recurring points that Frege makes in his discussions with colleagues is that the purpose of his investigations is very different from those of both Boole and Peano, and

[20] For a discussion of this and other metaphors for mathematics, see Schlimm (2016).

[21] See Łukasiewicz (1967); also Thiel (1968, 21) and Frege (1972, 73).

that this accounts in part for the differences between their notations. Other themes that are treated in these debates include the issues of familiarity with a notation, the choice of primitives and inference rules, and the two-dimensional layout.

3.1 Frege's aims with the Begriffsschrift

At the time of the publication of *Begriffsschrift*, the most recent advances in symbolic logic had their origin in Boole's *An Investigation of the Laws of Thought* (1854). It should, therefore, be no surprise that the Begriffsschrift was frequently compared with Boole's symbolic language or with some more recent developments of it. Moreover, in his book Frege himself did not refer explicitly to any of these developments in logic, which, understandably, earned him some criticism.[22] As a consequence, Frege made an effort to compare and contrast his approach with that of Boole and his followers in his replies.

The starting points for both approaches were identical, according to Frege: 'the first problem for Boole and me was the same: the perspicuous representation of logical relations by means of written signs' (Frege 1880/81, 14). However, their further aims led to strikingly different decisions regarding the details of their respective representations.

Throughout his texts, Frege is very clear about the fact that any assessment of a notation must be relative to its purpose, such that 'the same notation or stipulation can seem appropriate or inappropriate depending upon one's purposes' (Frege 1896, 1). Thus, since different notations could each serve best their respective goals (Frege 1880/81, 14), he notes:

> It would not be surprising and I could happily concede the point, if Boolean logic were better suited than my Begriffsschrift to solve the kind of problems it was specifically designed for, or for which it was specifically invented. (Frege 1880/81, 39; adapted from Frege 1979, 39)

To allow for a meaningful comparison of different notations, it is therefore crucial for Frege to stress that his goals in devising a formal language were quite different from those of Boole, with whom he had been compared unfavorably by Schröder. In Frege's view, Boole's aim was to 'present the logical form with no regard whatever for the content' (Frege 1882a, 47); in other words, Boole wanted to 'present an abstract logic in formulas', whereas his own concern was 'to express a content through written symbols in a more precise and perspicuous way than is possible with words' (Frege 1882/83, 97; quoted from Frege 1972, 90–91).[23] More concretely, Frege states:

> I wanted to supplement the formula-language of mathematics with signs for logical relations so as to create a Begriffsschrift which would make it

[22] For example, in the reviews by Michaëlis (1880, 218), Schröder (1881, 83), and Venn (1880).

[23] Both Frege and Schröder traced their notations back to Leibniz. However, we leave this part of the debate aside, because both sides claimed their own system to be a *lingua characteristica* and criticized the other system to be merely a *calculus ratiocinator*. For a historically informed discussion of this issue, see Peckhaus (2004).

> possible to dispense with words in the course of a proof, and thus ensure the highest degree of rigour whilst at the same time making the proofs as brief as possible. (Frege 1882a, 47; adapted from Frege 1979, 47)

We see here that Frege's main concern was to express the logical relations in mathematical texts that are usually presented in ordinary language with his Begriffsschrift. In this way the genuine mathematical content could be expressed more precisely and in a more clear and transparent way. Frege's emphasis on the rigour of mathematical inferences was addressed already in the very first paragraph of *Begriffsschrift*, where he explained that he was not able to attain the level of precision needed for establishing gap-free inferences (i. e., such that no inference step would not rely on any unstated assumptions or intuitions) as long as they were formulated in ordinary language. Since this difficulty is exacerbated as the expressions become more complex, it was exactly this inadequacy of ordinary language that led him to develop his Begriffsschrift (Frege 1879b, IV). Because the Begriffsschrift was intended to be used together with ordinary mathematics and not just with arbitrary propositions A and B, any use of common mathematical symbols would have led to ambiguities and thus had to be avoided from the start; for this reason Frege explicitly notes that his content stroke is longer than the minus symbol (Frege 1882b, 101). We shall come back to the interplay between logic and mathematics in the discussion below, in particular regarding the character of the logical primitives and inference rules (Sections 3.3 and 3.4), and the two-dimensional layout of the Begriffsschrift (Section 3.5).

In contrast to Frege, Boole could avail himself to ordinary mathematical symbols in his language, because for him *only* the logical form mattered. This also allowed him to draw attention to the fundamental similarities between logic and algebra:

> There is not only a close analogy between the operations of the mind in general reasoning and its operations in the particular science of Algebra, but there is to a considerable extent an exact agreement in the laws by which the two classes of operations are conducted. (Boole 1854, 6)

Accordingly, this approach is generally referred to as 'algebra of logic' (e. g., Burris and Legris 2015). These different aims of Frege and Boole have also been acknowledged by later logicians. For example, C. I. Lewis distinguished their approaches as yielding different kinds of logic, referring to Boolean logic as 'symbolic logic' and to Frege's approach as 'logistic'.[24] Echoing Frege's own characterization of the difference, he notes that the development of the latter is determined 'not from abstract logical considerations or by any mathematical prettiness, but solely by the criterion of application', in particular the application to mathematics (Lewis 1918, 116).

[24] The term 'logistic' was introduced in 1904 in French as 'logistique' by Gregorius Itelson, André Lalande, and Louis Couturat at the 2nd Congress of Philosophy at Geneva (Peckhaus 2009, 186). It figured prominently in a series of papers by Russell, Poincaré, and Couturat in the *Revue de Métaphysique et de Morale* in 1905–06. Couturat's contribution appeared in an English translation as Couturat (1912).

3.2 Familiarity with notations

Anybody familiar with logical systems in the 19th century would have realized at the very first glance that Frege's notation differed radically from all others. And although such an observation could stir a reader's curiosity, more often than not it sparked aversion. Frege himself considered the possibility that readers might be 'frightened off by the first impression of unfamiliarity' already in the Preface to his *Begriffsschrift*, but he also expressed the hope that this might not in the end lead to a rejection of his innovations (Frege 1879b, VII). Unfortunately, however, he seems to have overestimated the goodwill of his readers.

There is some disagreement among scholars on the immediate reception of Frege's book. While Bynum writes about the 'tragic' and 'unfavorable' reception (Frege 1972, 17, 76), Vilkko has recently argued that one should not speak of a generally negative or even hostile reaction to the *Begriffsschrift* (Vilkko 1998). Nevertheless, Vilkko points out that one issue that the six reviews and one brief discussion in a book have in common is a criticism of Frege's notation. For example, Michaëlis' generally positive review describes the notation as making 'a strange and chilling impression' (Michaëlis 1880, 232) and Tannery describes it as 'excessively complex' (Tannery 1879, 108). John Venn's short review, however, must have been the most infuriating for Frege. It concludes with the following words:

> I have not made myself sufficiently familiar with Dr. Frege's system to attempt to work out problems by help of it, but I must confess that it seems to me cumbrous and inconvenient. (Venn 1880, 237)

As a consequence, three years after its publication Frege felt that this original hope had not materialized and that this general distaste for his notation would make it more difficult to convince others about the usefulness of his Begriffsschrift. In a letter from August 29, 1882, he clearly formulates this dilemma:

> I find myself in a vicious circle: before people pay attention to my Begriffsschrift, they want to see what it can do, and I in turn cannot show this without presupposing familiarity with it. (adapted from Frege 1980, 102)

Indeed, even mathematicians who were favourably inclined towards reading the *Begriffsschrift* found the notation extremely daunting, as can be gleaned from the three following excerpts from Frege's correspondence. After following Russell's recommendation to read the book, Louis Couturat wrote to Frege:

> To tell the truth, on first approach they [i. e., your works] are not very inviting, and the symbolism they employ makes them difficult to read. I still cannot boast that I can read them fluently or that I understand them. (Letter to Frege, February 11, 1904; Frege 1980, 13)

Moritz Pasch, who shared Frege's concern for the foundations of mathematics and the need for rigorous expositions, did not get very far in reading Frege's *Grundgesetze*, because he did not find the necessary time to learn the Begriffsschrift:

> [...] I immediately picked up your book and read it, as much as this is possible without a mastery of your Begriffsschrift; the latter is impossible for me, given my age and the heavy demands on my time. (Letter to Frege, January 11, 1903; Frege 1980, 105)

Finally, similar remarks were also made by Hugo Dingler, who wrote more than another decade later, on June 27, 1917, to Frege:

> I bought a copy of your *Begriffsschrift* in 1905, but because of the way you presented your views I was unable to gain the degree of enlightenment about them I had hoped for. (Frege 1980, 26)

So, while some of Frege's contemporaries, like Peano and Russell, learned the Begriffsschrift notation well enough to translate it into their own symbolisms, it seems that for others the hurdle presented by Frege's original representation was just too high. We surmise that the unfamiliarity with formal systems in general and with Frege's idiosyncratic presentation in particular was indeed one of the main obstacles. Thus, we agree with Bocheński's assessment:

> The fate of Frege's work was in part determined by his symbolism. It is not true that it is particularly difficult to read, as the reader can assure himself from the examples given below; but it is certainly too original, and contrary to the age-old habits of mankind, to be acceptable. (Bocheński 1961, 268)

Note that, as I have argued in Section 2.3, the Begriffsschrift notation displays all the same advantages as the presentation of logical formulas in terms of syntax trees. Thus, the rejection of the Begriffsschrift seems to be in part due to a purely psychological factor, namely the deviation from more familiar modes of presentation.

Concrete aspects of Frege's notation that were also criticized by his contemporaries concern his choice of primitives and inference rules. To these we turn next.

3.3 Economy and character of primitives

Contemporary logicians are familiar with the possibility of formulating systems of propositional logic with different primitives and the notion of a *minimal set* of connectives has become standard in introductory textbooks.[25] At the time of Frege's writings, however, the situation was very different. Boole and his followers had always included conjunction and disjunction among the main connectives, and Carnap reports that he and his friend were fascinated by the idea that they learned in Frege's lectures of 1910, of being able to represent all logical expressions using only two connectives, negation and implication (Reck and Awodey 2004, 19).

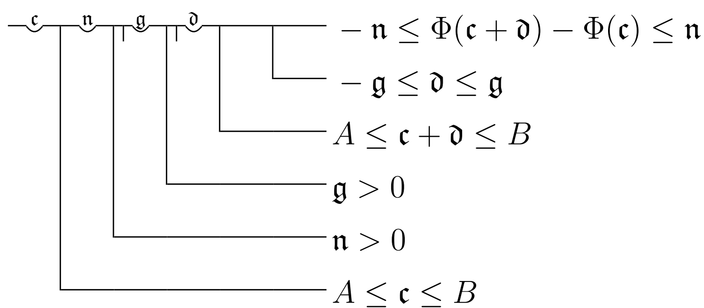
[25]It is well known, for example, that each of the following sets of connectives is sufficient to express all other connectives of classical propositional logic: $\{\neg, \rightarrow\}$, $\{\neg, \wedge\}$, $\{\neg, \vee\}$.

In general, the design for a notation for propositional logic must answer the questions of which notions should be taken as primitives and what symbols should be used to express them. Various criteria have been discussed as guides for answering them, like the number of symbols and the length of expressions. However, these criteria are often incompatible, so that trade-offs between them have to be taken into account.

3.3.1 Brevity of expressions

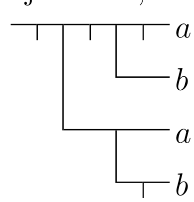
Reducing the length of an expression is one of the advantages that Frege himself brings up in order to argue for the use of formal symbols over formulations in natural language. After noting that logical relations can be expressed with a few basic symbols, he notes that the fact that thereby 'formulae are much briefer and more perspicuous than the equivalent definitions of the concepts in words' justifies their introduction (Frege 1880/81, 27). Frege points out that the demand for brevity is not for its own sake, but is a part of making the formulas 'more perspicuous'. As an example, let us consider the Begriffsschrift formula expressing the statement that 'The real function $\Phi(x)$ of a real variable x is continuous throughout the interval from A to B' (Frege 1880/81, 24):

(9)



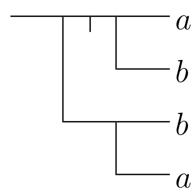
Frege is aware that the formula might at first appear to be 'longwinded' compared with the verbal statement, but he quickly points out that it also contains much more information. Instead of just naming the concept of continuity, it presents the result of a mathematical analysis of it: The formula expresses the conditions that must be satisfied for $\Phi(x)$ to be continuous throughout the interval. But even so, Frege adds, it still gets by with fewer symbols than the formulation in natural language.[26]

One of the criticisms that Schröder brought forward against the Begriffsschrift is that its formulas are 'definitively clumsy' in comparison with the Boolean notation. He gives the example of exclusive disjunction, which is represented by Frege as



[26] Because of the need for parentheses, the formula would also require more symbols in the modern notation: $\forall \mathfrak{c}((A \leq \mathfrak{c} \leq B) \rightarrow \forall \mathfrak{n}((\mathfrak{n} > 0) \rightarrow \neg \forall \mathfrak{g}((\mathfrak{g} > 0) \rightarrow \neg \forall \mathfrak{d}((A \leq \mathfrak{c}+\mathfrak{d} \leq b) \rightarrow ((-\mathfrak{g} \leq \mathfrak{d} \leq \mathfrak{g}) \rightarrow (-\mathfrak{n} \leq \Phi(\mathfrak{c}+\mathfrak{d}) - \Phi(\mathfrak{c}) \leq \mathfrak{n})))))))$.

but which can be rendered as $ab_1 + a_1b = 1$ or $ab + a_1b_1 = 0$ in Schröder's version of Boole's notation (Schröder 1881, 89; p. 227 of the English translation).[27] Since Schröder does not discuss any derivations involving this formula, the clumsiness seems to depend on the fact that Frege's formula is longer (i. e., involves more symbols) than those used in Boolean logic; that it is more spacious is a separate criticism of Schröder, to be discussed below. This is certainly how Frege understood the criticism, as he adduces the logically equivalent formula



in his reply (Frege 1882b, 103). This formula uses only a single negation stroke, instead of the four used in the formula mentioned by Schröder. Frege also points out that Boole himself used '+' as exclusive disjunction, instead of as the inclusive one, which is used by Schröder. Therefore, Boole's notation would actually be even shorter than Schröder's, namely $a + b = 1$. So, the fact that one notion allows for shorter expressions cannot be the only reason for preferring it. Moreover, even if the brevity of expressions is an explicit aim of a notation, this cannot be assessed by looking at one single example, but only by taking into consideration a wide range of formulas.

The best way of achieving the brevity of expressions would be to introduce a large number of primitive notions, e. g., by taking each of the 16 possible binary operations as primitive. However, Frege is quite clear that for him the desideratum of brevity of expressions is only of secondary importance: 'Precision and rigour are the prime aims of the Begriffsschrift; brevity will only be sought after if it can be achieved without jeopardizing those aims' (Frege 1880/81, 32). Frege's point here is that with an increased amount of primitives there would also be more assumptions that have to be explicitly made about them in terms of axioms and inference rules. This, in turn, would make it more difficult to keep track of the assumptions and easier to overlook some (Frege 1880/81, 35), thereby putting at risk the overall goal of gap-free derivations.

3.3.2 Number of primitive notions

Frege's argument for an economy of primitive notions clearly shows, on the one hand, the foundational — as opposed to practical — nature of his project, and on the other hand, his focus on a complete system in which all statements are either axioms or follow from them by chains of inferences. He considers it to be a basic principle of science 'to reduce the number of axioms to the fewest possible' and to achieve this, he is led to 'the basic principle of introducing as few primitives as possible' (Frege

[27]In Schröder's notation '+' stands for disjunction, juxtaposition for conjunction, and subscripts for negation.

1880/81, 35). Obtaining an overview and understanding of a discipline, he maintains, is easier if one has to deal with fewer primitive notions, 'for the fewer primitive signs one introduces, the fewer primitive laws one needs, and the easier it will be to master the formulae' (Frege 1882a, 50). Abbreviations for more complex notions can then be introduced by explicit definitions, and their laws should follow from these definitions together with the axioms that govern the primitive notions. Frege even hints at a quantitative assessment of the explanatory content of a systematization:

> Indeed the essence of explanation lies precisely in the fact that a wide, possibly unsurveyable, manifold is governed by one or a few sentences. The value of an explanation can be directly measured by this condensation and simplification: it is zero if the number of assumptions is as great as the number of facts to be explained. (Frege 1880/81, 35)

The economy that Frege is aiming at, however, is not reducible to a simple counting argument; the primitives and axioms must also be of a particular nature, as we shall see shortly. Moreover, while an increase in primitives also leads to an increase in axioms (see Frege 1882a, 48), and is thus to be rejected, a reduction of primitives does not necessarily lead to fewer axioms. This is evident from Frege's reaction to a suggestion by Russell to dispense with the negation symbol by defining $\neg p$ in terms of a universal formula that is always false (i. e., along the lines of $p \to \bot$, but without introducing a new symbol $\bot$). Frege replied that '[t]his would save a primitive sign; but we would probably need some new primitive laws' (Letter to Russell, November 13, 1904; Frege 1980, 166).[28] We see here the worry that a reduction of primitives might lead to an increase in the number of axioms, which is against Frege's intentions. In addition, he notes that the law that would have to be added to the system is 'really too complicated for a primitive law', thus implying that a statement also has to be relatively simple to count as an axiom.[29]

3.3.3 Content of primitives

Having established Frege's goals of minimizing the number of primitive notions and axioms, as well as the length of expressions, we turn now to the question: How did Frege determine the primitives for his Begriffsschrift? His reasoning seems to be based in part on the observation that '[i]n general it is always the sign with the simplest content which is the most widely applicable and leads to the clearest way of putting things' (Frege 1880/81, 36). This idea of taking into account the overall frequency with which certain notions are used in practice is elaborated further by Frege with an analogy to the notation used in chemistry.

[28]This issue is also discussed in Simons (1996, 290).

[29]We know that Sheffer had been in contact with Frege before publishing about the possibility of using a single symbol for a functionally complete system of propositional logic (Linsky 2011, 66–70), but we don't know about Frege's reactions to that. I presume Frege would not have found it very congenial to his own goals.

> E. g. Boole for his part has to use a more cumbersome expression for Schröder's $a + b$, the inclusive 'a or b'. But the exclusive 'or' perhaps only occurs once for every ten occurrences of the inclusive. So in chemistry everyone will regard it as more appropriate to represent the elements hydrogen and oxygen by single letters H and O, and to form OH from them, than to designate the hydroxyl complex OH by a single letter, while using a combination of signs to designate hydrogen as de-oxidized hydroxyl. (Frege 1880/81, 37)

To determine the connective with the simplest content, Frege begins by setting up a space of possibilities. Given two propositions A and B (or judgments, as Frege would say in 1879), there are four combinations depending on whether they are true or false (or, using the earlier terminology, affirmed or denied):

A	B	Judgment	Frege's implication	Boole's '+'	Conjunction
T	T	'A and B'	affirmed	affirmed	affirmed
T	F	'A and not B'	affirmed	denied	denied
F	T	'not A and B'	denied	denied	denied
F	F	'not A and not B'	affirmed	affirmed	denied

The above table shows the four possibilities using truth values for A and B, as well as using a judgment, which is Frege's way of presenting them. In principle, this yields 16 different choices for logical connectives, but Frege now considers the breadth of a connective's meaning for restricting the choices. For him, the 'simpler the content [of a symbol] is, the less it says' (Frege 1880/81, 35). He explains that a connective that denies only one of the four possibilities says less than one that denies two or even three of them, thus assuming that what a propositions 'says' are the possibilities that are being ruled out. From this perspective material implication is simpler than Boole's exclusive disjunction, which is again simpler than conjunction.[30] This still leaves four possible connectives and prima facie none of them is to be preferred over the others. Frege realizes that his choice 'may at first seem very artificial' (Frege 1882/83, 6), so that some additional criterion has to be taken into consideration and here Frege's aim of building a system that is adequate for representing mathematical inferences provides the decisive factor:

> I chose the denial of the third case, because of the ease with which it can be used in inference, and because its content has a close affinity with the important relation of ground and consequent. (Frege 1880/81, 37)

Additional support comes from the fact that even scientific inferences are based on this kind of inference:

[30] Frege discusses the four possible judgments and how they are denied or affirmed by the various operations also in (Frege 1879b, 5) and (Frege 1882a, 48–49).

> After all, the hypothetical judgment is the form of all natural laws, of all causal connections in general. (Frege 1882b, 6)

That Frege was fully aware that he could have chosen different connectives as primitives and that he also knew that conjunction and negation would form a minimal set of connectives, e. g., that $B \to A$ could be expressed by $\neg(B \wedge \neg A)$, becomes clear in the following passage from *Begriffsschrift*.

> Instead of expressing 'and' by means of the symbols for conditionality and negation, as is done here, conditionality could also be represented, conversely, by means of a symbol for 'and' and the symbol for negation. One might introduce, say,
>
> $$\begin{cases}\Gamma\\ \Delta\end{cases}$$
>
> as the symbol for the conjoined content of Γ and Δ, and then render
>
> 
>
>
> I chose the other way, since inference seemed to me to be expressed more simply that way. (Frege 1879b, 13)

This quotation also shows how Frege envisaged to represent conjunction as a primitive symbol in Begriffsschrift.[31] Notice how this symbol differs from that of implication: The two conjuncts are placed symmetrically above and below the main content stroke, presumably to indicate the symmetry (commutativity) of conjunction.

Other considerations regarding the primitive notions, e. g., that they should be dual, are missing in Frege. For Schröder, on the other hand, this seems to have been an advantage of using both conjunction and disjunction, as can be inferred from the fact that he presented rules for Boolean algebra in a dual-column format, like it was introduced by Gergonne for projective geometry (Schröder 1877).

3.3.4 Analogies to other systems

The criteria for the choice of primitives discussed so far have depended only on considerations internal to the system itself. However, the choice of primitives could also be guided by their relation to other systems. If a particular notion is similar to another notion that is better known, this will make it easier to learn the laws that govern it, because they just have to be adapted from those already known. In this way the new system will also look familiar from the start; and if exactly the same symbols are used in both systems, then the analogy will be even more prominent.

[31]Thus, the claim that '[a]nother disadvantage of Frege's notation is that it does not allow us to introduce abbreviations for the other connectives' (Gillies 1982, 80) is overstated.

We have mentioned above that Boole formulated his logical calculus to exhibit the 'close analogy' between the laws of algebra with those of logic. By using the symbols '+' and '×', some of the laws even look formally identical. In his review of *Begriffsschrift*, Schröder criticized Frege for not displaying this analogy. Overlooking Frege's groundbreaking innovations Schröder considers the Begriffsschrift only as a paraphrase of Boole's formal language, albeit a disadvantageous one that makes the connection to arithmetic unrecognizable. For Schröder, it is a clear disadvantage of Frege's notation that it does not exhibit the 'beautiful, real, and genuine analogies' between logic and arithmetic, despite its subtitle of 'a formula language modeled upon that of arithmetic' (Schröder 1881, 84; quoted from Frege 1972, 221).

Frege acknowledges the advantage of not having to learn new symbols and algorithms with Boole's notation, but objects to this way of proceeding on two grounds. First, the similarities between algebra and logic are of less importance than Boole and Schröder make them to be, in particular because the modes of reasoning in both disciplines are very different. Second, and more importantly, Frege maintains that logic is the more general discipline, which also underlies all reasoning. This, together with the general demand that 'the closest possible agreement between the relations of the signs and the relations of the things' should be reached, prohibits the use of symbols used in arithmetic or any other discipline for logic (Frege 1880/81, 12). After all, we should be able to apply the logical formalism in any other discipline. Thus, for Frege, it is

> more appropriate to develop for logic its own signs, derived from the nature of logic itself; we can then go on to use them throughout the other sciences wherever it is a question of preserving the formal validity of a chain of inference. (Frege 1880/81, 12)

Finally, Frege explains that the subtitle of *Begriffsschrift*, 'A formula language of pure thought modeled upon that of arithmetic', is not to be read in a narrow sense (as Boole and Schröder would have it), but in a wider sense, referring 'more to the fundamental ideas than to the detailed construction' (Frege 1879b, IV). The general ideas that Frege here alludes to are the use of variables and the analysis of expressions in terms of functions and arguments.

3.4 Economy of inference rules

We have seen that Frege's aim for his formula language was to provide a complete system of logic, i. e., one which would capture all possible mathematical inferences. In addition, he also wanted this system to be perspicuous, in the sense that the number of primitive elements should be as small as possible. Now, it would be possible to reduce the number of axioms by increasing the number of inference rules, but for Frege this would not constitute an improvement, since he considered both the axioms and the inference rules as parts of the system. For the sake of 'expediency' (Frege 1879b, 9) and to avoid redundancies (Frege 1880/81, 37–38), Frege adopted only a single rule of inference for his system, namely *modus ponens*: From A and $A \to B$, infer

B. Nevertheless, as in the case of his choice of implication and negation as primitive notions, Frege was aware that his analysis is not unique and that other inference rules could have been assumed (Frege 1879b, 25).

For Frege, the economy of a single rule of inference is not an end in itself, but serves the purpose of providing secure foundations that can easily be kept track of, because 'only what is finite and determinate can be taken in at once, and the fewer the number of primitive sentences, the more perfect a mastery can we have of them' (Frege 1880/81, 39). However, just as with primitive notions, where Frege held that more complex ones could be defined in the system, he was also explicit that additional inference rules could be added later for practical purposes.

> The restriction [...] to a single mode of inference is justified by the fact that in *laying the foundations* of such a Begriffsschrift the primitive elements must be as simple as possible if perspicuity and order are to be achieved. This does not rule out, *later*, transitions from several judgments to a new one, which are possible by this single mode of inference only in an indirect way, being converted into direct ones for the sake of abbreviation. In fact, this may be advisable for later applications. In this way, then, further modes of inference would arise. (Frege 1879b, VII; adapted from Beaney 1997, 51)

A consequence of Frege's decision to restrict himself to a single rule of inference is that proofs will turn out to be longer in his system, 'which might appear pedantic' (Frege 1880/81, 37–38). However, Frege's original aim was not to come up with a practical and convenient system, but with one that serves a foundational purpose. The aim was different with Frege's *Grundgesetze*, where he made some concessions by allowing additional modes of inference, like the rules of transposition and the removal of double negations, in order to avoid 'inordinate lengthiness' of the derivations (Frege 1893, 26).

3.5 The two-dimensional layout of the Begriffsschrift

Frege's original use of lines that are arranged in a two-dimensional layout for the Begriffsschrift notation was the main reason why the notation looked unfamiliar and unlike anything his readers had seen before. In this final section I would like to turn the attention to how Frege justified this unusual design decision.

When confronted with the criticism that the Begriffsschrift notation compares unfavorably with Boole's algebraic notation for logic, Frege replied by highlighting the difference between linear and graphical (or diagrammatic) representations. The latter, he explains, make full use of the possibilities of writing and they allow for a more flexible display of the structural information of logical relationships, which corresponds better to 'the multiplicity of logical relations connecting our thoughts with one another' (Frege 1882b, 159). They are 'laid out for the eye rather than for the ear' (Frege 1880/81, 13). Linear notations correspond more closely to spoken language and can only indicate 'by inessential marks or by imagery what a Begriffsschrift should spell out in full' (Frege 1880/81, 13). An example for such marks are parentheses, which are necessary to group subformulas together in Boole's one-dimensional notation, but which are not needed in the Begriffsschrift.[32]

In his review of *Begriffsschrift*, Schröder reproached Frege's notation for indulging in 'the Japanese practice of writing vertically' and thus for being a 'monstrous waste of space', which if all other things were equal 'should definitely decide the issue in favour of the Boolean school' (Schröder 1881, 91; quoted from Frege 1972, 229). While it is unquestionable that the Begriffsschrift uses more space, Frege points out that the use of the vertical dimension in writing is in fact commonly adopted by mathematicians when they write equations beneath each other (Frege 1880/81, 46). Here, each line corresponds to an individual proposition (the content of a possible judgment, in Frege's way of speaking) that is logically linked to the ones above and underneath. He writes:

> The Begriffsschrift makes the most of the two-dimensionality of the writing surface by allowing the assertible contents to follow one below the other while each of these extends [separately] from left to right. Thus, the separate contents are clearly separated from each other, and yet their logical relations are easily visible at a glance. For Boole, a single line, often excessively long, would result. (Frege 1882/83, 7–8; adapted from Frege 1972, 97)

Thus, by splitting up the individual components of a long formula into separate lines, the Begriffsschrift notation 'simplifies the recognition of that to which we wish to direct our attention in the given case' (Frege 1882b, 159).[33] Recall that Frege's main application of the Begriffsschrift is to mathematics.[34] Thus, expressions like Formula (9) on p. 20 above are more typical than any of the formulas (1)–(8) and if the more complex formulas would be presented in a purely linear notation, they surely would be more difficult to parse. As a consequence,

> [t]he disadvantage of the waste of space of the Begriffsschrift is converted into the advantage of perspicuity; the advantage of terseness for Boole is transformed into the disadvantage of being confusing. (Frege 1882/83, 7; adpated from Frege 1972, 97)

In this context Frege also points out the different degrees of granularity between his and Boole's notation. Every proposition is represented by Boole as a single letter, but in Frege's system it is intended to be completely spelled out. This is possible, because the full version of Frege's Begriffsschrift is a system of first-order logic, which has a

[32]For an example, compare the Begriffsschrift Formula (9) with the corresponding representation in modern notation in Footnote 26.

[33]Bynum gives a compelling example in his editorial comment to the passage from Frege quoted above (Frege 1972, 97). An anonymous reviewer has pointed out that this makes the Begriffsschrift *iconic* in Peirce's sense: the truth of A pictorially rests on or is founded on the truth of B.

[34]See also Frege (1879a).

greater expressive power than a system for propositional logic like Boole's. So, the greater degree of perspicuity is achieved to its fullest extent with the use of the first-order features of the notation. Indeed, Frege uses propositional variables for formulas only in Parts I and II of *Begriffsschrift*, i. e., the introduction to the notation and the derivation of some judgments for the purpose of illustrating the use of the Begriffsschrift, and in Part I of *Grundgesetze*. When the notation is actually applied to arithmetic, which — as we have seen above — is the main purpose of the Begriffsschrift, it is always mathematical or logical formulas that are connected by means of the notation. In the latter cases, the Begriffsschrift formulas lose the narrowness of their appearance and the advantage of the two-dimensionality of the notation becomes much more obvious.

The point about the improved readability of Begriffsschrift formulas is also made by Ebert and Rossberg in the 'Translators' Introduction' to the English translation of *Grundgesetze* (Frege 2013, xxx). To justify the use of the Begriffsschrift in the translation, they write that 'transforming Frege's notation into a more familiar formalism would generate the need for numerous parentheses which would hinder readability' and they support this claim by a nice example of a formula in the Begriffsschrift and in the corresponding modern formulation. As they show, using the left-association convention for embedded conditionals allows one to get rid of a number of parentheses, but without improving the readability of the formula.

Almost two decades after the publication of *Begriffsschrift*, the discussion about the notation was taken up again by Frege, now in a debate with Peano, who also employs a linear notation:

> In Peano's Begriffsschrift the writing of formulas on a single line is, so it seems, carried through as a fundamental principle which appears to me as a wanton renunciation of a major advantage of writing over speech. The convenience of the typesetter is not however the highest Good. For physiological reasons, a long line is harder to survey and its divisions are harder to grasp than shorter lines lying underneath each other, and created from the breaking up of the original line, provided that this partition corresponds to the division of the sense. (Frege 1896; translation adapted from Gillies 1982, 82).

Here, two novel considerations appear. First, the 'convenience of the typesetter' is dismissed as reason for one's choice of notation. Second, Frege now backs up his claims about the perspicuity of the Begriffsschrift by empirical research in the physiology of reading, without, however, giving any explicit reference. We can only surmise that he had in mind the work of some prominent author on this matter, like Javal (1879).[35]

[35]For other references to late 19th century literature, see Huey (1898).

4 Concluding remarks

The notation that Frege introduced in his 1879 *Begriffsschrift* stands out due to its two-dimensional layout with lines representing logical relations (implication and negation) on the left and the non-logical content on the right. While this might seem to be a disadvantageous design at first, it actually turns out to be quite powerful and perspicuous, in particular when considered for the application that Frege had in mind for it, namely a concise presentation of genuine mathematical content.

While Frege's innovation of using quantifiers and bound variables proved a major advance with regards to the expressivity of formal logic, the main features that distinguish his Begriffsschrift from other notations are already present in the propositional part of the system. For this reason, I have restricted the focus of the above discussions to the latter. In fact, Schröder's criticisms of the notation and Frege's subsequent replies also center around aspects of the propositional fragment of the Begriffsschrift.

In the first part of the this paper several idiosyncrasies of the notation, which allow an easy conversion of logically equivalent formulas, were discussed and its close connection to syntax trees was presented, arguing for the perspicuity and readability of the notation. In the second part, the aims that Frege pursued with his system together with his considerations regarding possible difficulties with the notation because of its unfamiliar look were presented. In addition, Frege's justifications for the design principles underlying the Begriffsschrift were discussed, about which he was very explicit in his replies to early criticisms and unfavorable comparisons with Boole's notation for propositional logic. Despite the fact that these discussions were mainly about the Begriffsschrift, they highlight some important trade-offs with regard to notations in general. The issue of familiarity is not about a notation *per se*, but relative to the historical and epistemological context in which it is introduced. The historical development of the Begriffsschrift shows that in some cases a notation that looks familiar has considerable advantages, e. g., readers are more likely to engage with it and previous knowledge can be transferred. In contrast, such familiarity might also lead to ambiguities and unjustified or misleading analogies. Another important theme that emerged in the discussions is the trade-off between brevity and perspicuity. At a very basic level, shorter expressions are easier to grasp, but when it comes to more complex subject matters the situation is not so clear-cut any more. Brevity can also be achieved by adopting additional implicit conventions (e. g., regarding the binding strength of connectives to avoid parentheses), but these in turn require additional effort to learn.[36] Frege's reflections demonstrate that he was well aware of many of these issues that surround the design of convenient notations. In sum, the above discussion has revealed that the Begriffsschrift is in fact a well thought-out and carefully crafted notation that intentionally exploits the possibilities afforded by the two-dimensional medium of writing like none other.

[36]A more detailed analysis of various cognitive and pragmatic trade-offs of notations is currently in progress by the author.

Acknowledgements. Thanks to Roy Cook, George Englebretsen, Ansten Klev, Bernard Linsky, Daniel Lovsted, Paolo Mancosu and three anonymous reviewers for comments on an earlier draft. Part of this research was supported by the Social Sciences and Humanities Research Council of Canada.

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