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Metaphors for Mathematics from Pasch to Hilbert

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Abstract

How mathematicians conceive of the nature of mathematics is reflected in the metaphors they use to talk about it. In this paper I investigate a change in the use of metaphors in the late nineteenth and early twentieth centuries. In particular, I argue that the metaphor of *mathematics as a tree* was used systematically by Pasch and some of his contemporaries, while that of *mathematics as a building* was deliberately chosen by Hilbert to reflect a different view of mathematics. By taking these metaphors seriously we attain a new vantage point for understanding historical changes in conceptions of mathematics.

1 Introduction

Mathematics changed profoundly around the turn of the 20th century. Jeremy Gray speaks of a 'decisive transformation of mathematical ontology' and a 'radical transformation in mathematical epistemology', which took place between 1890 and 1930 [Gray 2008, 2–3]. Leo Corry introduced the notion of *images of knowledge* in his account of the history of modern algebra to refer to a set of attitudes, implicit assumptions, and general views on mathematics. This enables him to characterize the historical developments as changes of the image of mathematics and, in particular, as a change towards a 'structural image' in the case of algebra [Corry 1996]. In the present paper, I explore these developments from the point of view of the metaphors that were used by mathematicians, with the aim of obtaining further insights about their images of mathematics.

According to Lakoff and Johnson, 'the essence of metaphor is understanding and experiencing one kind of thing in terms of another' and they have argued that metaphors are our main cognitive resource for dealing with abstract concepts [Lakoff and Johnson 2003, 5]. Indeed, since mathematics is generally conceived as the most abstract science, its nature is often conveyed metaphorically in terms of something more concrete and familiar. In addition, metaphors can also be employed as pedagogical means for conveying understanding about a specific subject matter (see, e. g., [Pimm 1981] and [McColm 2007]) and they have been proposed as mechanisms underlying mathematical reasoning by Lakoff and Núñez [2000]. In contrast to these latter uses of metaphors, however, the goal of the present investigation is to get a better grasp on mathematicians' own conceptions of mathematics by looking at those metaphors that they use systematically to talk about mathematics. Taken seriously, then, these metaphors can be interpreted as reflecting particular images of mathematics.

The time period under consideration in this paper is that between the end of the 19th and the beginning of the 20th century, when modern, abstract mathematics emerged, and I will focus on two specific metaphors for mathematics: Mathematics as a *tree* and mathematics as a *building*. Because people have been familiar with trees and buildings for several millennia, it is not surprising that they have been the source of numerous metaphors. The metaphor of a tree of knowledge is a very old one. As an illustration of a hierarchy of predicates it became known in the Middle Ages as the 'Tree of Porphyry' [Kneale and Kneale 1971, 232]. A little more recent examples can be found, for example, in Descartes' writings: The metaphor of a building underlies the description of his way of proceeding at the beginning of the First Meditation (1641) [Cottingham et al. 1985, vol. 2, 12] and it is also frequently employed in the *Discourse on Method* (1637). Descartes also uses the tree metaphor to illustrate the structure of the discipline of philosophy in the *Principles of Philosophy* (1644) [Cottingham et al. 1985, vol. 1, 186], and a tree is also used in Diderot and d'Alembert's *Encyclopédie* (1755) to represent the structure of a science [Diderot 1755]. These metaphors are indeed so prominent that 'Theories (and arguments) are buildings' and 'Ideas are plants' are listed as examples of common metaphors for our everyday concepts in [Lakoff and Johnson 2003, 47–48].

While there are no unique features of trees and buildings play a role in all of these metaphors, we can identify certain generalities. Both trees and buildings are structured in such a way that one part (the roots and trunk, and the foundations, respectively) carries the remaining structure. When it comes to scientific disciplines or theories, this role is played by first principles or axioms. Trees differ from buildings in that they are natural entities whose growth is largely independent from human intentions; in contrast, buildings are the result of human planning and workmanship, and, if constructed poorly, are in danger of collapsing. Note, however, that because an object has a large number of features that can be relevant for its use in metaphors, it is important to pay careful attention to how the metaphors are actually used in each specific instance. For example, just because trees typically form heartwood, which is a dead part of the tree, does not mean that everybody who uses a tree metaphor has also this particular feature of trees in mind; similarly, just because buildings typically have doors does not mean that the doors of a building must correspond to something in the realm to which the building metaphor is applied. Because it can be very tempting to over-interpret a metaphor, we should beware of doing so. In order to exhibit how the metaphors under consideration have actually been employed, I have frequently quoted larger passages of text below.

Since metaphors are constantly used in everyday language, it is no surprise to find them also in the writings of mathematicians. However, the present paper is not concerned with the occasional use of a metaphor in mathematical discourse, but, as I will argue, with the deliberate and systematic use of two metaphors with very specific features. That this use is indeed intentional is supported by the fact that both Pasch and Hilbert use their terminology in new, idiosyncratic ways, e. g., Pasch's use of '*Kern*' and '*Stamm*', and Hilbert's use of '*Fachwerk*'.

Pasch chose trees as main metaphor for the structure of mathematics (as did his contemporaries Frege and Klein)[1], while Hilbert frequently employed and elaborated the metaphor of mathematics as a building. As will become apparent in the discussion, these choices can be seen as reflecting different epistemological attitudes towards mathematics, as the origin and growth of plants are largely out of our control, but that of buildings is not. In the broader picture of philosophy of mathematics, I understand Pasch and Hilbert to be representatives of two different historical periods and philosophical attitudes. On the one hand, Pasch's thinking is firmly rooted in the 19th century (like that of Frege and Klein), despite the fact that he also contributed in important ways to the change that mathematics underwent at

[1]Pasch, Klein, and Frege were all born between 1843 and 1849, and died between 1925 and 1930.

the turn of the century. On the other hand, Hilbert is often considered to be one of the foremost representatives of the new, modern approach to mathematics. As will be shown, as part of this transition the picture of mathematics as a natural entity was replaced by that of mathematics as an artificially constructed one. This change in metaphors reflects a dramatic change in the way mathematics as a whole was conceived by these authors, i. e., before and after what Gray calls the 'modernist transformation' of mathematics [2008, 1].

2 Mathematics as a tree

2.1 Pasch's conception of mathematics

When Moritz Pasch (1843–1930) published his pioneering *Vorlesungen über neuere Geometrie* in 1882, it contained the first axiomatic presentation of projective geometry, the first axiomatization of the relation of betweenness, a careful discussion of logical gaps in Euclid's proofs, and a clear formulation of the role of logical deduction in geometry. Motivated in part by the duality of projective geometry Pasch also formulated the requirement that logical deductions should be independent of the meaning of the concepts involved [Pasch 1882, 98], an idea that became generally accepted as a fundamental principle of deductive reasoning. Pasch's book remained the foremost example of a rigorous axiomatic treatment of geometry until the publication of Hilbert's *Grundlagen der Geometrie* in 1899.[2]

An integral part of Pasch's conception of mathematics is the distinction between empirical and purely mathematical concepts [Schlimm 2010]. Pasch distinguishes between a philosophical, meaningful foundation of geometry, i. e., an axiomatization that is based on empirical geometric concepts, and a mathematical, formal foundation, i. e., an axiomatization from which all propositions of pure geometry can be deduced. For Pasch, the mathematical layer is to be obtained from the philosophical one by definitions and deductions, whereby the mathematical terms receive their meanings.[3] To emphasize this distinction, Pasch distinguishes systems of axioms terminologically, according to which of these two layers they belong to.

In Pasch's terminology, the axioms of the mathematical layer form a 'stem' (or 'trunk', in German: '*Stamm*') and he also speaks of 'stem' concepts and

[2] Among others, Pasch's book influenced Hilbert's own development, see Section 3.1, below, as well as that of Peano and the Italian school of geometry.

[3] A discussion of the connection between the two layers, which I called 'Pasch's Programme', can be found in [Schlimm 2010, 107–112].

propositions.[4] Originally, Pasch referred to the meaningful, empirical axioms as 'basic principles' ('*Grundsätze*'), but he later deliberately changed the terminology to 'core' propositions.[5] The corresponding German word is '*Kern*', which is often also translated as 'pip' or 'kernel'; another possible interpretation is that '*Kern*' is a short form for '*Kernholz*'[6], which means heartwood (i. e., the inner, dead part of the trunk that develops in some species of trees). Regardless of which of these interpretations is the intended one, Pasch's metaphor for mathematical knowledge is that of a tree, which grows from a core (the empirical basis) and develops into a stem (the mathematical theory). Using this terminology Pasch describes the development of the mathematical foundation from an empirical one as follows:

> Projective geometry arises from a mathematical 'stem', which is obtained on the ground of geometry before the splitting off of its projective part, through deduction from the core of geometry. [Pasch 1924b, 232][7]

It fits well into this picture that Pasch also refers to those propositions that are common to all sciences as an 'area of roots' ('*Wurzelgebiet*') [Pasch 1924a, 34]. From this idiosyncratic choice of terminology we can conclude that Pasch consciously chose the tree metaphor to reflect his conception of the nature of mathematics.

2.2 More trees: Klein and Frege

Pasch's contemporaries, Felix Klein (1849–1925) and Gottlob Frege (1848–1925) also employed the tree metaphor to express their views about mathematics. Klein used it explicitly when discussing the roles of intuition and logic in mathematics:

[4] Incidentally, also Desargues, who is credited for proving some important theorems of projective geometry, used idiosyncratic terminology based on botanical terms. For example, he called a line with points on it a 'trunk' and a line with three pairs of points on it a 'tree' (see [Poudra 1861, 99–102] for an overview of Desargues' terminology). However, Desargues employed this terminology for aspects of diagrams and constructions, and not, like Pasch, for the structure of mathematical theories. Moreover, as far as I know, Pasch never referred to Desargues' work, so that this connection seems to be purely coincidental. I am grateful for an anonymous reviewer for informing me about Desargues' terminology.

[5] The new terminology was introduced in [Pasch 1916, 276] and was incorporated into the second edition of Pasch's lectures [Pasch 1926].

[6] I am indebted to Felix Mühlhölzer for pointing this out to me.

[7] German original: 'Die projektive Geometrie entspringt aus einem 'Stamm', der auf dem Boden der Geometrie vor der Abspaltung ihres projektiven Teiles gewonnen ist, durch Deduktion aus dem Kern der Geometrie.'

> I compare the mathematical science with a tree, which drives its roots deeper and deeper into the soil, while it freely unfolds its shady branches upwards. Shall we consider the roots or the branches as the more essential parts? The botanists teach us that the question is ill-posed, but rather that the life of the organism is based *on the interaction of its parts*. [Klein 1895, 240; italics in original][8]

Here mathematics is again presented as a natural organism that grows both upwards and downwards; later, Klein also referred to the realm of mathematical ideas as a 'blooming tree' [Klein 1926, 150]. A very similar metaphor is also used by Frege:

> Can the great tree of the science of number as we know it, towering, spreading, and still continually growing, have its roots in bare identities? ([Frege 1884, § 16], quoted from [Frege 1980, 22])[9]

And again:

> If I compare arithmetic with a tree that opens out above into a multitude of methods and theorems, whilst the root pushes into the depths, then it seems to me that the growth of the root, at least in Germany, is weak. Even in a work that might be included in this movement, E. Schröder's *Algebra der Logik*, growth at the top soon regains the upper hand, before any greater depth is reached, causing a bending upwards and an opening out into methods and theorems. ([Frege 1893, XIII], quoted from [Beaney 1997, 200])[10]

In contrast to the empiricists Pasch and Klein, however, Frege emphasizes that the starting points for the development of mathematics, which he also called

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[8]German original: 'Ich vergleiche die mathematische Wissenschaft mit einem Baume, der seine Wurzeln nach unten immer tiefer in das Erdreich treibt, während er nach oben seine schattengebenden Äste frei entfaltet. Sollen wir die Wurzel oder die Zweige als den wesentlicheren Teil ansehen? Die Botaniker belehren uns, daß die Frage falsch gestellt ist, daß vielmehr das Leben des Organismus *auf der Wechselwirkung seiner Teile* beruht.' For a similar use of the tree metaphor, with free developments on the top and a grounding with thousand roots in intuition, see [Weyl 1910].

[9]German original: 'Soll dieser hochragende, weitverzweigte und immer noch wachsende Baum der Zahlenwissenschaft in blossen Identitäten wurzeln?'

[10]German original: 'Wenn ich die Arithmetik mit einem Baume vergleiche, der sich oben in eine Mannichfaltigkeit von Methoden und Lehrsätzen entfaltet, während die Wurzel in die Tiefe strebt, so scheint mir der Wurzeltrieb, in Deutschland wenigstens, schwach zu sein. Selbst in einem Werke, das man dieser Richtung zuzählen möchte, der Algebra der Logik des Herrn E. Schröder, gewinnt doch bald der Wipfeltrieb wieder die Oberhand, bevor noch eine grössere Tiefe erreicht ist, bewirkt ein Umbiegen nach oben und eine Entfaltung in Methoden und Lehrsätze.'

‘kernel’ in his 1914 lectures on logic [Reck and Awodey 2004, 137], are truths. In his *Grundlagen der Arithmetik* he notes that conclusions are contained in their definitions like plants are contained in their seeds and he explicitly contrasts this to the way beams are contained in a house [Frege 1884, 100].[11] Frege also introduced a novel aspect of the tree metaphor, according to which formalization is lignification:

> What originally was completely soaked with thoughts, hardens with time to a mechanism, which relieves the researcher in part from thinking. [. . .] I would like to compare this with the process of lignification. Where a tree lives and grows, it must be soft and juicy. But, if the juices would not lignify with time, then no considerable height could be achieved. However, if everything green is lignified, the growth ends. (Letter to Hilbert, October 1, 1895; [Frege 1976, 5])[12]

The formalized part of mathematics, according to Frege, is solidified, which allows for further, new developments. However, he warns that a complete formalization of mathematics would als mean the end of its development.

The view of mathematics that emerges from the metaphors used by all three authors discussed so far (Pasch, Klein, and Frege), is that of mathematics as a natural and (for the most part) living organism.

Madeline Muntersbjorn called attention to Frege’s botanical metaphors in her [2003], where she puts forward an interpretation according to which the conceptualization of mathematics as a plant allows one to consider it both as independent from human intentions and as dependent on external, environmental factors (that can include human interventions); that an oak tree develops from an acorn is predetermined, but the particular form and shape of the tree depends on its environment. Both nature and nurture, to continue with the metaphor, play a role in the growth of mathematics. Muntersbjorn concludes that according to this view mathematics is not subjective, but nevertheless amenable to creative influences, and she develops the metaphor further, characterizing mathematics as ‘cultivated’.

[11] This contrast may actually carry argumentative weight in connection with Frege’s epistemological claim that some inference extend knowledge; concepts that support extensions of knowledge are also said to have parts that are connected ‘more organically’ to each other [Frege 1884, 100]. On this matter, see [Tappenden 1995].

[12] German original: ‘Was ursprünglich ganz von Gedanken durchtränkt war, verhärtet sich mit der Zeit zu einem Mechanismus, der dem Forscher das Denken zum Teil abnimmt. [. . .] Ich möchte dieses mit dem Verholzungsvorgange vergleichen. Wo der Baum lebt und wächst, muss er weich und saftig sein. Wenn aber das Saftige nicht mit der Zeit verholzte, könnte keine bedeutende Höhe erreicht werden. Wenn dagegen alles Grüne verholtzt ist, hört das Wachstum auf.’

2.3 Towards new metaphors

Before turning to Hilbert's main metaphor for mathematics, let us briefly return to Pasch and look at some other metaphors that he employed. As we shall see, not only did his axiomatic work prepare the ground for Hilbert's later accomplishments, but also his use of metaphors foreshadowed later developments. What Pasch used as secondary metaphors were later emphasized and developed further by Hilbert.

In the Preface of his lectures on projective geometry, Pasch describes his view of the development of geometry from an empirical to a deductive science:

> the successful application that geometry is continuously subjected to in the natural sciences and in everyday life is based solely on the fact that the geometric concepts originally corresponded exactly to empirical objects, but that they were later covered by a *network of artificial concepts* in order to advance the theoretical development [Pasch 1882, V; emphasis by DS][13]

In using the metaphor of a network (or web) for the organization of the discipline of geometry, Pasch draws attention to the fact that the geometric concepts form a connected whole. However, the structure is artificial and a net usually does not have any privileged vertices in it.[14] In other words, while some of the empirical concepts of geometry might be more basic than others in terms of their content, this is not the case for the artificial concepts that are used in the deductive development of geometry, where the starting points can be chosen according to purely logical and pragmatic criteria.

The just mentioned freedom of choice in the construction of theories does not square well with the view of mathematics as a tree in which different parts play different roles. Thus, when it comes to the development of mathematical (as opposed to empirical) theories from a system of axioms, Pasch uses a third metaphor, namely that of a building. The German verb '*bauen*', which means to construct, is part of the words '*Unterbau*' and '*Lehrgebäude*', but these terms lose these constructive associations in their English translations of 'substructure' (sometimes '*Unterbau*' is also translated less literally as 'foundation') and 'theory'. '*Lehrgebäude*' is the common term in 19th century German for theory or doctrine,

[13] German original: 'Die erfolgreiche Anwendung, welche die Geometrie fortwährend in den Naturwissenschaften und im praktischen Leben erfährt, beruht jedenfalls nur darauf, daß die geometrischen Begriffe ursprünglich genau den empirischen Objekten entsprachen, wenn sie auch allmählich mit einem Netze von künstlichen Begriffen übersponnen wurden, um die theoretische Entwicklung zu fördern.'

[14] The distinction between the center and the fringes of the web that is prominent in Quine's famous 'web of belief' [Quine and Ullian 1970], is not made by Pasch.

so by itself, its use should not be over-interpreted. However, in combination with the other building metaphors, it becomes significant. In the following quote, Pasch combines this new metaphor with the one from botany discussed above. He explains:

> From the whole of geometry one can detach a substructure ('*Unterbau*'), which ascends from the physical point to the mathematical point, and build up a theory ('*Lehrgebäude*'), that knows only the mathematical point, not the physical one. [...] This theory develops from a stem, the substructure from a core. [...] Usually, only such a theory is presented as geometry, not a theory with a substructure. [Pasch 1917, 185][15]

While plants may die without light and nourishment, trees do not usually break without any external causes like a lightning stroke, flood, or earthquake. However, once mathematical theories are regarded as human constructions, the danger arises that human error can lead to faulty constructions. Pasch writes that gaps in the deductions cause a part of Euclid's 'building to totter'[16] [Pasch 1894, 28]. In a similar vein, the metaphor of mathematics as a building was also used by Dedekind, who explicitly speaks of a '*Gebäude*' (a building or edifice) when explaining that the replacement of the meaningful terms in Euclid's theory should not bring about its collapse (Letter to Lipschitz on July 27, 1876; [Dedekind 1932, 479]). As we shall see next, with Hilbert the mathematics as a building metaphor becomes the central metaphor for mathematics.

3 Mathematics as a building

3.1 Hilbert's *Fachwerk der Begriffe*

David Hilbert's (1862–1943) groundbreaking work on the foundations of geometry, his *Grundlagen der Geometrie* (1899), was preceded by a series of lectures on geometry he delivered in the 1890's.[17] It was only in 1894 that Hilbert read Pasch's

[15] German original: 'Aus der ganzen Geometrie kann man einen Unterbau absondern, der vom physischen Punkt zum mathematischen Punkt aufsteigt, und ein Lehrgebäude errichten, das nur den mathematischen Punkt kennt, keinen physischen. [...] Dieses Lehrgebäude entwickelt sich aus einem Stamm, der Unterbau aus einem Kern. [...] Als Geometrie wird herkömmlicherweise bloß ein solches Lehrgebäude hingestellt, nicht ein Lehrgebäude mit Unterbau.'

[16] German original: 'bringt einen Teil des Gebäudes zum Wanken'.

[17] See [Toepell 1988] and [Hallett and Majer 2004] for background and sources of the historical development of Hilbert's lectures on geometry.

book and the differences between Hilbert's lectures from 1891 and 1894 clearly show its influence: Many of Hilbert's axioms are taken from Pasch and a number of passages in Hilbert directly paraphrase Pasch.[18] In the opening pages of Hilbert's 1894 lectures, he makes some general remarks on the discipline of geometry, which echo those quoted above from the Preface of Pasch's *Vorlesungen*, but which also go beyond them in important ways:

> As every science aims at *ordering* the group of facts that lie in its domain, or to describe the phenomena [...] so does geometry with geometric facts. This ordering or describing happens with *certain concepts, that are to be connected by the laws of logic*. A science is more advanced, i. e., the *framework* ('Fachwerk') *of concepts* is more complete, the easier it is to accommodate any phenomenon or fact. Geometry is a science that essentially has advanced so far that *all of its facts can already be deduced from earlier ones by logical inferences*. [Hilbert 1894, 7; italics are underlined in the original][19]

Here Hilbert emphasizes the role of logic in connecting the geometric concepts, but, more importantly for the present paper, we see here that Pasch's 'network' ('*Netz*') of concepts becomes a '*Fachwerk*' for Hilbert. This term denotes the timber framework of a traditional building style in Germany ('*Fachwerkhaus*'); in English, houses that are built in this style are sometimes referred to as half-timbered or Tudor-style houses.

Although it was not unusual to use the building metaphor for theories (as we have seen in the case of Descartes, Pasch, and Dedekind, above), to use the term '*Fachwerk*' in this connection is unique to Hilbert. Not only that, but also that the notion of '*Fachwerk*' was indeed central for Hilbert's conception of mathematics is evidenced by the fact that he used it again in the discussion with Frege in 1899 [Frege 1976, 67 and 69] and also repeatedly in lectures on various subjects throughout his career. For example, it occurs in '*Grundlagen der Geometrie*' [1902], '*Logische Prinzipien des mathematischen Denkens*' [1905a], '*Grundlagen der*

[18] See Ulrich Majer's introduction and editorial footnotes to Hilbert's lectures on *Die Grundlagen der Geometrie* from 1894 in [Hallett and Majer 2004, 60–121].

[19] German original: 'Wie jede Wissenschaft dahin zielt, die in ihrem Bereiche liegende Gruppe von Thatsachen zu *ordnen*, oder die Erscheinungen zu beschreiben [...] so thut es die Geometrie mit eben jenen geometrischen Thatsachen. Dieses Ordnen oder Beschreiben geschieht vermittelst *gewisser Begriffe, die durch die Gesetze der Logik unter sich zu verknüpfen* sind. Die Wissenschaft ist desto fortgeschrittner, d. h. das Fachwerk der Begriffe ist desto vollkommner, je leichter jede Erscheinung oder Thatsache untergebracht wird. Die Geometrie ist eine Wissenschaft, welche im Wesentlichen so weit fortgeschritten ist, dass *alle ihre Thatsachen bereits durch logische Schlüsse aus früheren abgeleitet werden können*.'

Physik' [1916], '*Natur und mathematisches Erkennen*' [1920a], '*Grundsätzliche Fragen der modernen Physik*' [1921/23], '*Logische Grundlagen der Mathematik*' [1922/23a], and '*Wissen und mathematisches Denken*' [1922/23b]. In print, it appeared for the first time only in the publication of the lecture '*Axiomatisches Denken*' in 1918, 24 years after it had been employed for the first time by Hilbert. The published formulation reminds us strongly of the 1894 passage quoted above:

> When we assemble the facts of a definite, more-or-less comprehensive field of knowledge, we soon notice that these facts are capable of being ordered. This ordering always comes about with the help of a certain *framework of concepts* in the following way: a concept of this framework corresponds to each individual object of the field of knowledge, and a logical relation between concepts corresponds to every fact within the field of knowledge. This framework of concepts is nothing other than the *theory* of the field of knowledge. ([Hilbert 1918, 405], quoted from [Ewald 1996, 1107–1108])[20]

The 'framework of concepts' also appeared one of Hilbert's last publications, his 1930 talk in Königsberg [Hilbert 1930, 961].

3.2 Foundations: *Grundlagen* vs. *Fundamente*

The notion of a *Fachwerk* was employed by Hilbert to characterize the internal structure of mathematical theories, but he also elaborated the building metaphor further. In his famous *Zahlbericht* Hilbert characterizes the theory of number fields as a 'building of wonderful beauty and harmony' [Hilbert 1897a, 67][21]. In his lectures on '*Zahlbegriff und Quadratur der Kreises*' [1897b] from the same year Hilbert extends the metaphor of a building to also cover the relation between the size of the building and the need for stronger foundations. To appreciate this move it is necessary to note here that there are two German words that are commonly translated as 'foundations': '*Fundamente*', which is typically used for the foundations of a building, and '*Grundlagen*', used for foundations in a more general or abstract sense. In addition to the more commonly used term

[20] German original: 'Wenn wir die Tatsachen eines bestimmten mehr oder minder umfassenden Wissensgebietes zusammenstellen, so bemerken wir bald, daß diese Tatsachen einer Ordnung fähig sind. Diese Ordnung erfolgt jedesmal mit Hilfe eines gewissen *Fachwerks von Begriffen* in der Weise, daß dem einzelnen Gegenstande des Wissensgebietes ein Begriff dieses Fachwerkes und jeder Tatsache innerhalb des Wissensgebietes eine logische Beziehung zwischen den Begriffen entspricht. Das Fachwerk der Begriffe ist nichts Anderes als die *Theorie* des Wissensgebietes.'

[21] German original: 'Die Theorie der Zahlkörper ist wie ein Bauwerk von wunderbarer Schönheit und Harmonie.'

‘*Grundlagen*’, Hilbert also introduces the term ‘*Fundament*’ in his 1897 lectures. As far as I know, none of the authors discussed in the previous section used the term ‘*Fundament*’ in their writings. Hilbert writes:

> The more a science spreads out, the more diversified its material, the broader and more general its application reaches, the more necessary becomes a *rigorous examination of its foundations* (‘*Grundlagen*’), the farther one also must go back to investigate its principles. Naturally, the greater the building, the deeper and more solid the foundations (‘*Fundament*’) have to be. ⟨ The mightier a tree spreads its crown, the stronger its roots grow. ⟩ In the joy over the discovery of the differential calculus and its brilliant successes one went too far, one extended the building further than its foundation tolerated. Cracks formed, gaps, and even collapses. An examination of the foundations became necessary: Weierstrass, Cantor, Dedekind. [Hilbert 1897b, 1; emphasis by DS][22]

It is interesting to note that Hilbert deliberately employs the building metaphor in this passage in quite some detail. However, he also added a side remark in the margin of the lecture notes (indicated by the angled parentheses ⟨ . . . ⟩), which uses the tree metaphor. Perhaps he added it as an additional clarification for his newly introduced building metaphor, thinking that the tree metaphor would have been more familiar to his audience. I am aware of only two other occasions where Hilbert used the tree metaphor: [Hilbert 1617, 2] and [Hilbert 1917a, 42]. In the first occurrence the growth of a science is compared to the growth of a tree, both in the direction of the branches and of the roots; this metaphor is also picked up in the second occurrence, but in conjunction with the building metaphor (this passage is discussed below, p. 15).

After its introduction in 1897, also the term ‘*Fundament*’ became a regular staple in Hilbert’s lectures; e. g., it occurs in ‘*Ueber den Begriff des Unendlichen*’ [1898], ‘*Logische Prinzipien des mathematischen Denkens*’ [1905a], ‘*Probleme und Principienfragen der Mathematik*’ [1914], ‘*Prinzipien der Mathematik*’ [1917b], ‘*Natur und mathematisches Erkennen*’ [1920a], and ‘*Probleme*

[22] German original: ‘Jemehr sich eine Wissenschaft ausbreitet, je mannigfaltiger ihr Stoff, je weiter und allgemeiner ihre Anwendbarkeit reicht, desto nothwendiger wird eine *strenge Prüfung ihrer Grundlagen*, desto weiter muss man auch zur Untersuchung ihrer Principien zurückgehen und ausholen. Natürlich je grösser das Gebäude, desto tiefer und solider muss das Fundament sein. ⟨ Je mächtiger ein Baum sich mit seiner Krone ausbreitet, desto kräftiger wachsen auch seine Wurzeln.⟩ In der Freude über die Entdeckung der Differentialrechnung und ihre glänzenden Erfolge ging man zu weit, man baute das Gebäude mehr aus, als das Fundament vertrug. Es entstanden Risse, Lücken und es erfolgten sogar Einstürze. Es erwies sich nothwendig eine Prüfung der Fundamente: Weierstrass, Cantor, Dedekind.’

der mathematischen Logik' [1920b]. In print, it appears for the first time in '*Über die Grundlagen der Logik und Arithmetik*' [1905c].

Thus, in contrast to Pasch, with Hilbert the metaphor of a building with a foundation replaced the traditional metaphor of a tree with roots as the principal metaphor for a mathematical discipline. We have seen that some tenuous examples of this change of the image of mathematical theories appeared in Pasch, but it was Hilbert who carried it though all the way. While mathematics was previously regarded as something natural, independent of human constructions and with an inner logic to be discovered, it now became to be seen as something thoroughly artificial. Accordingly, mathematical developments, which were previously considered to be natural growth processes, were now seen as products of intentional constructions.

That mathematical theories are indeed carefully planned is also brought out in Hilbert's 1900 talk on mathematical problems, in which the 'master builder' or architect of theories is explicitly mentioned:[23]

> In fact, a thorough understanding of its special theories is necessary for the successful treatment of the foundations ('*Grundlagen*') of the science. Only that architect ('*Baumeister*') is in the position to lay a sure foundation ('*Fundamente*') for a structure ('*Gebäude*') who knows is purpose thoroughly and in detail. ([Hilbert 1900b], quoted from [Gray 2000, 258])[24]

That the metaphors of buildings and trees were seen also by their contemporaries as being intimately connected to Hilbert and Pasch, can be seen from the fact that in his discussion of Hilbert's work on geometry, Dehn contrasts their attitudes using the same metaphors that they used themselves: Pasch is described as aiming to 'root' ('*Verwurzeln*') the geometric axioms in the external world, while Hilbert research is described as 'constructive' ('*aufbauend*') [Dehn 1922, 79].

[23]The architect's perspective is used by Poincaré to criticize a view of mathematics that focuses solely on the logical relations: 'When the logician shall have broken up each demonstration into a multitude of elementary operations, all correct, he still will not possess the whole reality; this I know not what which makes the unity of the demonstration will completely escape him. In the edifices built up by our masters, of what use to admire the work of the mason if we cannot comprehend the plan of the architect? Now pure logic cannot give us this appreciation of the total effect; this we must ask of intuition.' (*Science et méthode* (1908), quoted from [Detlefsen 1996, 91–92].)

[24]German original: 'In der That bedarf es zur erfolgreichen Behandlung der Grundlagen einer Wissenschaft des eindringenden Verständnisses ihrer speziellen Theorien; nur der Baumeister ist im Stande, die Fundamente für ein Gebäude sicher anzulegen, der die Bestimmung des Gebäudes selbst im Einzelnen gründlich kennt.'

3.3 Disanalogies: Constructing buildings from the top and growing foundations

Another reason for considering the building metaphor as central to Hilbert's conception of mathematics is that he held onto it even if certain aspects of it are contrary to its domain of application. While both the tree metaphor and the building metaphor are compatible with a proportional relationship between a theory and its foundation, namely that as taller trees require deeper roots so do taller buildings require stronger foundations, these metaphors do not capture the historical order of things. That is, while trees start to grow at the roots and houses are built with their foundations first, the historical development in science need not follow this pattern. In 1905 Hilbert explicitly draws attention to this disanalogy:[25]

> It has happened repeatedly in the development of mathematics that in the process of moving forward, the greatest difficulties showed up in particular parts and serious contradictions came to the fore, which threatened to bring the entire edifice of science out of balance. In such cases one always had to turn back, halt the naive development of the discipline, and carefully investigate its principles. [. . .]
>
> Speaking metaphorically ('*im Bilde*'), the construction ('*Aufbau*') of science proceeds *inversely as the construction* of a house, where first strong and secure foundations ('*Fundamente*') are laid down and then one builds the living quarters on top of them. Since, in science one first builds the living quarters, i. e., the theories that are necessary for reaching the aim of the development, e. g., the theories that are demanded by the applications to which one is lead almost by itself. Only later, if it becomes necessary by to the unstableness of the edifice, one begins with a more solid formation of the foundations ('*Fundamente*'). And this seems to be the right way in this case, because if one would always want to start with the foundations ('*Fundamente*') without knowing where the superstructure ('*Oberbau*') is

[25]Corry discusses Volkmann's [1900] use of the building metaphor in connection with Hilbert [Corry 2004, 61–63]. In particular, Volkmann points out this disanalogy, writing: 'The conceptual system of physics should not be conceived as one that is produced bottom-up like an edifice' [Volkmann 1900, 3]; German original: '[. . .] dass *das physikalische Begriffssystem nicht etwa aufzufassen ist als ein System, welches nach Art eines Gebäudes von unten aufgeführt wird,* [. . .].' The passage quoted by Corry on p. 127 is from Max Born's notes of Hilbert's lectures [Hilbert 1905b], while the passage quoted in the present text is from the 'official' lecture notes by Ernst Hellinger [Hilbert 1905a], which contain annotations in Hilbert's hand. See also Corry's discussion of the empirical origins of Hilbert's axioms on pp. 123–127, and the brief discussion of this passage in [Peckhaus 1990, 51].

> supposed to go, a superfluous and an impractically distributed thoroughness would easily follow; one could then build too strong pillars where there is no weight to carry, and conversely too weak ones, where the superstructure urgently needs more support. — The best example for this development is given to us by the history of the infinitesimal calculus. [Hilbert 1905a, 191–193; emphasis by DS][26]

In addition to the difference in the natural order of growth of buildings and scientific theories, there is a further disanalogy, which is not present with the tree metaphor. While roots and branches grow continuously, buildings and their foundations usually do not. As a consequence, Hilbert adapts the idea of growth from plants and transfers it to buildings. Thus, we read in his 1917 lecture course on set theory the following.

> But, in the same way in which a growing tree spreads its branches wider and higher and at the same time drives its roots deeper and more branching into the soil, also the foundation ('*Grundlage*') of a science must, at the same time with its development, be made broader and stronger. It must be left to the tactfulness and understanding of the researcher to bring the foundation ('*Fundament*') in harmony with the size of the scientific edifice ('*Bau*'). [Hilbert 1917a, 42][27]

[26]German original: 'Es ist bisher in der Entwicklung der Mathematik schon wiederholt eingetreten, daß sich beim Vorwärtsschreiten in einzelnen Teilen die größten Schwierigkeiten zeigten, und schwerwiegende Widersprüche zu Tage traten, die das ganze Gebäude der Wissenschaft ins Wanken zu bringen drohten. Hier mußte dann stets eine Umkehr eintreten, ein Abschluß der naiven Fortentwicklung der Disciplin mußte erfolgen und eine genaue Untersuchung ihrer Principien [. . .]

Man kann etwa im Bilde sagen, daß der Aufbau der Wissenschaft gerade *umgekehrt wie der Bau eines* Hauses geschieht, wo man zuerst starke sichere Fundamente legt, und auf ihnen dann die Wohnräume erbaut. In der Wissenschaft nämlich errichtet man zuerst die Wohnräume, das heißt die für das Ziel der Entwicklung nötigen, etwa von den Anwendungen aus geforderten Theorieen, auf die man gewissermaßen von selbst geführt wird. Erst später, wenn es sich durch die Unsicherheit des Gebäudes erforderlich zeigt, beginnt man mit einer festeren Ausgestaltung der Fundamente. Das ist wohl auch hier der richtige Weg, denn würde man stets mit den ersten Fundamenten beginnen wollen, ohne zu wissen, wohin der Oberbau hinaus soll, so wäre eine überflüssige und unpraktisch verteilte Gründlichkeit leicht die Folge; man könnte dann zu starke Pfeiler setzen, wo keine Last zu tragen ist, und umgekehrt zu schwache, wo der Oberbau einer neuen Stütze dringend bedarf. — Das beste Beispiel für diese Entwicklung liefert uns die Geschichte der Infinitesimalrechnung.'

[27]German original: 'Aber so wie ein Baum im Wachsen seine Aeste immer weiter und höher ausbreitet und gleichzeitig seine Wurzeln immer tiefer und verzweigter in die Erde treibt, muss auch die Grundlage einer Wissenschaft gleichzeitig mit ihrem weiteren Ausbau, immer breiter und gefestigter werden. Es muss dem Takt und dem Verständnis des Forschers überlassen bleiben, das Fundament in Einklang zu bringen mit der Grösse des wissenschaftlichen Baues.'

In the famous talk on axiomatic thinking, which was published a year later, we find a more polished version of this statement; the tree metaphor is purged and the new phrase of 'deepening the foundations' is coined:

> The procedure of the axiomatic method, as it is expressed here, amounts to a *deepening of the foundations* of the individual domains of knowledge — a deepening that is necessary for every edifice ('*Gebäude*') that one wishes to expand and to build higher while guaranteeing its stability. [Hilbert 1918, 407; emphasis in original][28]

Hilbert illustrates the idea of deepening of the foundations with the example of the historical development of arithmetic. First, a number of central propositions of a theory are set up, which are enough to develop the theory from them. In arithmetic these are the laws for the basic operations. However, to justify these laws, or axioms, themselves, another layer of assumptions must be introduced. This leads to

> tracing things back to certain deeper propositions, which in turn are now to be regarded as new axioms instead of the propositions to be proved. ([Hilbert 1918, 407], adapted from [Ewald 1996, 1109])[29]

In other words, this process yields a more refined set of assumptions, 'the acutal ('*eigentlichen*') so-called *axioms* of geometry' ([Hilbert 1918, 407], quoted from [Ewald 1996, 1109]).

3.4 Finitary arithmetic outside of the metaphor

In the 1920s Hilbert published his ideas about what became known as 'Hilbert's Programme', namely the meta-mathematical study of formalized theories with the aim of establishing their consistency. Because arithmetic was frequently employed to show the consistency of other theories, the consistency of arithmetic itself posed a particular problem. For its solution, Hilbert proposed the use of 'finitary arithmetic', i. e., only a very restricted, contentful (as opposed to formal) part of arithmetic. It is interesting to note that Hilbert did not extend the building metaphor

[28] German original: 'Das Verfahren der axiomatischen Methode, [...] kommt also einer *Tieferlegung der Fundamente* der einzelnen Wissensgebiete gleich, wie eine solche ja bei jedem Gebäude nötig wird in dem Maße, als man dasselbe ausbaut, höher führt und dennoch für seine Sicherheit bürgen will.'

[29] German original: 'die Zurückführung auf gewisse tiefer liegende Sätze [...], die nunmehr ihrerseits an Stelle der zu beweisenden Sätze als neue Axiome anzusehen sind.'

to finitary arithmetic. Rather, he reserved it for particular theories, like the theory of number fields,[30] for physics[31] and, as we have seen above, for mathematics in general. Most frequently, however, he used it in connection with formal theories. The phrase 'Certain formulas that serve as the building blocks of the formal edifice of mathematics, are called axioms'[32] appears in [Hilbert 1922, 167], [Hilbert 1923, 152], [Hilbert 1928, 66]; later, the same phrase is used, but 'building blocks' is changed into 'foundations' ('*Fundamente*') in [Hilbert 1931, 489]. The duty of the mathematician, according to Hilbert, is 'securing mathematics in its foundations' [Hilbert 1929, 2][33] and her principal tool for this is the finitary standpoint.

Thus, just on the basis of the way Hilbert uses the mathematics as a building metaphor, we can see that he distinguished sharply between mathematics (including its expression in formal systems) and the finitary arithmetic employed in metamathematical investigations. After all, he could easily have described finitary arithmetic as the foundation on which the edifice of mathematics rests. Recall that for Pasch, what constitutes the starting point of mathematical thinking and what establishes its consistency is the 'area of roots', which is seamlessly and organically connected to the rest of mathematics. But, for Hilbert, mathematics and that which secures its consistency are not part of the same metaphor, which indicates that he thought about them in different ways. This observation is supported by Mancosu's argument that finitary arithmetic was initially seen only as a methodological tool by Hilbert, but that it was later also imbued with a special epistemological status [Mancosu 1998, 168–171].

3.5 More construction metaphors

In the spirit of Hilbert's conception of scientific theories and disciplines as something constructed is also his metaphor for the relations between different sciences, with which he opens the talk on axiomatic thinking. According to it the sci-

[30] 'The theory of number fields, e. g., is a gracefully built, sky high edifice [. . .]' ('Die Zahlkörpertheorie, z. B. ist ein feingegliedertes, himmelhoch errichtetes Gebäude [. . .]') [Hilbert 1931, 487].

[31] 'Finally, physics erected mathematical edifices of thought in front our eyes, whose halls are of impressing magnificience.' ('Die Physik endlich ließ vor unseren Augen mathematische Gedankengebäude erstehen, deren Hallen von imponierender Großartigkeit sind.') [Hilbert 1929, 2].

[32] German original: 'Gewisse Formeln, die als Bausteine des formalen Gebäudes der Mathematik dienen, werden Axiome genannt.'

[33] German original: 'die Mathematik in ihren Fundamenten zu sichern'.

ences are arranged like territorial states, with some of them being neighbors, like mathematics, physics, and epistemology [Hilbert 1918, 405].[34]

Let me briefly mention two other metaphors that were employed later and that carry on with highlighting some of the aspects of science and mathematics mentioned above. Bourbaki explicitly continues the trend exemplified by Hilbert by speaking of 'the architecture of mathematics' [Bourbaki 1950] and with Neurath's 'ship' [Neurath 1932/33, 205] — floating, without firm foundations — we see Pasch's idea a conceptual web without privileged vertices taken to sea.

4 Images and practices

After having argued that Pasch and Hilbert deliberately employed different metaphors to express their views of the nature of mathematics, the question arises whether these metaphors are also reflected in their mathematical practice.[35] While a detailed investigation along these lines would be beyond the scope of this paper, a few general observations can be made in support of a close connection between their images of mathematics and the way that they pursued their mathematical research.

An important aspect in which Pasch's and Hilbert's views differed concerns the way in which meanings are conferred to mathematical terms. For Pasch (and Klein), the ultimate sources of meaning are our everyday interactions with the world. In contrast, Hilbert maintains in his correspondence with Frege that the axioms alone, stated at the beginning of a theoretical development, determine the meanings of the primitives [Frege 1976, 12]. These radically opposed views accord well with their images of mathematical theories as something natural and artificial, respectively. Closely related to this issue are the different attitudes towards meta-mathematics. One the one hand, given the (in principle, but not in practice) arbitrary choice of axioms, establishing the consistency of theories became of paramount importance for Hilbert. The danger of the buildings collapsing due to faulty foundations was to be averted. As seen in Section 3.4, Hilbert thought that this could not be done by constructing another theory, but it needed a different approach, one for which the building metaphor no longer fit. For Pasch, on the other hand, the consistency of empirically grounded theories, i. e., those that were developed from a system of meaningful core axioms was as secure as our knowledge could be. Proofs of

[34] A few years later, Weyl employed the relation between citizens and a state as metaphor for the problem of space. See [Scholz 2004, 181–182]. I am grateful to Christophe Eckes for this reference.

[35] I'd like to thank Stephen Pollard to raising this issue.

consistency were only relevant for theories that are based on purely abstract stem axioms and this could be achieved by firmly connecting them to their empirical basis. Pasch's interest in foundational work was to exhibit the continuity between our everyday experiences and abstract, mathematical reasoning.

Both Pasch and Hilbert — the latter albeit to a greater extent and with much more sophistication — were interested in investigating which theorems follow from which axioms. However, while for Pasch these investigations were only aimed at clarifying, and possibly amending, a given body of knowledge, Hilbert encouraged and embraced the development of novel theories based on independence results. This is evidenced by the addition of results regarding 'new' geometries obtained by his student Max Dehn in the French translation of the *Grundlagen der Geometrie* [Hilbert 1900a, 106–110]. In these practices two different attitudes towards the development of new mathematics, which again are in accord with a natural and an artificial conception of mathematics, can be discerned. Like the work of a gardener or plant breeder, who tries to cultivate and bring out the best from the given organic material,[36] Pasch's investigations are always clearly connected to a previous body of mathematical knowledge, which is critically examined and steadily extended. Similarly, Klein's investigations are often aimed at drawing connections between and clarifying existing mathematical theories [Tobies 1981, 51]. Indeed, the botanical metaphor is also employed by Klein when talking about new theories of applied mathematics, which differ from pure mathematics in their levels of rigour; they are described as 'springing up like suckers next to the grafted plant' [Klein 1894–95, 497].[37] While mathematics might appear to develop arbitrarily to an outsider, Klein maintains that there is in fact a regulatory principle, namely *historical continuity* [Klein 1894–95, 488]. Hilbert's approach, in contrast, has been characterized in exactly the opposite way. Dehn describes it as 'unhistoric', 'disregarding prevailing forms', and 'radically exploiting the freedom of mathematical thought', which was of profound influence for the later generations of mathematicians and caused 'a revolution in their thinking that is comparable to the great revolutions in the history of mathematics' [Dehn 1922, 82].

[36]This is not understood in sense of genetic manipulation.

[37]German original: '[...] die wie ein Wurzelschößling neben der veredelten Planze emporschießt.'

5 Conclusion

In this brief survey of the systematic use of metaphors in the writings of Pasch, Klein, Frege, and Hilbert, we saw a shift with regard to the central metaphor for mathematics that was employed. On the one hand, Pasch, Klein, and Frege, who are deeply rooted in the mathematical traditions of the 19th century understood mathematical knowledge and mathematical disciplines as organic entities that follow a course of natural growth, as exemplified by the metaphor of mathematics as a tree. On the other hand, Hilbert replaced this by the building metaphor, which emphasizes the artificial nature of theories and sees their expansion as the result of human constructions. The latter can also lead to outright mistakes like paradoxical or inconsistent theories. The edifices can collapse — trees might grow old or be hit by lightning, but rarely do they fall on their own accord. Thus, the modernist transformation of mathematics identified by Gray is also reflected in the images of mathematics held by the mathematicians themselves, which find their expression in the metaphors they employed. Metaphors help us to conceptualize abstract phenomena, and by paying close attention and studying the metaphors that are used by the scientists themselves we attain a vantage point from which we get a glimpse into their ways of seeing and understanding the world.

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