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Axioms in Mathematical Practice

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Abstract

On the basis of a wide range of historical examples various features of axioms are discussed in relation to their use in mathematical practice. A very general framework for this discussion is provided, and it is argued that axioms can play many roles in mathematics and that viewing them as self-evident truths does not do justice to the ways in which mathematicians employ axioms. Possible origins of axioms and criteria for choosing axioms are also examined. The distinctions introduced aim at clarifying discussions in philosophy of mathematics and contributing towards a more refined view of mathematical practice.

1 Preliminaries

1.1 Motivation

Euclid's presentation of geometry in the *Elements* has been hailed as the paradigm of mathematics and it has served as a model for many theories, including scientific ones (e. g., Newton's). Euclid's presentation is *axiomatic* in that it puts a number of statements (the *axioms*, although he did not call them that) at the beginning and develops the rest of the theory from them via deductions and definitions. Thus, from antiquity onwards axioms have been regarded as an important ingredient in mathematics and many have commented on their special character, often in terms of some kind of privileged epistemological status. The *axiomatic method* was emphasized again at the dawn of the 20th century, most famously by Hilbert but also by many others, as being characteristic of mathematics. More recently, a renewed interest in the role of axioms in mathematics was sparked by Feferman's question 'Does mathematics need new axioms?' [1999]. In recent years philosophers of mathematics have also turned their attention towards mathematical practice (e. g., [Grosholz 2007] and [Mancosu 2008a]) and in this paper these two trends are brought together. To this end, a general framework for talking about the roles and functions of axioms is put forward and various considerations on the use of axioms in mathematical practice are discussed. Thus, instead of proposing just another particular point of view on the role of axioms, the present paper suggests a pluralistic approach.

1.2 The orthodox view and refinements

A general difficulty with discussing axioms in mathematical practice is that most readers will already have some conception of what axioms are and what they do. In particular, despite the fact that philosophers of mathematics have begun to include mathematical practice into their considerations, the role of axioms is still most often seen only in the so-called context of justification[1], where they serve as the foundation for a rigorous presentation of a theory. This view underlies the ideal conception of scientific theories that goes back to Aristotle.[2] According to this *orthodox* picture, axiomatizing consists merely in 'introducing *order* into an already developed field' [Copi 1958, 115]. The axiomatization of a discipline is the final step that concludes the development of a theory. Thus, axioms play no role in the creation of the 'substance' that must be present 'before you can generalize, formalize and axiomatize' [Weyl 1935][3], or, as Hanson put it: 'by the time a law gets fixed into a[n axiomatic] H-D system, the *original* scientific thinking is over' [Hanson 1958a, 1081]. In parallel to this methodological stance with regard to axioms one often finds a specific epistemological stance, where an axiom is considered to be an immediate truth, or as Feferman quotes the *Oxford English Dictionary:* 'A self-evident proposition requiring no formal demonstration to prove its truth, but received and assented to as soon as mentioned' [Feferman 1999, 100].

While various commentators have identified different roles of axioms (more on this below), other authors have insisted that there is one single role of axioms, or at least clearly one *main* role. For example, Easwaran claims that 'the *real* role of axioms' is 'to allow mathematicians to stay away from philosophical debates' [Easwaran 2008, 385 and 381; italics by DS], while Hintikka contends that '[t]he *basic* idea of the [modern axiomatic] method is the capture of a class of structures of the models of an axiomatic system' [Hintikka 2011, 69; italics by DS]. Against such one-dimensional characterizations, it will be argued below that the use of axioms is quite varied and rich.

1.3 Main claims and overview

The analysis of axiomatics presented here combines systematic and historical considerations, with an emphasis on the systematic side, and it is not tied to any particular case study or the views of any particular mathematician or philosopher. In the presentation and discussion of the more specific issues, namely the origins of axioms, criteria for their assessment, and advantages of axiomatic presentations (in Sections 3, 4, and 5, respectively), I will nonetheless frequently draw upon historical episodes in the development of mathematics as well as remarks made by mathematicians and philosophers. Particular conceptions of axiomatics can then be identified by taking some of the discussed aspects as the essential, defining characteristics of systems of axioms, or by adding further, more idiosyncratic considerations. However, any such choice comes at the cost of neglecting other aspects of axiomatics that have played a role in mathematical practice. Thus, one point that I argue for in this paper is:

[1] [Reichenbach 1938]; see [Schickore and Steinle 2006] for more recent discussions of the distinction between the contexts of justification and discovery.

[2] This conception has recently been articulated and discussed as the 'classical conception of science' by Pulte [2005, 30] and the 'classical model of science' by de Jong and Betti [2010, 186].

[3] Quoted from [Wilder 1959, 475].

(a) Axiomatics is an epistemic and methodological tool that can be employed in various ways. In other words, there is no single role that axioms play in mathematical practice; rather, the same set of axioms can be employed in different roles.

It is not the axioms themselves that determine their role, but their use (this point will be illustrated using a toy example in Section 2.1). Because of this, I contend that accounts of axiomatics that claim that there is one single role that axioms play in mathematics are misguided. Of course, an analysis might focus on a certain aspect or on a particular use, but this does not entail that other aspects or uses are of lesser importance for mathematical practice in general.

The historical uses of axioms that will be referred to in the discussion support two further claims, which are also opposed to the orthodox view of axioms sketched above.

(b) Popular criteria for axioms, e. g., that they should be as few, simple, and self-evident as possible, are highly idealized desiderata and by no means necessary conditions for systems of axioms.

The point here is not to deny that mathematicians often use certain epistemic and methodological criteria to guide their choices of axioms, but that these criteria may well stand in conflict with each other and can be given up in light of other considerations.

The history of mathematics does not back up the view that axiomatization is all but the final step in the development of a theory; for example, geometry did not end with Euclid and neither did the study of lattices end with the formulation of the axioms of lattice theory. These observations lead to my third main claim:

(c) The practical usefulness of axioms goes well beyond the context of justification and the aim of clarifying and providing foundations for mathematical theories; they are also engines for discovery in mathematics.

In this paper, I consider an *axiom system* to consist of a number of statements (the axioms) and a notion of consequence, which is employed to derive further statements from them; the resulting axiomatic *theory* includes all statements that are derivable from the axioms and which might also contain terms that are defined from the primitive terms that occur in the axioms (i. e., the *primitives* of the system). It is also useful on some occasions to distinguish further between the finite *known theory* of actually derived consequences and the infinite *logical theory* consisting of all derivable consequences.[4] In order to provide a framework for discussing the various uses of axioms, I identify (in Section 2) three dimensions (namely: presentation, role, and function) according to which axiom systems can be classified. To put my final claim concisely:

(d) The dimensions of *presentation*, *role*, and *function* allow us to characterize different uses of axioms and to compare them along these three axes, with the purpose of clarifying discussions of axiomatics in mathematical practice.

After briefly addressing some potential formal, philosophical, and historical concerns about my project, I will introduce and discuss the main dimensions of axioms in Section 2 with the help of an imaginary example. In mathematics the origins of any particular system of axioms lie in some

[4] Tennant calls a finite set of statements that includes the axioms and that is included by the logical theory a *development* [Tennant 2012, 85–86].

previously given body of mathematical knowledge, as will be argued for in Section 3, ‘Where axioms come from’. Frequently used criteria for assessing a proposed axiomatization are presented in ‘What are good axiom systems?’ (Section 4), and my discussion is closed by emphasizing the unique possibilities that axiomatics affords in Section 5, ‘What you can do with axioms that you couldn’t do without’.

1.4 Formal, philosophical, and historical concerns

Before proceeding to the main part of this paper, I would like to address some concerns that have been raised by readers of previous drafts of this paper regarding the use of formal logic, the distinction between philosophical and mathematical considerations, and the historical range of application of my analysis.[5]

Formal concerns. With the development of formal logic and the intensive study of properties of formal theories in the 20th century, immense theoretical insights have been obtained. In addition, it has become custom in philosophy to discuss mathematics in general from this modern perspective.[6] Without calling into question the usefulness of this approach with regard to some aspects of mathematics, there are however, other aspects that do not require a formal treatment or that even cannot be fully captured by such a treatment. The use of axioms in mathematical practice, I contend, is such an aspect. This view is supported by none other than Bourbaki, who writes that the essential aim of the axiomatic method ‘is exactly that which logical formalism by itself can not supply, namely the profound intelligibility of mathematics’, and that formalization is ‘but one aspect of this [axiomatic] method, indeed the least interesting one’ [Bourbaki 1950, 223]. Of course, I will avail myself of modern terminology, notation, concepts, and results, when appropriate. However, I do not want to use these as the starting point of the investigation — as the golden standard, so to speak, against which actual historical developments of mathematics are to be measured. The reasons for this are three-fold. First, most historical examples of axiomatics, even such paradigmatic ones as Euclid’s *Elements* or Hilbert’s *Grundlagen der Geometrie* [1899], would simply not meet this standard. Second, meta-logical notions that have a rigorous meaning in a formal setting, like proof, completeness, and categoricity, have all been employed outside of such a setting long before their formal counterparts were introduced. For example, Dedekind gave a proof of categoricity in [Dedekind 1888] and completeness is discussed in [Hilbert 1899].[7] Third, in contemporary discussions axiomatization is frequently conflated with formalization (e. g., in [Lakatos 1976, 142–143] and [Suppe 1977, 113]) and thus the differences between axiomatization and formalization are not brought out in full. In this paper I resist looking at mathematics exclusively through the lenses of formal logic and present axiomatics (both formal and informal) as a powerful methodological technique in its own right. Thus, I shall begin in Section 2

[5] I am particularly indebted to two referees of this journal for many insightful comments that prompted these reflections.

[6] Presentations of axiomatic theories in terms of formal logic can be found in any logic textbook, e. g., [Enderton 2001] and [Boolos et al. 2007]; for particular emphasis on axiomatics, see [Blanché 1962].

[7] For a more detailed discussion in connection with criteria for the choice of axioms, see Section 4.3.

by discussing the presentation of axiomatic theories, their roles and functions in general terms, with formal presentations as special cases.

Philosophical concerns. As mentioned in my brief presentation of the orthodox view of axioms, they are frequently described as some kind of immediate truths. As has been noted, however, for example by Maddy [1988, 483], mathematicians have willingly laid aside such strong philosophical commitments in light of more pressing mathematical considerations. In fact, some have openly renounced the importance of what they considered to be 'non-mathematical' considerations for mathematics, referring to them as 'metaphysical' (Leibniz, Veblen) or simply 'philosophical' (Klein). (Since this issue has to do mainly with criteria for selecting axioms, it will be discussed in more detail in Section 4.1.) Nevertheless, this alleged contrast between mathematical and philosophical issues neither implies that mathematicians do not pursue (broadly construed) epistemological goals themselves, nor that there are no genuine philosophical questions to be asked about mathematics (e. g., regarding the nature of the entities that are being described by axioms). Rather, the main target of the somewhat polemical remarks mentioned above are positions that are based on a special kind of insight into mathematical truth or the nature of mathematical objects, which were seen as hindering mathematical progress, or at least as not contributing to it in an obvious manner. For a contemporary mathematician it is difficult to imagine the force of the opposition that was once raised against the introduction of negative and complex (or imaginary) numbers.

There is also a lesson to be learned from these observations for contemporary philosophy of mathematics: An overly narrow understanding of epistemology that is focused on truth and certainty cannot do justice to mathematical practice. In other words, a philosophy of mathematics that takes mathematical practice seriously should be based upon a broader notion of epistemology, which includes methodological considerations (e. g., issues of clarity and intelligibility), as well as social factors and those currently studied by cognitive science. The present paper is intended as a contribution towards such a more refined view of mathematical practice and as a basis for further philosophical discussions.

Historical concerns. Examples of axiomatic work can be found throughout the history of mathematics, the most famous being Euclid's *Elements of Geometry*. This form of presentation, also known as 'more geometrico', soon came to be regarded as prototypical for all scientific theories. Works as different as Newton's *Principia* and Spinoza's *Ethics* were modeled after it. It was only in the nineteenth century, however, when axiomatic presentations were also employed in other areas of mathematics (e. g., in algebra and number theory). Slowly, the separation of syntax and semantics emerged, for example, in the duality of projective geometry, and the modes of inference themselves also became a focus of attention for some mathematicians. The development that was prepared by Dedekind, Frege, Pasch, and Peano found a high point in Hilbert's work on geometry and later on meta-mathematics. By the turn of the twentieth century, axiomatic presentations had become increasingly popular in many areas of mathematics and also in the more theoretically advanced of the natural sciences. Hilbert's 'Axiomatic thought' [1918] provides an informative historical account of these developments, and

Blanché's *Axiomatics* [1962] gives a presentation that also includes later developments, in particular of formal logic, model theory, and Gödel's limitative results. Nonetheless, it must be kept in mind that the development towards the modern view of axioms was neither as straightforward nor as smooth as the previous sketch might suggest. Individual mathematicians have held much more nuanced views, e. g., by identifying some areas of mathematics for which an axiomatic treatment is warranted, but rejecting it for others (Frege, Russell), or by entertaining different conceptions of axiomatics, depending on the purpose of the axiomatization (Pasch).

To be clear, my aim is not primarily to investigate how the term 'axiom' has been employed in mathematics, nor to explicate a common understanding of the term (indeed, there might not even be a significant common understanding that underlies all uses of axioms), but to discuss how and for what purposes mathematicians have employed axioms. For this it does not matter whether the historical actors were themselves aware of the dimensions I introduce, nor whether they used the term 'axiom' to refer to axioms in the sense of the present paper. Of course, each individual can also use other criteria to discuss and evaluate his or her own axioms. An example of such a more sophisticated view is Euclid's distinction between 'postulates' and 'common notions,' both of which I consider to be axioms, although Euclid himself did not refer to them as such. These criteria, however, can always be understood as refinements of the general framework.[8]

Given that the following discussion employs many notions that emerged explicitly only in the nineteenth century and were fully developed only in the twentieth century (e. g., the distinction between semantic and syntactic roles of axioms, Section 2.3), this might create the mistaken impression that my analysis of axioms covers only the mathematical practice of the past two centuries. Because of the popularity of axiomatics during this period of time, many of my historical examples will belong to it, but the analysis is intended to apply to all instances of axiomatics, from Euclid to the present. As mentioned above, it is irrelevant whether mathematicians were aware of the distinctions I refer to, or whether they used the term 'axiom' for their axioms: While individual mathematicians may not exploit all ways in which axioms can be put to use, the dimensions of presentation, role, and function introduced in the next section can nonetheless be used as a framework for the study of their achievements.

2 The main dimensions of axioms

2.1 Some imaginary axioms

Imagine you find an undated manuscript that reads: 'Axioms: Every gaal is copper to itself. If any gaal is copper to another gaal, and the second is copper to a third gaal, then the first gaal is also copper to the third one.' At first you are surprised by the word 'gaal', which you do not know, and by the unfamiliar use of the word 'copper'. These are the *primitive terms*, which are put into logical relations by the remaining words in the sentences.

[8]The same can be said for Pasch, Frege, Russell, Hilbert, who each had their own sophisticated conceptions of axiomatics, which deserve their own full-length treatments.

Given that you do not know anything about the origin of this manuscript, you surmise some possibilities about the *meanings* of 'gaal' and 'copper': The author had particular meanings of these words in mind, which are different from those in modern English; the author is giving definitions of gaals and the binary relation of copper; the author uses some made-up words, which in this connection had no more meaning for her than they have for you. With regard to the main *purpose* of the axioms (that they are axioms you know from the first word in the manuscript) you are also left in the dark: They might describe some properties of gaals and copper, or introduce the usage of the terms 'gaal' and 'copper'; it is also possible that the main interest of the author was to find out the relation between the axioms and other statements, like 'if a gaal is copper to another gaal, then the latter is also copper to the former'. Without further information, there is no way of determining which of these interpretations, or perhaps some combination of them or even some other possibility, is one that the author had in mind. In other words, the axioms themselves do not tell you how they were intended to be used. Indeed, it is easy to augment the story in such a way that any of the above interpretations becomes plausible. Without such additional information, however, there is no privileged understanding of 'axiom' that forces you to regard any one of the interpretations as correct and the others as incorrect. Moreover, regardless of the author's original intentions, you may now adopt any one of the proposed viewpoints and continue to study these axioms from that perspective.

The previous scenario illustrates some general features of axiomatic systems, which will be discussed in the following sections (2.2–2.4), whereby different ways of interpreting the mysterious manuscript are introduced systematically. To clarify discussions on the use of axioms in mathematics I distinguish three different independent dimensions of axiomatic systems: the presentation of the axioms (which includes the kinds of language and consequence relation employed), their role, and function (Table 1). Despite the fact that these dimensions are theoretically separable, in practice an

Presentation		*Role*	*Function*
Language	Consequence		
natural	implicit	semantic	descriptive
symbolic	explicit	syntactic	prescriptive
formal			

Table 1: Three dimensions of axiom systems.

axiomatic system is often not categorized clearly and unambiguously into any one of them. I will return to a discussion of the imaginary axioms according to these dimensions in Section 2.5 and then, in Section 3, I will employ these dimensions in a general discussion about where axioms come from.

What is important to realize at this point is that the observation made at the end of the second paragraph of this section, namely that one can adopt any viewpoint with respect to the role and function of a given set of axioms, also applies to the situation in which all terms used in axioms have meanings independently of the axiomatization, e. g., if the axioms are stated as 'every line segment is congruent to itself' and 'if any line segment is congruent to another line segment, and the second is congruent to a third line segment, then the first line segment is also congruent to the third one'. However, while such a presentation makes the axioms look more familiar, it also complicates the analysis by obscuring subtle conceptual distinctions.

2.2 The presentation of axioms

A system of axioms must be formulated in a language and have a notion of consequence associated with it.[9] These are the two components of the first dimension of axioms, which concerns their presentation. This dimension is independent of the other two, which both concern the usage of a system of axioms, but it differs in one important aspect from them. Although it might not always be straightforward to determine the kind of language and the notion of consequence that is employed, these are not completely up for reinterpretation as the aspects of role and function. In other words, we might take the language and consequence relation as specific characteristics that contribute to the individuation of a system of axioms. Moreover, given that a formal language is a specific kind of symbolic language, this means that the different aspects within one dimension are not independent in the way the aspects of the two dimensions of usage are.

2.2.1 The language of axioms: Natural, symbolic, formal

The kind of language in which axioms are presented can be a natural language or vernacular, it can contain certain symbols (or words) that are not taken to be part of a given natural language, or its grammar can be fully specified in a formal way.

The imaginary axioms from Section 2.1 are formulated in some variant of English, i. e., in a natural or informal language. With the increased use of axiomatic systems in 19th century mathematics a particular tension related to such presentations became more and more apparent. On the one hand, the use of familiar terms in the axioms and theorems could be advantageous from a psychological point of view, by calling forth unforeseen associations, aiding the imagination, and honing one's intuitions. On the other hand, such terms often suggest more attributes and relations than are actually specified by the axioms. For example, one might know that line segments have two distinct endpoints, but this does not follow from the two axioms about line segments listed above. If axioms are understood as being the only allowed starting points in the development of a theory, then inadvertently exploiting such background knowledge in a deduction amounts to a fallacy. A famous example is Euclid's proof of the first proposition in Book I of the *Elements*, where he uses the fact that two overlapping circles have two points of intersection — something that does not follow (according to our present logical standards) from the axioms.[10]

A strategy for averting the danger of unwarranted conclusions that result from unstated assumptions is to treat the primitives as if they were meaningless (this process has been referred to as 'emptying [...] the terms of contentual meaning' in [Detlefsen 1998, 422], and 'de-semantification' in [Krämer 2003]). Instead of treating familiar terms as if they were devoid of their usual meanings, one can

[9] In principle, if axioms are only used in a semantic role, no consequence relation would be necessary. However, as far as I know, there is always some notion of consequence at play in practice.

[10] Clearly, this depends on what counts as the allowed modes of inference. More on this issue in Sections 2.2.2 and 5.1.

introduce new terms (symbols) from the start and stipulate their usage with the axioms, in which case we speak of a *symbolic* axiomatization. This suggestion can be found for reasoning in geometry in Dedekind's 1876 letter to Lipschitz [Dedekind 1932, 479], and recent psychological experiments, which show that logical fallacies are spotted more easily if the statements involve unfamiliar terms, support this strategy [Sá et al. 1999]. Symbolization can come in various degrees and might not be as easy to detect as one might think. For example, for the author of our imaginary axioms the terms 'gaal' and 'copper' might not have been part of the vernacular and thus they could have been introduced as completely new symbols. A clearer instance of symbolization would use single letters, as in 'Every G is C to itself'. Finally, not only the primitives, but also the logical terms themselves could be expressed in symbols, as in '$\forall x(G(x) \supset C(x,x))$'. Early examples for symbolic presentations of mathematical theories are Peano's axioms for linear systems (vector spaces) and for the natural numbers [Peano 1888; 1889].

The demand for more rigorous mathematics in the 19th century led further to the desire of completely fixing the grammar of symbolic languages and thus to the development of formal languages, most famously by Frege [1879]. A *formal language* consists thus of a determined alphabet together with explicitly stated rules for constructing well-formed expressions. Typically one distinguishes between symbols for logical connectives, quantifiers, variables, constants, predicates, etc.[11]. Closely related to the kind of language in which the axioms are formulated is the notion of consequence that underlies deductions.

2.2.2 Implicit and explicit notions of consequence

Since axioms are used as starting points for the development of theories, some notion of consequence is presupposed to determine what follows from them. In general, however, this notion is not formulated explicitly. Euclid relied on the notion of consequence implicit in the mathematical practice of his day, as did Hilbert in his *Grundlagen der Geometrie* [1899].[12] Famous attempts at making explicit the principles underlying inferences are Aristotle's syllogisms, the identification of the rule of *modus ponens*, Boole's algebraic treatment of logic, and of course Frege's *Begriffsschrift* [1879]. Note that these often presuppose some regimentation of the language. In connection with formal languages, a variety of methods have been developed to specify rigorously what counts as a valid formal inference.[13] If the rules of inference by which theorems are derived from the axioms are stated explicitly, we speak of a *formalization* or *formal system* [Sieg 1995]. Thus, a formal axiomatization is expressed in a formal language and employs only explicitly stated inference rules.

Hardly any theory is presented in a completely formalized way in mathematical practice, since this would require, in addition to a formal presentation of the main notions of the theory, also a formalization of the background theory (e. g., arithmetic and set theory in case one wants to refer to

[11] See any textbook on formal logic, e. g., [Enderton 2001] and [Boolos et al. 2007].

[12] An interesting attempt to explicate Euclid's reasoning is [Manders 2008a;b].

[13] To mention a few: syntactic rules, such as modus ponens and substitutions [Frege 1879, Hilbert and Ackermann 1928], natural deduction and sequent calculus [Gentzen 1935], semantic tableaux [Beth 1955], and resolution [Robinson 1965]; semantic notions of consequence were developed by Tarski [1956 (1936] and Hintikka [2011].

numbers and sets in the theory). However, in cases where a theory has been formulated very carefully, as in Hilbert's geometry, it is often considered rather straightforward to translate a given axiomatization into a formal language and to apply a logical calculus to it, i. e., to formalize it.[14]

It is important to notice that formalization can be useful for particular purposes (in particular, the study of meta-theoretical properties, see Section 5.4), but that it is *not* a necessary ingredient of axiomatics, which can be seen from the already mentioned axiomatizations of geometry by Euclid and Hilbert. Van Heijenoort notes in his introduction to Peano's *Arithmetices Principia* [1889] that the way Peano typically formulates his proofs brings out 'the whole difference between an axiomatization, even written in symbols and however careful it may be, and a formalization', since Peano does not explicitly formulate rules of inference [van Heijenoort 1967, 84].[15] Thus, with regard to the presentation, Peano employs a symbolic language and an implicit notion of consequence. Nevertheless, axiomatization and formalization have been frequently conflated, as one can find criticisms that are directed at formalization being leveled against axiomatization in general.[16]

2.3 Semantic and syntactic roles

Based on the brief discussion in Section 2.1 of some possible purposes of the imaginary axioms introduced above, we can distinguish two very different aspects of axioms. On the one hand, we can focus on their meanings and then the axioms play a *semantic* role by characterizing (describing or defining, see Section 2.4) one or more domains (models, structures). Examples of axiomatizations that are frequently understood in this way are Dedekind's and Peano's axiomatizations of the natural numbers and axiomatic presentations of algebraic groups and fields. On the other hand, we can focus on axioms as sentences, in which case they serve a *syntactic* role of systematizing and representing the propositions pertaining to a theory. Here, the axioms — together with some notion of consequence (Section 2.2.2)[17] — determine the sentences (theorems) that can be derived from them. An axiomatization is thus a stipulation of the starting propositions of a science and at the same time an initial regimentation of the scientific language, which can be extended by the definition of new terms. Euclid's and Hilbert's axiomatizations of geometry are often understood mainly in this sense.

These roles must not be understood as excluding each other. Indeed, any system of axioms can play both roles and in many cases they are considered simultaneously. Thus, this distinction pertains to the focus of attention when the axioms are used. For example, in the early developments of non-Euclidean geometry the axioms were employed mainly in their syntactic role (with the aim of deriving a contradiction from the negation of Euclid's parallel postulate), but later, when models for these theories were sought, the semantic role of the axioms was at play. There seems to be a tendency to consider axiom systems with very few and simple axioms as playing mainly a semantic role (e. g., axiomatizations of algebraic structures), and axiom systems with more and complex axioms as playing a syntactic role (e. g., axiom systems for geometry or Hull's theory of rote learning with 18 postulates

[14] On difficulties of formalization, see [Rav 1999].

[15] See also Section 5.1 for a discussion of the separation of statements and inferences.

[16] See, e. g., [Lakatos 1976] and [van Fraassen 1980].

[17] Axioms can play a syntactic role, even if the notion of consequence that is used to obtain theorems is a semantic one.

and 16 primitives [Hull et al. 1940]). Nevertheless, every system of axioms can be viewed both from a syntactic and semantic point of view, since it is not the axioms themselves that determine the role, but the person who uses them. For example, we can make deductions from the axioms of group theory and read axioms of geometry as defining geometric spaces; our two imaginary axioms from Section 2.1 can be used syntactically as the starting point for the deduction of theorems, or semantically to characterize systems of objects ('gaals') with a binary relation ('copper') on them. This double role of axioms might indeed be a characteristic of mathematical notation in general.[18] For example, the symbol i can be understood to represent an imaginary number (or object), but it can at the same time be manipulated syntactically according to the usual rules of algebra with the additional restriction that $i^2 = -1$.

2.4 Descriptive and prescriptive axiomatizations

For both the semantic and syntactic roles of axioms it is possible to draw an orthogonal theoretical distinction depending on the function of the axiomatization. Along the semantic dimension, a system of axioms can be intended to *describe* one or more given domains, like natural numbers or substitution groups, or to *prescribe* the conditions for certain domains (i. e., to define them). In the language of modern logic there is only the relation of *satisfaction* between a system of axioms and its models, but this obscures the difference between describing and defining a model. The term 'intended model' is used to single out the axiomatizer's intentions, but again this terminology does not allow for a differentiation between a model that is being described and one that is defined. However, logic is not to blame for this, since these relations do not pertain to logic per se, but to the pragmatics of axiom systems. Along the syntactic dimension, the same distinction can be drawn. We can say that a particular set of axioms describes a certain theory, i. e., a set of sentences that are given independently, for example, by making repeated observations of triangles. Alternatively, we can say that the axioms define the theory as consisting of only those propositions that belong to its consequences. Again, this distinction cannot be expressed in logical terms alone.

The difference between descriptive and prescriptive axiomatizations is often cast in terms of the *meanings* of the primitive terms involved (as I did in Section 2.1). If the primitives are assumed to have a meaning outside of and independent from the system of axioms, the axiomatization is commonly considered to be descriptive. In this case, Bernays speaks of the axioms being 'embedded in the conceptuality of the theory in question' [Bernays 1967, 189] and Hilbert and Bernays refer to the axiomatization as *material* or *contentful* [Hilbert and Bernays 1934].[19] The paradigmatic example of a material axiom system is Euclid's *Elements*, where the terms 'points' and 'lines' supposedly refer to points and lines.[20] Understood in this way, the axioms in our running example express some actual properties of gaals and copper (possibly ones that are self-evident, but not necessarily so, see

[18] See [Sfard 1991] and [De Cruz and De Smedt 2010].

[19] Weyl also uses the term 'sachlich,' which can be translated as 'realistic' or 'matter-of-fact' [Weyl 1930, 17–18]. Bernays distinguishes between two 'kinds' of axiomatics: *material* or *pertinent axiomatics*, and *definitory* or (contrary to the terminology used in the present article) *descriptive axiomatics* [Bernays 1967]. The term 'descriptive axiomatization' is also used in [Detlefsen 2013], where the attention is restricted to the syntactic role of axioms.

[20] What Euclid thought points and lines were is a separate question.

Section 4.1). A connection is frequently drawn between the dimension of presentation and that of the function of an axiomatization, such that informal axiomatizations are often interpreted materially, but this is, again, by no means necessary. Recall that the three dimensions of axioms shown in Table 1 are independent from each other.

If the meanings of the primitives of an axiomatic system depend entirely on the relations that are expressed by the axioms, then the axioms are considered to be prescriptive (or normative), i. e., they determine the subject matter under consideration and one also speaks of them as 'implicit definitions'. If the meanings of the terms are presumed fixed, then the axioms simply determine what can be said about the referents of the primitive terms, in our example, about gaals and copper. However, since no information about gaals and copper is deemed relevant for the axiomatic presentation that goes beyond what is stated in the axioms, the axioms could also be seen as being about any other domain that satisfies them. This leads to the view that the referents of the primitive terms are variable and the axioms are not statements that can be true or false; rather, they represent *schemata* for a variety of different statements.[21] In analogy to Russell's propositional functions, Carmichael refers to axiom systems as 'doctrinal functions,' which yield a doctrine (or theory) if the primitive terms are given an interpretation [Carmichael 1930]. For example, our imaginary axioms become meaningful if we consider 'gaals' to refer to rational numbers and the relation 'copper' to hold between any two rational numbers whose difference is an integer, e. g., between $3\frac{1}{2}$ and $5\frac{1}{2}$. Considered in this way, a system of axioms generally cannot determine a single interpretation for its terms, but only the structure of the relations that hold between them; understood in this way, axiom systems are also called *relational*, *structural*, *algebraic*, or *abstract*. In other words, such a system of axioms does not define one single model, but a class of models. This state of affairs has also been expressed by saying that the axioms determine a predicate that is satisfied by their models; Carnap speaks in this connection of the *explicit predicate* of an axiom system [Carnap 1958, 175] and Suppes, who considers models to be purely set theoretic, speaks of axioms as defining a *set theoretic predicate* [Suppes 1957].

If an axiom system is able to determine all sentences of a given target theory, it is called 'complete' (or 'relatively complete', if the target consists of a set of sentences that is not necessarily deductively closed [Awodey and Reck 2002, 4]).[22] In this case the differences between the extensions of a descriptive and a prescriptive axiomatization vanish, which helps to understand the following quote by Hilbert, who is well aware of the double function that axioms can play, i. e., that they can serve as definitions and descriptions. Without batting an eyelash he speaks of the role of axioms as being both descriptive and prescriptive:[23]

[21] Historically, intermediate positions were also held, in which the primitives were considered to have fixed meanings, but where they could be replaced systematically by others, thus yielding a different system of axioms. See, e. g., [Pasch 1882] and [Enriques 1903].

[22] For other notions of completeness in mathematics, see [Awodey and Reck 2002] and [Detlefsen 2013]. These more sophisticated analyses of completeness can be understood as refinements of the framework presented here. We will return to the issue of completeness in the discussion on the origins of axioms in Section 3.1, below.

[23] Hilbert's notion of *existential axiomatics* can be understood as an attempt to capture both aspects of axiomatizations just discussed. In existential axiomatics, one first posits a system of objects, which the axioms are then intended to describe. For a discussion, see [Sieg 2002].

> When we are engaged in investigating the foundations of a science, we must set up a system of axioms which contains an exact and complete *description* of the relations subsisting between the elementary ideas of that science. The axioms so set up are at the same time the *definitions* of those elementary ideas [. . .] (Hilbert 1900a, 298–299; quoted from Ewald 1996, 1104; italics by DS)

2.5 The imaginary axioms revisited

To illustrate the three main dimensions of axioms introduced above, let me briefly return to the imaginary axioms from Section 2.1 and discuss them in terms of the terminology introduced above. The first dimension concerns the presentation of the axioms. If the language in which the axioms are formulated is a natural language, then we assume that the terms 'gaal' and 'copper' are meaningful terms in the language of the axioms' author and that their referents are clear to anybody who speaks the author's language, even though we might not know what these are. If the language is symbolic, then the primitive terms are understood as symbols, whose referents must be specified by other means. To recognize the language of the axioms as a formal language, we would need to be presented with explicit rules for forming well-formed expressions in that language. If we have no information about the notion of consequence that is to be employed, we must consider it to be left implicit.

Once there is agreement on the mode of presentation, there are four different combinations of the dimensions of role and function. Any axiomatization can be used in any of these four ways, regardless of the original intentions of the author or the way in which they have been used traditionally. The possibility of taking these different perspectives accounts for the multiple ways in which axioms can be employed in mathematics and is reminiscent of the notion of 'productive ambiguity' discussed by Grosholz [2007].

If we consider the main role of the axioms to be semantic, then we focus on understanding their referents. Are the axioms intended to be used syntactically, the referents are much less relevant, but the relations between the axioms and other sentences is what matters most. If the are axioms viewed as being 'semantic descriptive', then they express known relations between gaal and copper; if they are 'syntactic descriptive', they represent already accepted statements about them. If the axioms are 'syntactic prescriptive', they are stipulations of statements are to be taken as genuine truths about gaal and copper; in their 'semantic prescriptive' role they determine certain relations in which the referents of 'gaal' and 'copper', whatever they may be, must stand in order to count as constituents of a model of the axioms.

2.6 Discussion

My claim that the dimensions of axioms introduced above help to clarify discourse in philosophy of mathematics can be demonstrated by taking a brief look at other classifications of axiom systems that have been proposed in the recent literature. In his article on 'Does mathematics need new axioms?' Feferman distinguishes between *structural axioms* and *foundational axioms*. The former are '*definitions of kinds of structures*,' such as 'groups, rings, vector spaces, topological spaces, Hilbert spaces, etc.', and they reflect the common understanding of axioms of the working mathematician.

Foundational axioms are axioms in one sense in which they are used by logicians, namely 'axioms for such fundamental concepts as number, set, and function that underlie *all* mathematical concepts' [Feferman 1999, 100; italics in original].[24] Prima facie, Feferman's distinction is based on the function that the axioms play in their semantic role: structural axioms are prescriptive, while foundational axioms seem to be descriptive (indeed, the two examples that Feferman discusses 'started with an informal "naive" system'). This is, for example, how Easwaran interprets Feferman's distinction, considering it to be one between 'two distinct types of axioms' [2008, 382]. However, other criteria also appear to underlie Feferman's distinction. First, foundational axioms are about one particular model (up to isomorphism), while structural axioms also allow non-isomorphic models. Second, the domains that foundational axioms are about, are particular in the sense that they are 'concepts that underlie *all* mathematical concepts'. Moreover, after introducing the distinction Feferman presents the development of two foundational axiom system, namely the Dedekind-Peano axioms for number theory and the Zermelo-Fraenkel axioms for set theory, focusing on the latter in his careful discussion of the reasons for accepting new axioms. In this discussion, however, Feferman mentions that the axioms have different models, i. e., now considering their function to be prescriptive. While this is reasonable from the perspective of mathematical practice and corresponds with the attitude towards axioms that I am advancing in this paper, it does not fit neatly into Feferman's own classification of axiomatic systems. After all, as becomes clear in his discussion, foundational axioms are sometimes employed as structural axioms. To this Feferman could simply reply that when logicians asses their axioms they become working mathematicians or that the distinction between structural and foundational axioms was never meant to be an exclusive one; this, however, seems to be more in line with the main tenets of the present paper than with the way Feferman frames his discussion.

More recently, Shapiro introduced a similar classification of axioms into *algebraic* and *assertory* ones. An algebraic understanding of axioms implies that 'any given branch is "about" any system that satisfies its axioms', whereas axioms are assertory if they 'are meant to express propositions with fixed truth values' by referring to a single mathematical structure (up to isomorphism) [Shapiro 2005, 67]. This distinction also seems to rely on the difference between the prescriptive and the descriptive functions of axioms. Indeed, Shapiro here also explicitly allows for the same axioms being 'algebraic from one perspective, and assertory from another', correcting his 'potentially misleading' earlier discussion of arithmetic and real analysis as being non-algebraic [Shapiro 2005, 67, footnote 6]. However, given the restriction that assertory axioms must be categorical, it follows that all axiom systems that can be viewed as assertory can also be viewed as algebraic, but not vice versa, which is a consequence that Shapiro does not point out and for which there does not seem to be any independent reason for accepting it. Moreover, axioms that are used descriptively, but that are not categorical (e. g., Dedekind's axioms for 'number modules', which developed into the prescriptive axioms for lattices [Schlimm 2011, 53–57]) do not fit nicely into Shapiro's classification.

[24]The second sense in which logicians speak of axioms is for 'laws of valid reasoning that are supposed to apply to *all* parts of mathematics', but Feferman does not discuss this usage further.

One general difficulty that arises with such classifications of axioms as those by Feferman and Shapiro is that they really are classifications about the intended use of axioms, not about the axioms themselves. For example, for all we know Peano might have intended his axioms to describe the structure of the natural numbers, i. e., they are 'foundational' or 'assertory' in Feferman's and Shapiro's terms. However, assuming that Peano's language is a first-order language, then the axioms are not categorical, thus they do not uniquely determine this structure, which would make them 'structural' or 'algebraic'. So, what are they? Using the terminology introduced in this paper, we can say that Peano intended his axioms to be descriptive, but that when they are used in a prescriptive role, they have non-isomorphic models.

3 Where axioms come from

Let us briefly return to the short story about the mysterious manuscript from Section 2.1. Related to the question regarding the author's intentions when writing down the axioms is the following: Where do the axioms come from in the first place? Or, in other words: Why did the author write down these axioms and not others?

If mathematics were nothing but a game with formal systems, it would be sensible to write down a number of unmotivated statements and take them as axioms for a new mathematical theory. In practice, however, I am not aware of any introduction of a new system of axioms that was not related in some rational way to a previously given body of knowledge. Recall that geometry existed before Euclid, non-Euclidean geometry grew out of the axioms for Euclidean geometry, number theory was practiced before Dedekind's and Peano's axiomatizations, and set-theoretic theorems were proved before ZFC became the canonical system of axioms for set theory. Given the two different roles of axioms identified in Section 2.3, the connection between new axioms and existing mathematics can be established on the basis of syntactic or semantic factors. For example, the links to a given body of knowledge can be a number of statements that are accepted about the subject matter (e. g., about possible constructions with straightedge and compass) or a previously given system of axioms; alternatively, the previous mathematical knowledge can also involve one or more particular domains that the axioms are intended to characterize (e. g., natural numbers, groups of substitutions). Finally, as the examples make clear there is certainly more than one single way to arrive at new axioms. In the remainder of this section I shall discuss what I take to be the main considerations that are involved in the formulation of new axiom systems, treating syntactic (Sections 3.1–3.2) and semantic (Section 3.3) origins of axioms separately.

3.1 Reasoning from theorems

According to the orthodox picture, axiomatizing consists merely in 'introducing *order* into an already developed field' [Copi 1958, 115], where a substantial number of statements that are part of a theory have already been obtained and the axiomatizer judiciously picks out some of them from which all others can be deduced. Seen in this way, axiomatizing might require some skill and experience, but is not a truly creative process. Nevertheless, axiomatics has a certain critical function, since it allows for the discovery of circular arguments and hidden assumptions, and for the exposure of previous flaws or inaccuracies in our reasoning. One strategy for arriving at the axioms is to reason one's way backwards from certain accepted propositions in such a way that the latter become theorems that are provable from the axioms. This procedure goes back to ancient Greek mathematicians and has been discussed in the literature under various names, such as *analytical*, *abductive*, *regressive*, *critical*, *retroductive*, or *axiomatic methods*.[25]

Historically, one of the earliest and most extensive accounts of a heuristic device for finding suitable axioms to prove given conjectures and for solving construction problems, called the *method of analysis and synthesis*, was given by Pappus of Alexandria (ca. 290–330 AD). According to a popular reading, this procedure is roughly as follows. The *analysis* consists in assuming a statement s to be proved and deriving from it a series of consequences ending in c, until either c is recognized as false (in which case s is also false) or it is recognized as being true. In the latter case, the *synthesis* consists in deriving s from c. In strictly logical terms this method is problematic, because it is not the case in general that if one can deduce c from s, one can also deduce s from c. Thus, Greek analysis guarantees success only in cases in which convertible inferences are used in the deduction of c.[26] Alternatively, possible candidates for axioms can be obtained by a judicious procedure of trial and error.

Regardless of how the new axiom c has been obtained, we can distinguish between the following variants of this method: (a) The method aims at reducing conjectures *to already accepted axioms*. (b) The method labels certain already *accepted statements* as axioms. (c) It yields completely new, *previously unknown statements* as axioms.[27]

If analysis is understood in the sense of (a), an independent criterion is necessary for determining the membership in the set of axioms, to enable us to know when to stop our quest. The most uncontroversial ideas seems to be that axioms should be *simple*, but what that amounts to in practice is difficult to make precise. This and other possible criteria for axioms are discussed in Section 4, below. The view that axiomatizations are guided by independent criteria for what counts as an axiom is expressed, for example, by Bocheński [1954, 82], where the 'regressive method' is said to proceed from theorems to known premises, but without specifying how they came to be known.

In the absence of an independent criterion for axioms, axioms could be singled out exclusively on the basis of being the starting points of derivations. This leads to variants (b) and (c) above. If the axioms belong to the already accepted statements, then axiomatization amounts to a reorganization of the statements in question. Such an understanding is expressed, for example, in Corcoran's description of the axiomatic method:

> originally, a method for reorganizing the accepted propositions and concepts of an existent science in order to increase certainty in the propositions and clarity in the concepts. [Corcoran 1995, 57][28]

[25] See [Hintikka and Remes 1974], [Lakatos 1978b], [Otte and Panza 1997], [Peckhaus 2002], and [Grosholz 2007, Ch. 2].

[26] An inference from a to b is said to be *convertible*, if the inference from b to a is also valid.

[27] The distinction between (b) and (c) is also discussed in [Detlefsen 2013].

[28] See also [Hilbert 1903, 50] for a similar statement.

For a more detailed discussion on how to use the character of proofs and definitions to assess different axiomatic systems, see Section 4.4.

In variant (c), analysis is understood as a creative activity that yields hitherto unknown statements as axioms. Such statements might at first be considered only hypothetically, before they are accepted as axioms. Indeed, they could well be surprising and obscure, and might even contain new primitives. For example, in order to prove c, one might be led to consider the statements 'a' and '$a \supset c$' as axioms, where a contains new primitives. In practice, further criteria are often also appealed to in this case to restrict what can be taken as an axiom. C. S. Peirce considered the form of inference of kind (c) to be a logical inference on a par with induction and deduction, called *abduction*.[29] According to Peirce, abduction 'consists in examining a mass of facts and in allowing these facts to suggest a theory. In this way we gain new ideas; but there is no force behind the reasoning' [Peirce 1958–1966, 8.209].[30]

According to Copi's and Corcoran's views of axiomatization mentioned above, we seek axioms in order to be able to derive not only particular theorems, but *all* statements of a given domain, i. e., we want axioms to be complete.[31] Of course, this presupposes a descriptive conception of axioms (see Section 2.4) together with some independent access to the body of knowledge that is being axiomatized. The latter is by no means trivial, as can been seen from the fact that at the Second International Congress of Mathematicians in Paris (1900), Hilbert considered one of the great open tasks of the axiomatic approach in geometry to be to show 'that the system of axioms is adequate to prove all geometric statements' [Hilbert 1900b, 257]. In practice one often has only a finite set of known statements that one wants to deduce from the axioms, in which case the axioms are 'relatively' or 'quasi-empirically' complete [Awodey and Reck 2002, 4].

3.2 Manipulation of axioms

Another quite common — though rarely discussed — source for axiom systems is other axiom systems that have been developed previously. New systems can be obtained relatively easily by deleting, negating, or otherwise modifying given axioms, or by adding axioms to a given set. In reference to the early 20th century, Kline comments on this practice as follows:

> The opportunity to explore new problems by omitting, negating, or varying in some other manner the axioms of established systems enticed many mathematicians. This activity and the erection of axiomatic bases for the various branches of mathematics are known as the axiomatic movement. It continues to be a favorite activity. [Kline 1972, 1027]

Most famously, the axioms for non-Euclidean geometries were obtained in this way, with the original goal of showing the necessity of Euclid's parallel postulate. Modifying a given set of axioms is an exploratory process that reveals some game-like aspects of mathematics. However, for accepting a system obtained in this way, mathematicians usually demand other independent motivations — for example, that the system have genuine mathematical models, as opposed to models that are concocted

[29] See [Peirce 1990 (1896–1908] and [Peirce 1902].

[30] This idea was later taken up and further elaborated by Hanson [1958b, 85–86].

[31] For a discussion of various notions of completeness, see [Awodey and Reck 2002].

only for the sake of satisfying the axioms. Two illustrative historical case studies for this kind of proceeding are Schröder's and Dedekind's axiomatic introductions of the notion of lattices [Schlimm 2011].[32]

The modification of a given system of axioms might appear as the least genuinely creative mathematical process and more akin to investigations based on blind trial and error. Nevertheless, *some* mathematics was developed in this way, which refutes the orthodox view about mathematics that claims that all axiomatizations describe some previously given 'substance'.

3.3 Conceptual analysis of one or more domains

The previous two sections were mainly concerned with the syntactic role of axioms; we now turn to the semantic one. Mathematicians often have a more or less clear conception of the objects of their investigations and sometimes they employ axioms to characterize these objects as sharply as possible. If practiced in this sense, axiomatization is a method of conceptual analysis, broadly understood. The natural numbers, some of the oldest and most basic mathematical objects, are a prime example. How can the natural numbers be described best? This was one of the leading questions behind Dedekind's and Peano's famous axiomatizations [Dedekind 1888, Peano 1889]. Dedekind's letter to Keferstein [Dedekind 1890] is an illuminating testimony of the deliberations that led Dedekind to his axioms. The result, he explains, did not come about

> in one day; rather, it is a synthesis constructed after protracted labor, based upon a prior analysis of the sequence of natural numbers just as it presents itself, in experience, so to speak, for our consideration. [van Heijenoort 1967, 99]

Note Dedekind's terminology of 'synthesis' and 'analysis', discussed in Section 3.1 above, but where analysis is now applied to 'the sequence of natural numbers just as it presents itself', not to statements about them. In modern terminology, Dedekind began with a careful analysis of an intended model, '[t]he number sequence N [which] is a *system* [or set] of individuals, or elements, called numbers' and he asked himself: 'What are the mutually independent fundamental properties of the sequence N, that is, those properties that are not derivable from one another but from which all others follow?' [van Heijenoort 1967, 99–100]. Notice that already at the very starting point of his investigations, the syntactic aspects of axioms are considered together with the semantic ones: derivability on the one hand and properties of N on the other.[33] In addition to the categorical axiom system for the structure of natural numbers, Dedekind also formulated a system of axioms to characterize substitution groups, thus using axioms to describe a family of non-isomorphic structures. Incidentally, his axiomatic attitude allowed him to recognize similarities among various mathematical structures, like rotations and quaternions, and to identify them as instances of the abstract notion of group [Dedekind 1932,

[32] According to the cognitive psychologist Kevin Dunbar, 'conceptual change [. . .] is the result of tinkering' [Dunbar 1997, 488], which lends support to the claim that tinkering with axiom systems can be conducive to conceptual innovation.

[33] For an detailed discussion of Dedekind's axiomatization of the natural numbers, see [Sieg and Schlimm 2005].

439]. As these examples of Dedekind's axiomatic work show, both a single domain (like the natural numbers) and various analogous domains (like groups) can be the source for a system of axioms.[34]

Various ways of spelling out the relation between the given domain(s) and the result of the analysis have been introduced by philosophers. In the following, I shall employ Carnap's notion of *explication*. A stronger notion, according to which the description is understood as uncovering essential constituents of the analysandum, is sometimes referred to as 'conceptual analysis'; this term is then used in a narrower sense that I have been using it in the present section.[35] If an axiomatization is understood to be the presentation of a domain that is given non-axiomatically in the sense of an explication or rational reconstruction of the theory of that domain, this amounts to the transformation of informal or not yet satisfactorily axiomatized notions into more exact ones [Carnap 1950, 3].[36] Despite one's best intentions of being descriptive, however, it may happen that the description and what is being described do not fully agree. This is often the case because the source domain is not yet very sharply conceived and is only a vague notion.[37] This divergence is also pointed out in Hilbert, von Neumann, and Nordheim's paper on the foundations of quantum mechanics: 'The concepts that previously were a bit vague, like probability and so on, lose their mystical character through the axiomatization, since they are then implicitly defined by the axioms' [Hilbert et al. 1927, 3]. In many cases an axiomatization does not simply exhibit the commonly accepted meanings of the terms that occur as primitives, but rather 'proposes a specified new and precise meaning' for them [Hempel 1952, 11]. In other words, the original conception of the source domain and its expression in the axiomatic theory do not coincide (for formal axiomatizations certain limitations are even insurmountable, as is evidenced by Gödel's first incompleteness theorem). Hempel concludes that a descriptive axiomatization 'cannot be qualified simply as true or false; but it may be adjudged more or less adequate according to the extent to which it attains its objectives' [Hempel 1952, 12]. Thus, not only does he point out the divergence between source and target just mentioned, but Hempel also emphasizes that axiom systems can have different objectives. (The descriptive and prescriptive aims were introduced in Section 2.4; other goals of axiomatizations will be discussed in Section 4.)

The axiomatization of set theory provides a good illustration of the possible tension between a source theory and an axiomatization.[38] To highlight the difference between the source theory (mainly Cantor's) and the axiomatic theory (his own), von Neumann introduced the term 'naive set theory' to refer to the former. He describes the relations between the source theory, the axioms, and the new axiomatic theory as follows:

> To replace this notion [i. e., the naive notion of set] the axiomatic method is employed; that is, one formulates a number of postulates in which, to be sure, the word 'set' occurs but without any

[34] See also [Schlimm 2011] for similar observations with regard to the introduction of lattices. On the use of axioms to characterize analogies, see [Schlimm 2008].

[35] For a discussion of various notions of conceptual analysis and explication, see [Beaney 2000; 2004]; see [Beaney 2012] for an excellent overview of various philosophical notions of 'analysis', which includes also the ancient Greek one mentioned earlier.

[36] See also [Kreisel 1967].

[37] An illustrative example: When asked about the notion of set, Cantor is reported to have answered 'A set I imagine as an abyss' [Dedekind 1932, 449].

[38] See [Maddy 2001] for a discussion of various motivations behind the axiomatization of set theory.

> meaning. Here (in the spirit of the axiomatic method) one understands by 'set' nothing but an object of which one knows no more and wants to know no more than what follows about it from the postulates. The postulates are to be formulated in such a way that all the desired theorems of Cantor's set theory follow from them, but not the antinomies. In these axiomatizations, however, we can never be perfectly sure of the latter point. ([von Neumann 1925, 36]; quoted from [van Heijenoort 1967, 395])

In this connection it is no surprise that Poincaré, when discussing Zermelo's axiomatization, leaves the German term for set, 'Menge', rather than translating it into the French 'ensemble'. He did so on purpose and explains:

> It is because I am not sure that the word *Menge* in these axioms retains its intuitive meaning, without which it would be difficult to reject Cantor's definition; now, the French word *ensemble* suggests this intuitive meaning too forcefully for us to use it without inconvenience when the meaning is altered. [Poincaré 1963 (1913, 57]

Here Poincaré points out that the term used in the axiomatization might have a different meaning than the term 'Menge' as it is used in Cantor's theory.

Whether a proposed set of axioms is adequate, i. e., whether they are necessary and sufficient to capture the given body of knowledge, is not a straightforward issue, and so additional arguments are often put forward to justify and guide the formation of axioms. Purely semantic considerations often involve model-theoretic investigations of the axioms, e. g., Dedekind's arguments for the satisfiability and categoricity of his axiomatic characterization of the natural number structure. More often, however, syntactic arguments are also adduced to support axioms. Zermelo and von Neumann argued for the adequacy of their axiomatizations of set theory by showing how central theorems of Cantor's theory follow from them, while the paradoxes seemed to be avoided.

Since axiomatization can be employed to contribute to the clarification of concepts and the construction of foundations for a discipline, many mathematicians who pursue the axiomatization of concepts or theories can be said to be engaged in philosophical work [Suppes 1968, 653]. Thus, it is no coincidence that many of the more philosophically inclined remarks made by mathematicians come from those who held axiomatics in general in high esteem, like Frege, Dedekind, Peano, Hilbert, Bernays, and Russell (despite disagreeing on more specific issues of axiomatization).

3.4 Proofs and refutations

The possible origins of axiom systems mentioned above reflect the different roles and functions that axioms can play in mathematics and they by no means exclude each other. Rather, in practice they are often employed together. Consider for example the strategy suggested by Russell for 'regressive investigations,' which are motivated by the appearance of a contradiction:

> first, to make a kind of hierarchy of obviousness among the results to which our premises ought to lead, then to isolate, if possible, the premises from which contradictions flow, and the kind of reasoning which gives rise to the contradictions, and then to *invent various modifications* of the guilty premises, applying to such modifications the two tests (1) that they must yield the more

> obvious of the results to be obtained, (2) that they must not yield any demonstrably false results. [Russell 1973 (1907, 280; italics by DS]

Here Russell suggests to begin with ranking the statements of a given body of knowledge, in order to assess possible axiomatizations. Since he considers the investigation to be motivated by the occurrence of an inconsistency, he suggests to isolate those axioms or forms of reasoning that led to the contradiction. Then, these axioms are to be modified and then assessed with respect to their consequences. Thus, all three main sources of axiomatizations discussed above, namely reasoning from theorems, manipulation of given axioms, and conceptual analysis, are appealed to.

The road to axioms is certainly not a one-way street. When presented in textbooks, we might find the axioms stated at the very beginning, but the situation is very different during the process in which a set of axioms is first formulated and consolidated.[39] In practice axioms are tentatively formulated to describe some notion or to capture some of its characteristic properties; then their consequences and meta-mathematical properties are investigated, the axioms are reformulated, and so on. The experimental nature of axiomatic investigations is illustrated in Givant's description of the atmosphere at the seminar on mathematical logic led by Łukasiewicz:

> The participants viewed the seminar as a kind of logico-mathematical laboratory where they could conduct experiments in assessing the expressive and deductive powers of various theories. [Givant 1999, 52]

This (in part empirical) process is similar to that of finding the 'right' definition of polygons, as well as to the proof of Euler's formula, as it is presented in Lakatos' *Proofs and Refutations* [1976]. In both cases, we find a continuous alternation of initial conjectures, deductive arguments, semantic considerations, and various kinds of refinements.

4 What are good axiom systems?

Some criteria for assessing individual axioms as well as systems of axioms are discussed in this section. These criteria can be used to assess whether to accept a proposed axiomatization or to choose between logically equivalent systems of axioms. It should be noted, however, that these criteria may very well conflict with each other. As will become evident in the discussion, although many criteria seem to be quantitative at first (e. g., one should have few axioms) it is in the end almost always qualitative considerations that matter most.

Traditionally, axioms have been considered to be somehow epistemologically or metaphysically privileged. As I shall argue in Section 4.1, however, this has often been only an ideal that was not achieved in practice. Similarly, the desire that axioms should be somehow simple has been repeated frequently, but is difficult to make precise, and is readily jettisoned in light of other considerations. In other words, 'intrinsic' [Maddy 1988, 482] criteria for justifying axioms play only a secondary role in

[39] On the distinction between textbook mathematics and mathematical practice, see [Schlimm 2013], and on the role of concepts in these investigations, see [Schlimm 2012].

mathematical practice; some of the more relevant, extrinsic criteria for axiom systems are addressed in Sections 4.2–4.4.

4.1 Narrow philosophical vs. mathematical concerns

We have seen that in an axiomatic presentation the axioms are logically prior to the other statements of a theory, but this does not necessarily mean that they also enjoy a privileged epistemological or ontological status. While the term 'axiom' has traditionally been reserved for those propositions of a theory that enjoy a particular epistemological status, this understanding has not been the most important for mathematical practice. In particular, many mathematicians increasingly objected to metaphysical views that were considered irrelevant for mathematics, and to narrow epistemological views, that were based on an ability of immediately grasping mathematical truths. This is not to say that broader epistemological concerns might not be relevant for guiding an inquiry or helping to decide between alternative axiomatizations, but rather that, in case of disagreement, mathematical concerns ultimately determine which propositions are accepted as axioms and which are not.[40] Before addressing such concerns in the following sections, I briefly present some historical views regarding the status of axioms.

Aristotle considered axioms to be evident necessary truths (*Posterior Analytics*, I.5; McKeon 1947, 20), but this ideal had been violated in mathematical practice already in his own times [Mueller 1969, 294]. Most famously, the obviousness of Euclid's fifth postulate had been called into question from the very early commentators onwards. Still, regarded as an ideal, the criterion of self-evidence has continued to be very popular in philosophical discussions of axioms. Pascal, for example, in what he calls the 'method of the geometrical proofs or the art of persuading,' lists as the only necessary rule for axioms: 'Not to demand, in axioms, any but things that are perfectly evident of themselves' [Pascal 1909–1914]. And over three centuries later, in 1918, Hermann Weyl writes that among the assertions of mathematics there 'are a few which are immediately recognized as true, the axioms' [Weyl 1994 (1918, 17].[41]

The independence of mathematical progress from metaphysical assumptions was also demanded fairly early on. As Leibniz remarked, 'there is no need to make mathematical analysis dependent on metaphysical controversies'.[42] Decisive developments, however, took place during the 19th century. Kant regarded the axioms of geometry as evident synthetic judgments a priori, which are given in intuition. But, in the wake of the emergence and acceptance of non-Euclidean geometries, the a priori character of axioms was seriously questioned. As a result, empiricist or conventionalist views became en vogue. For example, the axioms of geometry were considered as (idealized) facts by Helmholtz and as conventions by Poincaré. The latter famously argued that Euclidean geometry is not synthetic

[40] There are exceptions: Brouwer's intuitionism, for example, is one of them. In general, however, nothing is sacred, not even the venerated law of the excluded middle of Aristotle, as Russell admits: 'I once attempted a partial denial of the law of excluded middle as an escape from the contradictions; but plainly it is better to attempt other denials first' [Russell 1973 (1907, 279].

[41] For a recent analysis of notions of self-evidence, see [Shapiro 2009].

[42] 'On n'a point besoin de faire dependre l'analyse mathematique des controverses métaphysiques', [Leibniz 1859, 91], quoted from [Krämer 2003, 534].

a priori — since it is not the only conceivable geometry — and neither an empirical science, and concluded:

> [T]he geometrical axioms are therefore neither synthetic a priori intuitions nor experimental facts. They are conventions. [...] What then are we to think of the question: Is Euclidean geometry true? It has no meaning. [...] One geometry cannot be more true than another; it can only be more convenient. [Poincaré 1952 (1901, 50]

The above developments slowly led to the emergence of the relational view of axioms, in which the question regarding the truth of the axioms became relativized to particular models. At the same time mathematicians increasingly began to separate mathematical from narrow epistemological questions concerning axioms, relegating the latter to philosophy.[43] An early formulation of this tendency is given by Felix Klein:

> But the question [regarding the justification of the parallel postulate] is obviously a philosophical one, which concerns the most general foundations of our knowledge. The mathematician *as such* is not interested in this question and wishes that his investigations are not regarded as depending on the answer that might be given to the question from one side or the other. [Klein 1893, 493–494; italics in original]

According to the geometer Schmidt, it was exactly the neglect of epistemological concerns that allowed Hilbert to gain such a high degree of logical clarity in his 1899 *Grundlagen der Geometrie* [Schmidt 1932, 406]. The separation between foundational or logical questions from epistemological, psychological, and empirical ones is expressed explicitly at the turn of the 20th century by the Italian school of mathematicians around Peano and also by the American postulate theorists Huntington and Veblen.[44] They all reject discussions regarding the status of axioms as non-mathematical and make this clear by a deliberate choice of terminology. For example, Huntington speaks of 'the method of postulates' [Huntington 1937, 482], while Veblen writes:

> We shall not enter into the metaphysical questions as to whether these assumptions [i. e., the starting points of deductions] are self-evident truths, axioms, common notions, experimental data or what not, but shall try to keep within the realm of mathematics by using the non-committal word assumptions. [Veblen 1914, 4–5]

Paul Bernays gave a nuanced analysis of these developments in his reflections on 'Hilbert's significance for the philosophy of mathematics' [1922]. As he explains, it was not philosophy in general that was rejected by mathematicians, but only the traditional view, heavily influenced by Kant, according to which mathematical knowledge is established somehow a priori, i. e., without recourse to mathematical investigations. He writes:

> The framework which earlier philosophical views, and even Kantian philosophy, had marked out for mathematics was burst. Mathematics no longer allowed philosophy to prescribe the method and the bounds of its research; rather it took the discussion of its methodological problems into its own hands. ([Bernays 1922, 94]; quoted from [Mancosu 1998, 190])

[43] See [Bernays 1922, 94].
[44] See, e. g., [Padoa 1901, 121] and the account in [Scanlan 1991, 988].

Thus, we do not get an elimination of philosophy of mathematics, but a re-orientation, such that practical and methodological considerations became part of it.

Parallel to the developments in mathematics, axiomatizations of physical theories also contributed to the abandonment of the view that axioms represent a priori truths.[45] Instead, by showing that provisional hypotheses could lead to empirically tested consequences, the truth of the premises was held to be established a posteriori, if at all. Such a view is expressed, e. g., in Newton's fourth rule of philosophizing in Book III of the *Principia*, where he demands that facts should determine the theory.[46] To explain 'in what sense a comparatively obscure and difficult statement may be said to be a premise for a comparatively obvious proposition' [Russell 1973 (1907, 272] Russell distinguished, both in science and mathematics, between *empirical* (epistemically prior) and *logical* (logically prior, i. e., simpler) premises.[47] For him, the method for obtaining and believing in the principles of mathematics or the general laws of science is 'substantially the same', namely induction: 'we tend to believe in the premises because we can see that their consequences are true' [Russell 1973 (1907, 273–274]. While scientists might feel uneasy accepting axioms on logical grounds alone,[48] at the end of the day they will accept them, if no viable alternatives are at hand.[49] Thus, just as in mathematics, metaphysical and epistemological criteria for axioms in physical theories were increasingly replaced by logical and pragmatic ones. The psychologist Clark Hull put this quite succinctly:

> The history of scientific practice so far shows that, in the main, the credentials of scientific postulates have consisted in what the postulates can *do*, rather than in some metaphysical quibble about where they come from. [Hull 1935, 511]

In sum, both metaphysical and narrow epistemological concerns might play a role in the motivation and assessment of axioms, but they do not impose incontrovertible conditions for axiomatizations that are used in scientific and mathematical practice. This observation is confirmed also by Maddy's most recent investigation of set theory. The final sentence of her book reads 'What does matter, all that really matters, is the fruitfulness and promise of the mathematics itself' [Maddy 2011, 137].

4.2 Number of axioms and primitives

For a set of axioms to be of practical use for human mathematicians it must be presented in such a way that makes it easy to survey, in particular to be able to ascertain what are the axioms. For this reason the number of axioms is the desideratum for axiomatic systems that is mentioned most often in

[45] This development is presented in detail in [Pulte 2005].

[46] See [Densmore 1996, 242].

[47] Similarly, Reichenbach introduced a distinction between theories according to the empirical character of their axioms, distinguishing between *deductive* axiomatics, which begins with general statements that are not subject to direct empirical tests, and *constructive* axiomatics, where only empirically testable statements can be used as axioms [Reichenbach 1925].

[48] Consider for example Einstein's remark that the introduction of conceptual elements that are not assumed to be real, but determine the behavior of real objects without being affected by them (e. g., the hypothesis of a center of the universe, or of an inertial system), 'though not exactly inadmissible from a purely logical point of view, is repugnant to the scientific instinct' [Galilei 1967 (1632, xiii]. See also Friedman's notion of 'independently acceptable sentences' as a criterion for scientific premises [Friedman 1974, 16].

[49] In terms of the 'Classical Model of Science' presented in [de Jong and Betti 2010, 186], this means that conditions (4)–(7) lost more and more of their grip.

the literature. For example, Bernays speaks of 'a few' [1922, 94] and Blanché of 'a small number' of axioms [1973, 163]. Many famous axiomatizations consist in fact of only a small number of axioms, like Zermelo's seven axioms of set theory, Kolmogorov's three axioms of probability theory, or Euclid's initial five axioms for geometry (although more are added as the need arises). Other than just having a 'direct aesthetic appeal,' a minimal set of assumptions has also been proposed as an indicator for 'the seriousness of our understanding of the theory, and a measure of depth of its development' [Suppes 1968, 656].

Of course, any finite set of axioms can be reduced to a single axiom by conjunction. However, in practice, mathematicians appeal to some notion of simplicity (see below) that rules out such trivial reductions, e. g., by tacitly assuming that axioms do not have conjunction as the main sentential connective.[50] Nonetheless, one cannot impose the general restriction to a finite number of axioms, since this would rule out, for instance, the Zermelo-Fraenkel axioms for set theory, which include infinitely many axioms of separation and replacement; or, the standard first-order axiomatization of the natural numbers, which has an induction axiom for every predicate; or, the elementary theory of geometry, which has infinitely many continuity axioms.[51] These axioms, however, because they share a common form, can be encoded by an axiom *schema*, which allows for a concise presentation.[52]

The formal property of *recursiveness* may be required to ensure easy surveyability, but only as a necessary, not as a sufficient condition. For example, the recursive axiomatization resulting from Craig's [1956] procedure of eliminating a certain set of primitives from a given system of axioms surely would not be considered to be a practically useful axiomatization by any means. These worries are reflected in Corcoran's observation that 'one feels that there is something illegitimate about taking an infinite number of sentences as axioms' [Corcoran 1973, 42].[53]

In addition to limiting the number of axioms, reducing the number of primitives is also frequently mentioned as one of the objectives of axiomatizations.[54] However, mathematicians also value a certain familiarity with the axioms and notions that are employed. The tension between these two desiderata is illustrated in the development of axiomatizations of propositional logic, where we can first observe a trend of reducing the numbers of axioms, followed again by an increase, while retaining similar rules of inference[55] (Table 2).

This development was also accompanied by the reduction of the primitives of propositional logic to a single one, *rejection* [Sheffer 1913].[57] In the introduction to the second edition of *Principia*

[50] This is explicitly mentioned in [Moore 1902, 488–489].

[51] See [Tarski 1959, 167].

[52] The notion of axiom schemata was introduced in [von Neumann 1927, 13]. In this connection it is worth pointing the reader to Tennant's discussion of a recent interesting result by Friedman, according to which 'In every first order theory axiomatized by axioms and axiom schemes, subject to mild conditions, every theorem follows logically from at most finitely many logically minimal sets of its axioms and axiom-scheme instances' [Tennant 2012, 276–285].

[53] See also [Adams et al. 1970, 381] with regard to the use of infinite axiom systems for empirical theories.

[54] See, e. g., [Parsons 1972, 190].

[55] The relation between axioms and inferences rules will be addressed in Sections 4.4 and 5.1, below. It is also worth mentioning that the authors mentioned often considered different axioms and inference rules themselves.

[56] See [van Heijenoort 1967, 29], [Russell and Whitehead 1910, 13], [Nicod 1917], [Łukasiewicz 1970], [Hilbert and Ackermann 1928, 22], and [Kleene 1952, 82].

[57] This connective is attributed to Sheffer and called the 'stroke' in [Nicod 1917, 32].

1879	Frege	6 axioms
1910	Russell & Whitehead	5 axioms
1918	Nicod	1 axiom
1925	Łukasiewicz	1 axiom
1928	Hilbert & Ackermann	4 axioms
1952	Kleene	10 axioms

Table 2: Historical development of the number of axioms for sentential logic.[56]

Mathematica (1927) this was hailed by Russell as the most important contribution in logic since the appearance of the first edition over a decade earlier.[58] Clearly, if this were the only connective, proofs would become much longer, and the logical formulas would become more difficult to parse. Accordingly, Hilbert and Ackermann refer to this reduction only as a 'curiosity' [Hilbert and Ackermann 1928, 9]. However, the search for particularly parsimonious axiomatizations for various theories continues to this day, now with the help of automated reasoning techniques.[59]

Another aim that can stand in conflict with the goal of reducing the number of axioms is to exhibit the relation of one axiomatization to another theory (see also Section 4.4). For example, Kleene's axiomatization of propositional logic consists of ten axioms, but by modifying a single one of them, it can be transformed into an axiom system for intuitionistic propositional logic [Kleene 1952, 82 and 101]. Both the tendencies towards logical economy on the one hand and towards conceptual clarity and intuitiveness on the other, are identified by Schmidt in Hilbert's formulation of the axioms for geometry, and he notes that if they stand in conflict, for Hilbert it is the logical economy that has to give [Schmidt 1932, 406].

Mathematicians seem to have developed some stable notion of simplicity, but one that is very difficult to make precise. One attempt is to resort to logical simplicity, which can be characterized as having fewer logical constituents or fewer types of non-logical elements.[60] Other criteria of simplicity for the axioms can be their logical form, or the number and the character of the primitive concepts employed. The latter has been studied extensively by Nelson Goodman in connection with explicating the notion of the measurement of formal simplicity of predicate bases of scientific theories.[61] Several syntactical, language related, and semantical criteria of simplicity for axiom systems in first-order logic are discussed in [Pambuccian 1988], and a formal criterion for 'natural axiomatizations' is put forward in [Gemes 1993]. However, the numerous attempts at formally characterizing what it means for a statement to be logically simple have not yielded a generally agreed upon proposal.

4.3 Meta-mathematical properties: Consistency, independence, and categoricity

In addition to the various notions of completeness, which were discussed earlier in connection with the origins of axioms, and to the surveyability of the axioms the only desideratum for a good axiomatization

[58] See [Linsky 2011] for the background regarding the second edition of *Principia Mathematica*.
[59] See, e. g., [Ernst et al. 2002] and [McCune et al. 2002].
[60] This is mentioned, e. g., in [Cohen and Nagel 1934, 213]; see also [McCune et al. 2002].
[61] [Goodman 1958]; see also [Rudner 1961] and [Bunge 1962].

on which there is almost unanimous agreement is that of *consistency*, i. e., the impossibility of deriving a pair of statements one of which is the logical negation of the other. For, under most commonly accepted notions of consequence, any statement whatsoever follows from a contradiction. This renders an inconsistent system of axioms utterly useless from a logical point of view, because the theory that is determined by it comprises *all* well-formed statements that can be formulated in the underlying language, and because there can be *no* model that satisfies such a set of axioms.[62]

However — and the reader who has followed the dialectic of the previous discussions may already suspect a certain tension arising even from such a seemingly unproblematic constraint — a word of caution is in order: In practice it may not be known at a certain point in time whether a system of axioms is consistent or not, and an inconsistent system may well serve as a promising starting point for the further development of a science. For example, despite the fact that early conceptions of sets turned out to be inconsistent (which was known to Cantor himself, who distinguished between consistent and 'absolutely infinite or inconsistent' sets to get around this [Hallett 1984, 166]), many ideas and proofs were not affected by this and set theory has developed nonetheless into a well-respected mathematical discipline. A somewhat similar story can be told about certain aspects of Frege's work on logic. Thus, even inconsistent theories may play a crucial role in scientific development and cannot be dismissed right away if the practice of science is under investigation.[63]

Due to Gödel's fundamental limiting result that it is not possible to prove the consistency of many formal systems by means of weaker systems [Gödel 1931], relative consistency proofs are the best we can aim for in most cases.[64] A famous example of such a proof is Gödel's own result establishing that if ZF is consistent, then ZFC+GCH is consistent too.[65] A point I would also like to emphasize in this connection is that relative consistency proofs are not limited to formal axiomatizations, as can be seen from Beltrami and Klein's interpretations of non-Euclidean geometry (although they might not have seen their work in this light themselves). While it has been common to show relative consistency by semantic means, this is not the only possible way. For example, Gentzen and Gödel independently developed the 'double negation translation' between formulas of classical and intuitionistic logic and showed that a formula is provable in classical logic, if and only if its translation is provable intuitionistically. As a consequence, a contradiction in one theory also entails a contradiction in the other.

An important relation between axioms, which is related to consistency, is that of *independence*.[66] An axiom is said to be independent from some other axioms if neither it nor its negation follows

[62] Since inconsistent theories have no model, the semantic view of theories ([van Fraassen 1980]) considers them all as equivalent, i. e., it cannot distinguish between Cantor's inconsistent theory of sets and Frege's inconsistent theory of arithmetic. Thus, this approach to theories is not well suited as a basis for discussing the role of theories in mathematical practice.

[63] The existence of non-trivial but inconsistent theories has been a motivating factor for the development of paraconsistent logics, e. g., [Priest et al. 1989].

[64] See [Sieg 1990] for a discussion of the philosophical implications of relative consistency proofs.

[65] [Gödel 1940]. ZF stands for the theory defined by the Zermelo-Fraenkel axioms for set theory, while ZFC+GCH is the theory with the additional axiom of choice and the generalized continuum hypothesis.

[66] One way to exhibit the independence of an axiom a from a set of axioms S is to show that S is consistent with the addition of the negation of a. The consistency of S and $\neg a$ implies that a does not follow from S, because if it followed, then both a and $\neg a$ would be consequences of S and $\neg a$, i. e., the latter would constitute an inconsistent set of sentences.

from them. Conversely, a dependent axiom is thus redundant from a deductive point of view and can be eliminated from the set of axioms without affecting the set of consequences. Considerations of independence have led to the desire to reduce the number of axioms to a minimum and this has been an important methodological motivation for many investigations of axiomatic systems (see Section 4.2). Nonetheless, other aims of axiomatizations, like providing a convenient basis for one's theory, providing an interesting systematization of a theory such that certain subtheories can be marked off easily by removing particular axioms (see Section 4.4), or pedagogical reasons[67] can call for giving up the criterion of independence. Practitioners of axiomatics are well aware of this issue, as we can see, for example, in the following quote by Tarski:

> Often, however, one does not insist on these methodological postulates for practical, didactical reasons, particularly in cases where the omission of a superfluous axiom or primitive term would bring about great complications in the construction of the theory. [Tarski 1995 (1936, 132]

This explains Huntington's remark on the value of independence for mathematical activity as 'something of a luxury' [Huntington 1937, 490].

While consistency and independence can be expressed as syntactic notions, *categoricity* is a purely semantic notion. An axiom system is categorical if all of its models are isomorphic, i. e., structurally indistinguishable. If the aim of an axiomatization is the characterization of a particular structure, like that of the natural numbers, categoricity is a desideratum, as formulated clearly in Dedekind's *Was sind und was sollen die Zahlen?* [1888]. Formal investigations have revealed that a first-order axiomatization that has an infinite model cannot be categorical. Second-order logic, which allows for categorical axiom systems, however, is not complete, i. e., the syntactic and semantic notions of consequence do not coincide. These results thus point to a trade-off between expressive power and deductive strength of the underlying logic. The expressive limitations of first-order logic and the incompleteness of second-order logic have led Bernays to 'warn against an overestimation' of formal axiomatics [Bernays 1967, 188]; for

> there seems to be a limit to strict axiomatization in the sense that we either have to admit a certain degree of imprecision, or to be unable adequately to characterize what we mean, for instance, by well ordering, continuity, and number series. [Bernays 1967, 191]

In practice, however, such considerations rarely come up explicitly, even in cases where the theory of numbers is introduced as a 'deductive science, based on the laws of arithmetic' [Davenport 2008, 1].

4.4 Character of proofs, definitions, and models

Another popular criterion for axioms is the number and character of their consequences. This is sometimes referred to as their *fertility* [Cohen and Nagel 1934, 143; 213–215] or *fruitfulness* [Carnap 1950, 7]. Since most axiom systems allow the inference of infinitely many theorems,[68] it is the quality rather than the quantity of the consequences that is crucial in evaluating the axiomatization. From

[67] This is explicitly mentioned in [Pickert 1971, 18].

[68] Trivially, from an axiom a one can infer the sentences 'a & a', 'a & a & a', etc.

a purely logical perspective all theorems stand on equal footing, but in practice some theorems are considered to be more central to a theory than others (e. g., the 'fundamental theorems' of algebra and arithmetic). Thus, what is usually wanted is an axiom system that provides the basis for a theory in such a way that the proofs of central theorems or the definitions of central concepts or canonical objects of the theory are particularly intelligible and explanatory.[69] These notions are notoriously difficult to analyze and cannot be discussed further in this paper. For understanding the practice of mathematics, however, they are crucial ingredients.

The conciseness of proofs depends on the number and character of axioms and primitives as well as the number of allowed inferences. The separation between axioms and inferences that is afforded by an axiomatization (see Section 5.1) brings to light a trade-off between the number of axioms and the number of inference rules. Simply put, the more inferences we admit, the fewer axioms we need. Again, the axiomatizations of propositional logic provide a good example. In addition to substitution rules, Russell and Whitehead, as well as Hilbert and Ackermann, admit only one rule of inference, namely modus ponens. Their calculi require five and four axioms, respectively. Gentzen's natural deduction calculus for classical propositional logic [Gentzen 1935], on the other hand, gets by with only one axiom (which itself can be easily replaced by an inference rule, as Gentzen noted), but has two rules (introduction and elimination) for each sentential connective.

In addition to the previously discussed criteria, which are predominantly internal to a particular theory, the relation of a theory to other areas of mathematics are also often taken into consideration. According to the two roles of axioms, this relation can again be of a semantic or syntactic nature. The latter is discussed in Section 5.3.

Intuitively there is an inverse relation between the complexity of the axiomatization and the number of interesting models: On the one hand, the axioms of the example in Section 2.1 are very few and very simple, so it should be no surprise that many different models are easy to find for them. On the other hand, an axiomatization of Euclidean geometry is much more involved, and thus it is more difficult to come up with different models for it that are of genuine interest independently of the axiomatization. Notice the qualification: Any consistent theory can in principle have infinitely many models and many first-order theories even have infinitely many structurally different models. So again, it is not the absolute number of different models, but rather their specific character that matters in mathematical practice.

5 What you can do with axioms that you couldn't do without them

In traditional philosophy of mathematics, which does not pay much attention to history and practice, it is often taken for granted that theories are presented axiomatically, and no particular importance is attributed to this fact. To counter this attitude, I shall now address some features that are germane to axiomatic theories, i. e., that are not to be found in theories that are not presented axiomatically. To be clear, however, most of these features do not depend on a presentation in a formal language.

[69] See [Grosholz 2007, Ch. 1 and 2] and [Mancosu 2008b].

Theories can be given non-axiomatically in several ways. For example, (i) as a set of statements, e. g., those contained in a series of textbooks, (ii) as the true statements pertaining to their particular subject matter, e. g., the natural numbers, and (iii) as the judgments of a particular community, e. g., the geometers who were contemporaries of Euclid. In the present section, a comparison between these non-axiomatic ways of presenting theories and axiomatic presentations will make apparent the usefulness of axiomatics. In particular, I shall take a closer look at the role of axioms for clarifying the distinction between theory and reasoning, for theory demarcation, i. e., clarifying what belongs to a theory and what not, for systematization, i. e., clarifying the internal structure of a theory, for providing means to evaluate theories and determine their meta-theoretical properties, and for promoting intersubjectivity and clarifying communication.

5.1 Separation of theory and reasoning

The importance of agreement on the admitted inferences has been mentioned several times above and axiomatics is indeed a vehicle for achieving this goal. The fact that all statements of an axiomatic theory are encoded in the axioms and that decoding is effected through definitions and the notion of consequence allows for a clear-cut distinction between the *statements* of the theory and the *inferences* used to connect them. To justify the claim that a particular statement belongs to an axiomatic theory, a chain of inferences must be exhibited that displays how that statement follows from the axioms. By requiring these chains of inferences to be made explicit, the single inferential steps (which are not necessarily formal) are also made explicit as the links between two statements. In this way they are put out in the open and can themselves be made objects of study.[70] In other words, each of these steps can be examined and it can be discussed whether it is acceptable or not. Of course any two sentences *a* and *b* can be said to be connected by an inference, namely by '*b* follows from *a*.' But the point here is that once such an assertion is made, the question as to whether this inference should be accepted can then be discussed in public. The infamous debate surrounding the principle of choice, isolated first by Zermelo, illustrates this observation. Another example is given by Euclid's first proof in Book I of the *Elements*, in which he infers the existence of a point where two constructed circles intersect. This step is not licensed by contemporary standards of logical reasoning and is thus interpreted as a flaw in the presentation [Pasch 1882, 45]. Whether Euclid himself would have accepted such a verdict is a matter of speculation and clearly depends on his own notion of consequence.

Discussions regarding acceptable inferences help in promoting agreement among mathematicians or at least help in clarifying the points of disagreement. The formulation of formal rules of inference has made a tremendous contribution towards this goal. However, as has been shown by the debate between proponents of intuitionistic and classical logic at the beginning of the 20th century, or by the current disagreement about whether first- or second-order logic is the appropriate formal framework for modeling mathematical reasoning, a formalization of the admissible rules of inference does not by itself settle the disputes about their acceptability—but it certainly helps in understanding what these arguments are about and in investigating the consequences and limitations of the positions involved.

[70] On more general philosophical reflections about the use of logic to make explicit related facts, see [Brandom 1994].

Once there is agreement on the admitted inferences (formal or not) further disagreements about statements of an axiomatic theory must have to do with the axioms. Thus, an axiomatization provides a kind of localization of problem areas within a theory in a way that other forms of presentation do not.

5.2 Theory demarcation

A system of axioms makes explicit a body of statements in a systematic manner. Because the axioms, together with an implicit or agreed upon notion of consequence, determine the set of statements that follow from them, they give a clear, though not necessarily effectively computable criterion as to what belongs to the theory and what does not. Hence, axioms provide a means for the *demarcation* of a theory via the notion of consequence: A statement belongs to the axiomatic theory if and only if it follows from the axioms of the theory.

If a theory is presented only as an unstructured set of statements, no systematic criterion other than the inclusion in this set could be given to determine whether a particular statement s is part of the theory T or not, i. e., all we have is the triviality $s \in T$ if and only if $s \in T$, or that x is in T if it follows from some subset of T, but without any hints about the nature of this subset.

Alternatively, we could appeal to the subject matter that the statements of the theory are intended to be about to characterize the theory. So, in modern terminology, if M is a particular model of the theory T, then $s \in T$ if and only if s is true in M. The difficulty with this account is that we need an independent way of accessing M to determine which statements are true in it. For example, it is one thing to say that arithmetic consists of all true statements about the natural numbers, but it is another to determine whether a statement belongs to arithmetic or not. For this we have to be able to decide what can truthfully be said about the natural numbers, which is clearly a difficult task, as is evidenced by the famous open problems of arithmetic, such as the Goldbach conjecture. Moreover, it might be a psychological fact that some people believe that the natural numbers are colored,[71] but a statement like 'the number three is red', even if it were true, would still not belong to arithmetic. Thus, appealing to a subject matter, or to models, does not free us from the need of regimenting our language in a way that is deemed appropriate for mathematics, and we are brought back to the problem of theory demarcation that we started out with.[72] When it comes to more abstract theories the problem of independently accessing the subject matter only becomes worse. For example, recall the debate concerning the axioms of set theory in the early 20th century. Here the leading experts could not agree about which principles were true of sets. The subject matter just did not provide enough substance to settle the controversy. In fact, the subject matter itself was still in the process of being constituted properly and early set theorists had only vague intuitions about it.[73] The contemporary discussion about new axioms for set theory is a further case in point.

These debates also illustrate the difficulties involved if a theory is given as the collection of statements to which a community of experts agrees. In this case the question whether a statement

[71] See [Galton 1880] and [Seron et al. 1992].

[72] This issue is closely related to arguments against the semantic view of theories discussed in [Schlimm 2006, 240–243].

[73] Compare the quotation by Cantor in footnote 37 with von Neumann's bleak, but precise, conception of sets mediated through axiomatics expressed in the quotation on page 19.

belongs to the theory in question can only be answered by asking these experts, which might not yield a unique answer. Just imagine asking both Hilbert and Brouwer whether the law of excluded middle belongs to logic or not. Moreover, in case the experts disagree, no independent and objective criteria are ready at hand for deciding the issue or for making precise the cause of the disagreement. It was only after Heyting's axiomatization of intuitionistic logic, which had previously only been given through not well-understood remarks by Brouwer, that this theory became accessible to the public and it became possible to articulate and investigate in a precise manner how it differed from classical logic. The relative consistency proof by Gödel and Gentzen mentioned above followed briefly afterwards.

Agreement about how to determine which statements belong to a theory and which do not can also be used to identify those notions that are extraneous to the theory. This is sometimes referred to as ensuring *methodological purity*.[74] The relation between purity of method and the use of axiomatics was pointed out over a century ago by Bernstein, who maintained that the question of the purity of certain proofs had been objectivized by the question of whether certain statements can be proved from certain axioms [Bernstein 1909, 391]. For this reason it is no surprise that an increased attention to axiomatics went together in the 19th century with the view that mathematical demonstrations, in particular in arithmetic and analysis, should be independent of intuitions, in particular geometric ones.[75] But geometry itself was also to be developed without recourse to anything not stated explicitly in the axioms [Pasch 1882], and the distinction between analytic and synthetic geometry reflects different views on the allowed means for reaching geometric conclusions. Those in the synthetic camp were often motivated by an ideal of methodological purity that rejects the use of non-geometric notions in geometry and which has been upheld in some form or other by a large number of mathematicians throughout the centuries. The axiomatizations of geometry in the 19th century have been interpreted as a continuation of this ideal [Epple 1997]. Along similar lines, the algebraic geometer Hartshorne has recently bemoaned the use of real numbers in introductory classes on geometry, which he regards as belonging to analysis, but not to geometry proper. To retain 'the pure spirit of geometry', he suggests to teach geometry according to Euclid, starting with the first four books of the *Elements* (presented in a modernized, rigorous fashion). That the axiomatic approach of Euclid is indeed a crucial ingredient of Hartshorne's suggestion to avoid any appeal to real numbers and continuity becomes clear when he mentions the ease with which other geometries can later be introduced by dropping or adding axioms [Hartshorne 2000, 464].

5.3 Internal and external systematization

By encoding a theory into a set of axioms a particular logical structure is imposed on the statements and concepts of the theory, which can then be analyzed further in order to gain insight into the theory itself. According to Mueller, who argues that Euclid's aims for mathematics were similar to those that Socrates had for philosophy, '[t]he evolution of the axiomatic method is explicable solely in terms of

[74] See [Detlefsen and Arana 2011] for a recent discussion. However, here the purity of methods of proof is discussed for particular theorems, not in relation to systems of axioms.

[75] See [Frege 1879] and also the discussion in [Stein 1988].

the desire for clarity and order in geometry' [Mueller 1969, 293]. To emphasize the latter aspect of axiomatics, Dedekind speaks of the 'systematizer,' whose greatest art lies in the careful and deliberate formulation of definitions and assumptions for the sake of bringing out the logical connections within a body of knowledge [Dedekind 1854, 429–430]. Hilbert used the famous phrase of a 'scaffolding of concepts' [Hilbert 1918, 146] to illustrate the systematizing effects of axiomatics. If theories are presented by a set of statements or a class of models, no such logical structure is automatically imposed on the statements.

Since axiomatization provides a canonical form in which the assumptions and primitives are listed explicitly, it facilitates the comparison among axiomatic theories and the discovery of syntactical similarities. For examples of fertilization across theories in logic, algebra, meta-mathematics, and topology that are mediated by common axioms see [Schlimm 2009]. In case two theories share common axioms this leads to a great practical advantage, namely an economy of proof, which makes the transfer of proofs from one theory to another a matter of routine. After all, once a theorem is derived from a set of axioms it also holds in other theories in which these same axioms hold.[76] It is important to notice that while the contribution of axiomatics to this kind of theory unification is most apparent when the axioms are understood abstractly, this is by no means necessary. One can easily find relations between two axiomatizations based purely on syntactical considerations, i. e., when the axioms are understood materially. This is evidenced by Aristotle's remark on the possibility of a single proof for the law that proportionals alternate for such different domains as numbers, lines, solids, and durations (*Posterior Analytics*, I.5; [McKeon 1947, 20]).

Additional layers of systematization of an axiomatic theory can be achieved by grouping particular axioms and primitives together. This amounts to the formation of subtheories or extensions of a theory and it allows for obtaining results about the relations between them. A wonderful example is Hilbert's *Grundlagen der Geometrie* [1899]. Here the axioms of geometry are arranged into five groups: Axioms of incidence (determining the relations between points and lines), of order (about the betweenness relation of different points on a line), of parallels (determining the number of parallels to a given line through a given point), of congruence (regarding the congruence relation between line segments, angles, and triangles), and of continuity (assuring that the axioms are categorical). This grouping has proved to be particularly fruitful for investigating the assumptions upon which particular theorems depend — one of Hilbert's explicit goals [Hilbert 1903, 50]. Imposing this fine-grained hierarchy on the geometrical theorems enhanced our knowledge about the structure of Euclidean geometry. Indeed, according to Gray,

> Hilbert became attracted to the axiomatic approach earlier than had been thought, and [. . .] what caught his interest was the opportunity to discover new results. The (self-appointed) task of giving a good starting point to geometry was a chore; the hint of new results was a challenge. [Gray 1992, 235]

[76] Remarks regarding the economy of proof effected by axiomatics can be found, for example, by [Pasch 1882, 100], [Enriques 1903, 344], [Dingler 1906, 583], and [Haupt 1929, 39].

Moreover, Hilbert's subdivision also allowed for various aspects of Euclidean plane geometry to be studied separately and certain classes of mathematically interesting models were subsequently found and studied that satisfy only some of the groups of axioms.[77] The grouping of theories on the basis of their axioms also allows for a unified presentation of hierarchies of theories and models. Huntington regards axiomatics as 'a tool for saving mental effort, through the simple device of classifying systems according to their logical structure' [Huntington 1937, 486]. Most famously, to provide such an organization of mathematics has been the explicit aim of the efforts of the Bourbaki group, which considered the 'axiomatic method' as their main tool [Bourbaki 1950].

5.4 Theory evaluation

An axiomatic theory is distinguished from other presentations of theories by the singling out of particular statements as axioms and of particular terms as primitives. These axioms and primitives not only determine the theory in question, but they also play a crucial role in its assessment. Following Quine, the bound variables that are used in quantified statements involving the primitives tell us what the ontological commitments of a theory are and our belief in the theory can be evaluated through our beliefs in the axioms. Moreover, properties of the theory can be determined on the basis of the axioms, in particular meta-theoretic properties (see Section 4.3). In the absence of axioms and primitives the entire set of terms and statements of a theory must be taken into consideration when the theory is evaluated, which is, for all practical purposes, clearly an unfeasible task. According to Suppes, the possibility of directing our attention towards the axioms of a theory is one of the great lessons in formal logic of the 20th century [Suppes 1979]. A clear example of this is the research program of 'reverse mathematics', which aims at investigating which axioms are necessary for proving specific mathematical theorems [Simpson 1999].

The grounded logical connections among statements of an empirical theory are arguably also necessary for distributing the blame in case of the falsification of a particular statement[78] and for allowing a theory as a whole to be tested against empirical data. With logical connections alone we can trace the antecedents of a falsified statement, but the axioms will tell us where to stop.

5.5 Intersubjectivity and pedagogy

All of the features that distinguish axiomatic theories from non-axiomatic presentations discussed above can be said to facilitate communication. Through its role in the demarcation of theories axiomatics helps in clarifying what theories are being considered in the first place; and by imposing an internal structure on the theories the attention of the researchers is focused upon the axioms and primitives. We have also seen that theories can be rigorously evaluated and compared with each other if they are presented axiomatically and that this also contributes to the identification and standardization of the inference processes. Thereby, axiomatics makes communication easier and at the same time it achieves

[77] This was pointed out to me by John Mumma.

[78] See [Duhem 1954 (1914] and [Glymour 1980].

a higher level of objectivity at which controversies can be resolved.[79] Let me now add a few words on how axiomatics facilitates the communication of theories at different levels of discourse.

Once a theory is presented axiomatically and the inferential machinery is agreed upon, the semantic content of the primitives and statements can be completely neglected. This insight is so fundamental that it has been called the 'radicalization of the axiomatic method' [Sieg 1990, 267]. Hence, axioms provide a linguistic representation of mathematical content that is independent of philosophical views regarding mathematical objects. Two mathematicians can agree on certain axioms and derivations from them, while holding incompatible views about what the primitives refer to. Whether the natural numbers live in a platonic heaven or only in the minds of mathematicians, whether Galois discovered or invented mathematical groups, and innumerable other such metaphysical questions, do not affect a discourse that is based solely on a set of axioms.[80] In this sense Bernstein speaks of using axiomatics to replace the subjective element of the various mathematical schools [Bernstein 1909, 391].

This independence of axiomatics from other views about the subject matter facilitates not only synchronic communication among contemporaries, but also diachronic 'communication' across different periods in the history of mathematics. The mathematical content conveyed by Euclid's axioms is accessible to us even though we can only speculate about what Euclid's views were regarding what he was doing. As discussed in Section 4.1, the views regarding the epistemological status of axioms have changed considerably over time, but this did not lead in general to the abandonment of previous mathematical achievements. The effect of axiomatics in retaining older mathematics is in accord with Kitcher's account of the cumulative character of mathematics, which he explains as being based on reinterpreting older theories [Kitcher 1983], and with Weinberg's analysis regarding the constancy of the linguistic presentations of 'great equations' even in times when their interpretations were radically changing [Weinberg 2002].

Finally, a further contribution of axiomatics to the development of mathematics is that it provides clear-cut entry points into the various mathematical disciplines, so that axioms can be used as the starting point for studying and learning a theory. The development of abstract group theory, for example, can be traced back to various implicit uses of group-theoretic methods followed by the study of groups with specific types of elements. The contemporary student, however, does not have to go through this whole development to learn about group theory: She can start with the axioms and take it up from there. This also allows for a fresh view on the subject matter that is not bound to tradition, which, as has been pointed out by Pasch, can be very useful for the development of mathematics:

> When a new domain is being developed or a new kind of question is attacked, it often happens that the view on the consequences of a claim and on unfamiliar details is obstructed; it reveals itself only to those who approach the object in a fresh manner and free themselves from the spell of tradition. [Pasch 1914, 137]

[79] See [Suppes 1968, 654–655] for a formulation of very similar views.

[80] See also von Neumann's remark on axiomatic set theory, quoted on page 19. Very similar views with regard to theories in psychology have been expressed by Hull [1935, 491, 510–511]. These considerations might also be a reason why 'the typical working mathematician is a [. . .] formalist on Sundays', according to Davis and Hersh [1981, 321].

To know what a theory is about can surely also be psychologically advantageous for the beginner, but this is by no means ruled out by an axiomatic presentation. However, a non-axiomatic presentation of a theory does not provide anything like the clearly identified entry points into the theory that an axiomatization does.

6 Conclusion

In the preceding pages I have discussed various dimensions of axioms that play a role in mathematical activities. In part, the power of axiom systems stems from the possibility of changing our perspective and using them in different ways. Thus, the same axioms can play semantic or syntactic roles, and can be intended to be descriptive or prescriptive. Putting forward an axiomatization does not commit mathematicians to one particular perspective. This becomes especially clear when looking at the origins of axioms, where a conceptual analysis of one or more domains, reasoning from theorems, and manipulation of axioms are often all present together. Thus, analyses that argue for *one* particular role of axioms[81] can only be regarded as idealizations that do not capture the richness of mathematical practice. Similarly, what I have referred to above as one aspect of the 'orthodox view' of axiomatization, namely that it is a purely cosmetic and expository enterprise, highlights some a particular use of axioms, but fails to do justice to the creative role that they can play in the development of new mathematics. I have also discussed various criteria that can be, and have been, employed in assessing a system of axioms and concluded that there must be trade-offs, because more often than not these criteria stand in conflict with each other. This is one reason why axiomatizing a theory is not as straightforward a matter as some would like it to be and why there is no single set of criteria for what makes a good axiomatization. Nevertheless, there are advantages of axiomatic presentations, like the separation of theory and reasoning, theory demarcation, internal and external systematization, and intersubjectivity that cannot be obtained by other forms of presentations of theories. Finally, because of its degree of explicitness, axiomatics is a vehicle for clarifying a discourse and for reducing verbal ambiguities. Thus, there are good reasons why axioms are employed in mathematical practice that go well beyond simply providing a rigorous foundation for a discipline.

Many of the important philosophical issues of mathematical practice have only been touched upon briefly in this paper, but by providing a general framework for the discussion of axiomatics I hope to have given a fruitful starting point for further investigations and to have contributed to a better understanding of mathematics — and thus to a philosophy of mathematics that takes both past and present practices of mathematics seriously.

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[81] See, e. g., [Copi 1958, 115], [Easwaran 2008] and [Hintikka 2011].

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References

Adams, E. W., Fagot, R. F., and Robinson, R. E. (1970). On the empirical status of axioms in theories of fundamental measurement. *Journal of Mathematical Psychology*, 7:379–409. 25

Amsel, Abram and Rashotte, Michael E. (eds.) (1984). *Mechanisms of Adaptive Behavior. Clark L. Hull's Theoretical Papers, with Commentary*. New York: Columbia University Press. 41

Awodey, Steve and Reck, Erich (2002). Completeness and categoricity, part I: Nineteenth-century axiomatics to twentieth-century metalogic. *History and Philosophy of Logic*, 23:1–30. 12, 17

Beaney, Michael (2000). Conceptions of analysis in early analytic philosophy. *Acta Analytica*, 15(25):97–115. 19

—— (2004). Carnap's conception of explication: From Frege to Husserl? In Steve Awodey and Carsten Klein (eds.), *Carnap Brought Home*, pp. 117–150. Chicago: Open Court. 19

—— (2012). Analysis. In Edward N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy (Summer 2012 Edition)*. URL = `http://plato.stanford.edu/archives/sum2012/entries/analysis`. 19

Bernays, Paul (1922). Die Bedeutung Hilberts für die Philosophie der Mathematik. *Die Naturwissenschaften*, 10:93–99. English translation: Hilbert's Significance for the Philosophy of Mathematics, by Paolo Mancosu in Mancosu [1998], pp. 189–197. 23, 25

—— (1967). Scope and Limits of Axiomatics. In Mario Bunge (ed.), *Delaware Seminar in the Foundations of Physics*, volume 1 of *Studies in the Foundations Methodology and Philosophy of Science*, chapter 11, pp. 188–191. Springer Verlag, Berlin, Heidelberg, New York. 11, 28

Bernstein, Felix (1909). Über die axiomatische Einfachheit von Beweisen. In Guido Castelnuovo (ed.), *Atti del IV congresso internazionale dei matematici, Roma, 6-11 aprile 1908*, pp. 391–392. Roma: Tipografia della R. Accademia dei Lincei. 32, 35

Beth, E.W. (1955). Semantic entailment and formal derivability. *Mededelingen van de Koninklijke Nederlandse Akademie der Wetenschappen, Afdeeling Letterkunde, Nieuwe Reeks*, 18(13). 9

Blanché, Robert (1973). Axiomatization. In Philip P. Wiener (ed.), *Dictionary of the History of Ideas—Studies of Selected Pivotal Ideas*, volume 1, pp. 162–72. New York: Scribner. 25

Blanché, Robert (1962). *Axiomatics*. Monographs in Modern Logic. Dover Publications. Translated by G. B. Keene from the French *L'Axiomatique*, 1955. 4, 6

Bocheński, I. M. (1954). *Die Zeitgenössischen Denkmethoden*. Berlin: Francke Verlag. 16

Boolos, George, Burgess, John P., and Jeffrey, Richard C (2007). *Computability and Logic*. Cambridge: Cambridge University Press, 5th edition. ISBN 9780521877527 (hbk.). 4, 9

Bourbaki, Nicholas (1950). The architecture of mathematics. *American Mathematical Monthly*, 57(4):221–232. 4, 34

Brandom, Robert (1994). *Making it Explicit: reasoning, representing, and discursive commitment*. Cambridge, Mass.: Harvard University Press. 30

Bunge, Mario (1962). The complexity of simplicity. 59(5):113–135. 26

Carmichael, Robert D. (1930). *The Logic of Discovery*. Chicago: Open Court Publishing. 12

Carnap, Rudolf (1950). *Logical Foundations of Probability*. Chicago, IL: University of Chicago Press. 19, 28

—— (1958). *Introduction to Symbolic Logic and its Applications*. Dover Publications Inc., New York. Translated by William H. Meyer and John Wilkinson from the German *Einführung in die symbolische Logik mit besonderer Berücksichtigung ihrer Anwendungen*, Wien, 1954. 12

Cohen, Morris Raphael and Nagel, Ernest (1934). *An Introduction to Logic and Scientific Method*. Harcourt, NY: Brace and Co. 26, 28

Copi, Irving M. (1958). Artificial languages. In Paul Henle (ed.), *Language, Thought, and Culture*, chapter 4, pp. 96–120. Ann Arbor: The University of Michigan Press. 2, 15, 36

Corcoran, John (1973). Gaps between logical theory and mathematical practice. In Mario Bunge (ed.), *The Methodological Unity of Science*, pp. 23–50. D. Reidel, Dodrecht, Holland. 25

—— (1995). Axiomatic method. In Robert Audi (ed.), *The Cambridge Dictionary of Philosophy*, p. 57. Cambridge University Press, Cambridge. 16

Craig, William (1956). Replacement of auxiliary expressions. *The Philosophical Review*, 65(1):38–55. 25

Davenport, H. (2008). *The Higher Arithmetic. An introduction to the theory of numbers*. Cambridge University Press. 28

Davis, Martin (ed.) (1965). *The Undecidable. Basic Papers On Undecidable Propositions, Unsolvable Problems And Computable Functions*. Raven Press, Hewlett, New York. 40

Davis, Philip J. and Hersh, Reuben (1981). *The Mathematical Experience*. Houghton Mifflin Company, Boston, MA. 35

De Cruz, Helen and De Smedt, Johan (2010). Mathematical symbols as epistemic actions. *Synthese*. Online First. URL = `http://dx.doi.org/10.1007/s11229-010-9837-9`. 11

de Jong, Willem R. and Betti, Arianna (2010). The Classical Model of Science: A millennia-old model of scientific rationality. *Synthese*, 174(2):185–203. 2, 24

Dedekind, Richard (1854). Über die Einführung neuer Funktionen in der Mathematik. Habilitationsvortrag. In: Dedekind [1932], pp. 428–438. English translation *On the introduction of new functions in mathematics* by William Ewald in Ewald [1996], pp. 754–762. 33

—— (1888). *Was sind und was sollen die Zahlen?* Vieweg, Braunschweig. Reprinted in Dedekind [1932], pp. 335–391. English translation by Wooster W. Beman, revised by William Ewald, in Ewald [1996], pp. 787–833. 4, 18, 28

—— (1890). Letter to Keferstein. Published in Sinaceur [1974], pp. 270–278. English translation in van Heijenoort [1967], pp. 98–103. 18

—— (1932). *Gesammelte mathematische Werke*, volume 3. F. Vieweg & Sohn, Braunschweig. Edited by Robert Fricke, Emmy Noether, and Öystein Ore. 9, 18, 19, 38

Densmore, Dana (1996). *Newton's* Principia: *The Central Argument. Translation, Notes, and Expanded Proofs*. Santa Fe, NM: Green Lion Press. 24

Detlefsen, Michael (1998). Hilbert's programme and formalism. In E. Craig (ed.), *Routledge Encyclopedia of Philosophy*, volume 4, pp. 411–438. London: Routledge. 8

—— (2013). Completeness and the ends of axiomatization. In Juliette Kennedy (ed.), *Interpreting Gödel*. Cambridge University Press. Forthcoming. 11, 12, 16

Detlefsen, Michael and Arana, Andrew (2011). Purity of methods. *Philosophers' Imprint*, 11(2):1–20. 32

Dingler, Hugo (1906). Zur Methodik in der Mathematik. *Jahresbericht der Deutschen Mathematiker Vereinigung (JDMV)*, 14(5):581–584. 33

Duhem, Pierre (1954 (1914)). *The Aim and Structure of Physical Theory*. Princeton, NJ: Princeton University Press. Translation by Philip P. Wiener from the second edition, published in 1914 by Marcel Rivière & Cie., Paris, under the title *La Théorie Physique: Son Objet, Sa Structure*. 34

Dunbar, Kevin (1997). How scientists think: On-line creativity and conceptual change in science. In T. B. Ward, S. S. Smith, and J. Vaid (eds.), *Creative Thought. An investigation of conceptual structures and processes*, pp. 461–493. Washington D.C.: American Psychological Association. 18

Easwaran, Kenny (2008). The role of axioms in mathematics. *Erkenntnis*, 68:381–391. 2, 14, 36

Enderton, Herbert B. (2001). *A Mathematical Introduction to Logic*. San Diego: Harcourt Academic Press, 2nd edition. 4, 9

Enriques, Federigo (1903). *Vorlesungen über projektive Geometrie*. Leipzig: B. G. Teubner. Original in Italian 1898. 12, 33

Epple, Moritz (1997). Syles of argumentation in late 19th century geometry and the structure and mathematical modernity. In Otte and Panza [1997], chapter 7, pp. 177–198. 32

Ernst, Zachary, Fitelson, Branden, Harris, Kenneth, and Wos, Larry (2002). Shortest axiomatizations of implicational S4 and S5. 43(3):169–179. 26

Ewald, William (1996). *From Kant to Hilbert: A Source Book in Mathematics*. Clarendon Press, Oxford. Two volumes. 13, 38, 41, 43

Feferman, Solomon (1999). Does mathematics need new axioms? *American Mathematical Monthly*, 106(2):99–111. 1, 2, 14

Frege, Gottlob (1879). *Begriffsschrift. Eine der arithmetischen nachgebildeten Formelsprache des reinen Denkens*. Halle a/S.: Verlag Louis Nebert. English translation: *Begriffsschrift, A formula Language, Modeled upon that for Arithmetic*, in van Heijenoort [1967], pp. 1–82. 9, 32

Friedman, Michael (1974). Explanation and scientific understanding. *The Journal of Philosophy*, 71(1):5–19. 24

Galilei, Galileo (1967 (1632)). *Dialogue Concerning the Two Chief World Systems. Translated by Stillman Drake, foreword by Albert Einstein*. Berkeley, CA: University of California Press, second revised edition. 24

Galton, Francis (1880). Visualized numerals. *Nature*, 21(533):252–256. 31

Gemes, Ken (1993). Hypothetico-deductivism, content, and the natural axiomatization of theories. *Philosophy of Science*, 60(3):477–487. 26

Gentzen, Gerhard (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431. English translation: *Investigations into logical deduction* in Szabo [1969] pages 68-131. 9, 29

Givant, Steven (1999). Unifying threads in Alfred Tarski's work. 21(1):47–58. 21

Glymour, Clark (1980). *Theory and Evidence*. Princeton University Press. 34

Gödel, Kurt (1931). Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I. *Monatshefte für Mathematik und Physik*, 38:173–198. Engl. translation: On formally undecidable propositions of Principia Mathematica and related systems I, in Davis [1965], pp. 4–38. 27

—— (1940). *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis*. Princeton: Princeton University Press. Reprinted with additional notes in 1951 and 1966. 27

Goodman, Nelson (1958). The test of simplicity. 128(3331):1064–9. 26

Gray, Jeremy (1992). The nineteenth-century revolution in mathematical ontology. In Donald Gillies (ed.), *Revolutions in mathematics*, pp. 226–248. Claredon Press, Oxford. 33

Grosholz, Emily R. (2007). *Representation and Productive Ambiguity in Mathematics and the Sciences*. Oxford: Oxford University Press. ISBN 9780199299737 (alk. paper). 1, 13, 16, 29

Hallett, Michael (1984). *Cantorian Set Theory and Limitations of Size*. Oxford: Claredon Press. 27

Hallett, Michael and Majer, Ulrich (eds.) (2004). *David Hilbert's Lectures on the Foundations of Geometry 1891–1902*. Berlin, Heidelberg, New York: Springer. 40

Hanson, Norwood Russell (1958a). The logic of discovery. 55(25):1073–89. 2

—— (1958b). *Patterns of Discovery*. London: Cambridge University Press. Reprint 1969. 17

Hartshorne, Robin (2000). Teaching geometry according to Euclid. *Notices of the American Mathematical Society*, 47(4):460–465. 32

Haupt, Otto (1929). *Einführung in die Algebra*, volume 1. Leipzig: Akademische Verlagsges. m.b.H. 33

Hempel, Carl G. (1952). *Fundamentals of Concept Formation in Empirical Science*, volume II/7 of *International Encyclopedia of Unified Science*. Chicago, IL: University of Chicago Press. 19

Hilbert, David (1899). *Grundlagen der Geometrie*. Teubner, Leipzig. Reprinted in Hallett and Majer [2004], 436–525. English *The Foundations of Geometry* by E. J. Townsend, Open Court, Chicago, 1902 and by L. Under, Open Court, La Salle, Ill., 1971. 4, 9, 33

—— (1900a). Sur les problèmes futurs des mathématiques. *Compte Rendu du Deuxième Congrès des Mathématiciens*, pp. 59–114. German: Mathematische Probleme. Vortrag gehalten auf dem Internationalen Mathematiker-Kongreß zu Paris 1900, *Nachr. königl. Ges. Wiss. Gött. Math.-Phys. Klass.*, pp. 253–297. Reprinted in Hilbert [1935], pp. 290–329. English translation: Mathematical problems, Ewald [1996], pp. 1096–1105. 13

—— (1900b). Über den Zahlbegriff. *Jahresbericht der Deutschen Mathematiker Vereinigung (JDMV)*, 8:180–194. Reprinted as Anhang VI in *Grundlagen der Geometrie*, 3rd ed., Leipzig, 1909. pp. 256–261. English translation *On the concept of number*. 17

—— (1903). Über den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck. *Proceedings of the London Mathematical Society*, 35:50–68. 16, 33

—— (1918). Axiomatisches Denken. *Mathematische Annalen*, 78:405–415. Reprinted in Hilbert [1935], pp. 146–156. English translation: Axiomatic thought, Ewald [1996], pp. 1105–1115. 5, 33

—— (1935). *Gesammelte Abhandlungen*, volume 3. Springer Verlag, Berlin. 41

Hilbert, David and Ackermann, Wilhelm (1928). *Grundzüge der theoretischen Logik*. Springer Verlag, Berlin. 9, 25, 26

Hilbert, David and Bernays, Paul (1934). *Grundlagen der Mathematik*, volume 1. Springer Verlag, Berlin. 2nd edition 1968. 11

Hilbert, David, von Neumann, Johann, and Nordheim, Lothar (1927). Über die Grundlagen der Quantenmechanik. *Mathematische Annalen*, 98:1–30. 19

Hintikka, Jaakko (ed.) (1969). *The Philosophy of Mathematics*. Oxford Readings in Philosophy. Oxford University Press. 42

Hintikka, Jaakko (2011). What is the axiomatic method? *Synthese*, 183:69–85. 2, 9, 36

Hintikka, Jaakko and Remes, Unto (1974). *The Method of Analysis. Its Geometrical Origin and Its General Significance*, volume XXV of *Boston Series in the Philosophy of Science*. Dordrecht/Boston: D. Reidel. 16

Hull, Clark L. (1935). The conflicting psychologies of learning—A way out. *Psychological Review*, 42(6):491–516. Reprinted in as no. 9 in Amsel and Rashotte [1984], pp. 265–290. 24, 35

Hull, Clark L., Hovland, Carl I., Ross, Robert T., Hall, Marshall, Perkins, Donald T., and Fitch, Frederic B. (1940). *Mathematico-Deductive Theory of Rote Learning. A Study in Scientific Methodology*. New Haven, CT: Yale University Press. 11

Huntington, Edward V. (1937). The method of postulates. *Philosophy of Science*, 4(1):482–495. 23, 28, 34

Kennedy, Hubert C. (ed.) (1973). *Selected works of Giuseppe Peano*. Toronto University Press, Toronto. 43

Kitcher, Philip (1983). *The Nature of Mathematical Knowledge*. Oxford University Press, Oxford. 35

Kleene, Stephen C. (1952). *Introduction to Metamathematics*. Amsterdam: North-Holland. 25, 26

Klein, Felix (1893). Vergleichende Betrachtungen über neuere geometrische Forschungen. *Mathematische Annalen*, 43(1):63–100. Original: Erlangen, October 1872 (*Erlangen Program*). Reprinted in Felix Klein, *Gesammelte mathematische Abhandlungen*, vol. 1, Berlin, 1921, ch. XXVII, pp. 460–497. 23

Kline, Morris (1972). *Mathematical Thought from Ancient to Modern Times*. Oxford University Press, New York. 17

Krämer, Sybille (2003). Writing, notational iconicity, calculus: On writing as a cultural technique. *Modern Language Notes – German Issue*, 118(3):518–537. Johns Hopkins University Press. 8, 22

Kreisel, Georg (1967). Informal rigour and completeness proofs. In Imre Lakatos (ed.), *Problems in the Philosophy of Mathematics*, volume 1 of *Proceedings of the International Colloquium in the Philosophy of Science, London, 1965*, pp. 214–269. North–Holland, Amsterdam. Partially reprinted in [Hintikka 1969, 78–94]. 19

Lakatos, Imre (1976). *Proofs and Refutations*. Cambridge University Press, Cambridge. Edited by John Worrall and Elie Zahar. 4, 10, 21

—— (1978a). *Mathematics, Science and Epistemology*, volume 2 of *Philosophical Papers*. Cambridge University Press. Edited by John Worrall and Gregory Currie. 42

—— (1978b). The method of analysis-synthesis. In Lakatos [1978a], chapter 5, pp. 70–103. 16

Leibniz, Gottfried Wilhelm (1859). *Mathematische Schriften*, volume 4. Halle: H. W. Schmidt. Edited by C. I. Gernhardt. 22

Linsky, Bernard (2011). *The Evolution of Principia Mathematica: Bertrand Russell's manuscripts and notes for the second edition*. Cambridge: Cambridge University Press. 26

Łukasiewicz, Jan (1970). Comments on Nicod's axiom and on "generalizing deduction". In L. Borkowski (ed.), *Jan Łukasiewicz: Selected Works*, pp. 179–196. Amsterdam: North-Holland. 25

Maddy, Penelope (1988). Believing the axioms I. 53(2):481–511. 5, 21

—— (2001). Some naturalistic reflections on set theoretic method. *Topoi*, 20:17–21. 19

—— (2011). *Defending the Axioms. On the Philosophical Foundations of Set Theory*. Oxford, UK: Oxford University Press. 24

Mancosu, Paolo (1998). *From Brouwer to Hilbert: The debate on the foundations of mathematics in the 1920s*. Oxford University Press, Oxford. 23, 37

Mancosu, Paolo (ed.) (2008a). *The Philosophy of Mathematical Practice*. Oxford University Press. 1

Mancosu, Paolo (2008b). Mathematical explanation: Why it matters. In Paolo Mancosu (ed.), *The Philosophy of Mathematical Practice*, pp. 134–150. Oxford University Press. 29

Manders, Kenneth (2008a). Diagram-based geometric practice. In Paolo Mancosu (ed.), *The Philosophy of Mathematical Practice*, pp. 65–79. Oxford University Press. 9

—— (2008b). The Euclidean diagram (1995). In Paolo Mancosu (ed.), *The Philosophy of Mathematical Practice*, pp. 80–133. Oxford University Press. 9

McCune, William, Veroff, Robert, Fitelson, Branden, Harris, Kenneth, Feist, Andrew, and Wos, Larry (2002). Short single axioms for boolean algebra. *Journal of Automated Reasoning*, 29(1):1–16. 26

McKeon, Richard (ed.) (1947). *Introduction to Aristotle*. New York: Random House. 22, 33

Moore, Eliakim Hastings (1902). A definition of abstract groups. *Transactions of the American Mathematical Society*, 3(4):485–492. 25

Mueller, Ian (1969). Euclid's Elements and the axiomatic method. *British Journal for the Philosophy of Science*, 20(4):289–309. 22, 33

Nicod, Jean G. P. (1917). A reduction in the number of the primitive propositions of logic. 19:32–41. 25

Otte, Michael and Panza, Marco (eds.) (1997). *Analysis and Synthesis in Mathematics. History and Philosophy*, volume 196 of *Boston Studies in the Philosophy of Science*. Dordrecht, Boston, London: Kluwer Academic Publishers. 16, 39

Padoa, Alessandro (1901). Essay d'une théorie algébrique des nombres entiers, précédé d'une introduction logique à une théorie déductive quelconque. In *Premier congrès international de philosophie, 1900*, pp. 309–365. Paris: Armand Colin. Partial translation in van Heijenoort [1967], pp. 118–123. 23

Pambuccian, Victor (1988). Simplicity. *Notre Dame Journal of Formal Logic*, 29(3):396–411. 26

Parsons, Charles D. (1972). Mathematics, Foundations of. In Paul Edwards (ed.), *Encyclopedia of Philosophy*, volume 5–6, pp. 188–213. Macmillan Publishing Company, New York. 25

Pascal, Blaise (1909–1914). The art of persuasion. In *Minor Works*, volume 48 of *The Harvard Classics*. New York: P. F. Collier. Translated by O.W. Wright. 22

Pasch, Moritz (1882). *Vorlesungen über Neuere Geometrie*. Leipzig: B. G. Teubner. 12, 30, 32, 33

—— (1914). *Veränderliche und Funktion*. Leipzig and Berlin: B. G. Teubner. 35

Peano, Giuseppe (1888). *Calculo Geometrico secundo l'Ausdehnungslehre di H. Grassmann e preceduttto dalle operazioni della logica deduttiva*. Torino: Fratelli Bocca. English translation by Lloyd C. Kannenberg, *Geometric calculus, According to the Ausdehnungslehre by H. Grassmann*, Birkhäuser, Boston, 2000. 9

—— (1889). *Arithmetices Principia, nova methodo exposita*. Bocca, Torino. English translation by Hubert C. Kennedy *The principles of arithmetic, presented by a new method* in Kennedy [1973], pp. 101–134. Partial translation by Jean van Heijenoort in van Heijenoort [1967], pp. 81–97. 9, 10, 18

Peckhaus, Volker (2002). Regressive analysis. In Uwe Meixner and Albert Neven (eds.), *Philosophiegeschichte und logische Analyse. Logical Analysis and History of Philosophy*, volume 5, pp. 97–110. Paderborn: Mentis. 16

Peirce, Charles S. (1958–1966). *Collected papers*. Cambridge, MA: Belknap Press of Harvard University Press. Edited by Charles Hartshorne and Paul Weiss. Vols. 7–8 edited by A.W. Burks. 17

Peirce, Charles Sanders (1902). The essence of mathematics. In *Peirce 1931–58*, volume iv, pp. 189–204. Introductory section of Chapter 3 of the manuscript *Minute logic*. Reprinted in Ewald [1996], pp. 637–648. 17

—— (1990 (1896–1908)). Abduction and induction. In Justus Buchler (ed.), *Philosophical Writings of Peirce*, chapter 11, pp. 150–6. New York: Dover Publications. 17

Pickert, Günter (1971). *Ebene Inzidenzgeometrie*. Frankfurt am Main: Diesterweg Salle, 3rd edition. 28

Poincaré, Henri (1952 (1901)). *Science and Hypothesis*. Dover Publications, New York. Original in French 1901. First English translation 1905. 23

Poincaré, Henri (1963 (1913)). *Mathematics and Science: Last Essays*. New York: Dover Publications. Original in French, *Dernières Pensées*, Ernest Flammarion, 1913. Translated by John W. Bolduc. 20

Priest, Graham, Routley, Richard, and Norman, Jean (eds.) (1989). *Paraconsistent Logic. Essays on the inconsistent*. München: Philosophia Verlag. 27

Pulte, Helmut (2005). *Axiomatik und Empirie: eine wissenschaftstheoriegeschichtliche Untersuchung zur Mathematischen Naturphilosophie von Newton bis Neumann*. Darmstadt: Wissenschaftliche Buchgesellschaft. ISBN 3534158946 (pbk.). 2, 24

Rav, Yehuda (1999). Why do we prove theorems? *Philosophia Mathematica (Series III)*, 7(1):5–41. 10

Reichenbach, Hans (1925). Über die physikalischen Konsequenzen der relativistischen Axiomatik. *Zeitschrift für Physik*, 34:32–48. 24

—— (1938). *Experience and Prediction. An analysis of the foundations and the structure of knowledge*. Chicago: University of Chicago Press. 2

Robinson, John Alan (1965). A machine-oriented logic based on the resolution principle. *Journal of the Association for Computing Machinery*, 12(1):23–41. Reprinted in: Siekmann and Wrightson [1983], pp. 397–415. 9

Rudner, Richard (1961). An introduction to simplicity. *Philosophy of Science*, 28(2):109–119. 26

Russell, Bertrand and Whitehead, Alfred North (1910). *Principia Mathematica*, volume 1. Cambridge: Cambridge University Press. 25

Russell, Betrand (1973 (1907)). The regressive method of discovering the premises of mathematics. In Douglas Lackley (ed.), *Essays in Analysis*, pp. 272–283. New York: George Braziller. Read before the Cambridge Mathematical Club, 9 March 1907. 21, 22, 24

Sá, Walter C., West, Richard F., and Stanovich, Keith E. (1999). The domain specificity and generality of belief bias: Searching for a generalizable critical thinking skill. *Journal of Educational Psychology*, 91(3):497–510. 9

Scanlan, Michael (1991). Who were the American postulate theorists? *Journal of Symbolic Logic*, 56(3):981–1002. 23

Schickore, Jutta and Steinle, Friedrich (eds.) (2006). *Revisiting Discovery and Justification: Historical and philosophical perspectives on the context distinction*, volume 14 of *Archimedes*. Dordrecht: Springer. 2

Schlimm, Dirk (2006). Axiomatics and progress in the light of 20th century philosophy of science and mathematics. In Benedikt Löwe, Volker Peckhaus, and Thoralf Räsch (eds.), *Foundations of the Formal Sciences IV*, Studies in Logic Series, pp. 233–253. London: College Publications. 31

—— (2008). Two ways of analogy: Extending the study of analogies to mathematical domains. *Philosophy of Science*, 75(2):178–200. 19

—— (2009). Bridging theories with axioms: Boole, Stone, and Tarski. In Bart van Kerkhove (ed.), *New Perspectives on Mathematical Practices. Essays in Philosophy and History of Mathematics*, pp. 222–235. London: World Scientific. 33

—— (2011). On the creative role of axiomatics. The discovery of lattices by Schröder, Dedekind, Birkhoff, and others. *Synthese*, 183(1):47–68. 14, 18, 19

—— (2012). Mathematical concepts and investigative practice. In Friedrich Steinle and Uljana Feest (eds.), *Scientific Concepts and Investigative Practice*, volume 3 of *Berlin Studies in Knowledge Research*. Berlin: De Gruyter. 21

—— (2013). Conceptual metaphors and mathematical practice. *Topics in Cognitive Science*. (Forthcoming). 21

Schmidt, Arnold (1932). Zu Hilberts Grundlegung der Geometrie. In *David Hilbert. Gesammelte Abhandlungen*, volume 2, pp. 404–414. Springer Verlag, Berlin. 23, 26

Seron, Xavier, Pesenti, Mauro, Noël, Marie-Pascale, Deloche, Gérard, and Cornet, Jacques-André (1992). Images of numbers, or "When 98 is upper left and 6 sky blue". *Cognition*, 44:159–196. 31

Sfard, Anna (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22:1–36. 11

Shapiro, Stewart (2005). Categories, structures, and the Frege-Hilbert controversy: The status of meta-mathematics. 13(1):61–77. 14

—— (2009). We hold these truths to be self-evident: But what do we mean by that? 2(1):175–207. 22

Sheffer, Henry Maurice (1913). A set of five postulates for Boolean algebras, with application to logical constants. *Transactions of the American Mathematical Society*, 14(4):481–488. 25

Sieg, Wilfried (1990). Relative consistency and accessible domains. *Synthese*, 84:259–297. 27, 35

—— (1995). Formalization. In Robert Audi (ed.), *The Cambridge Dictionary of Philosophy*, p. 273. Cambridge University Press, Cambridge. 9

—— (2002). Beyond Hilbert's reach? In David Malament (ed.), *Reading Natural Philosophy. Essays in the History and Philosophy of Science and Mathematics*, pp. 363–405. La Salle, Il.: Open Court. Reprinted in Sten Lindström et al. (eds), *Logicism, Intuitionism, and Formalism*, Springer, pp. 499–483. 12

Sieg, Wilfried and Schlimm, Dirk (2005). Dedekind's analysis of number: Systems and axioms. *Synthese*, 147(1):121–170. 18

Siekmann, Jörg and Wrightson, Graham (eds.) (1983). *Automation of Reasoning: Classical Papers on Computational Logic 1957–1966*, volume 1. Springer Verlag, Berlin. 44

Simpson, Stephen G. (1999). *Subsystems of Second Order Arithmetic*. Berlin: Springer. 34

Sinaceur, M.A. (1974). L'infini et les nombres—Commentaires de R. Dedekind à 'Zahlen'—La correspondance avec Keferstein. *Revue d'historie des sciences*, 27:251–278. 39

Stein, Howard (1988). *Logos*, Logic, and *Logistiké*: Some Philosophical Remarks on Nineteenth-Century Transformation of Mathematics. In William Aspray and Philip Kitcher (eds.), *History and Philosophy of Modern Mathematics*, volume 11 of *Minnesota Studies in the Philosophy of Science*, pp. 238–259. University of Minnesota, Minneapolis. 32

Suppe, Frederick (ed.) (1977). *The Structure of Scientific Theories*. Urbana, IL: University of Illinois Press, 2nd edition. 4

Suppes, Patrick (1957). *Introduction to Logic*. D. Van Nostrand, New York. 12

—— (1968). The desirability of formalization in science. *The Journal of Philosophy*, 65(20):651–664. Sixty-Fifth Annual Meeting of the American Philosophical Association Eastern Division. 20, 25, 35

—— (1979). Intellectual autobiography. In Radu J. Bogdan (ed.), *Patrick Suppes*. Dordrecht: D. Reidel. 34

Szabo, M. E. (ed.) (1969). *The Collected Papers of Gerhard Gentzen*. North–Holland, Amsterdam. 40

Tarski, Alfred (1956 (1936)). On the concept of logical consequence. In *Logic, Semantics, Metamathematics*, pp. 409–420. Oxford University Press, Oxford. First published in Polish as *O pojciu wynikania logicznego*, Prezglad Filozoficzny 39:97–112 (1936) and in German as *Über den Begriff der logischen Folgerung*, Actes du Congrès International de Philosophie Scientifique, vol. VII, Paris, pp. 1–11 (1937). 9

—— (1959). What is elementary geometry? In Leon Henkin, Patrick Suppes, and Alfred Tarski (eds.), *The Axiomatic Method, with Special Reference to Geometry and Physics*, pp. 16–29. North–Holland, Amsterdam. 25

—— (1995 (1936)). *Introduction to Logic and to the Methodology of the Deductive Sciences*. Dover Publications Inc., New York. Translated by Olaf Helmer. Reprint of the second revised edition, 1946. Original in German, 1936. 28

Tennant, Neil (2012). *Changes of Mind. An essay on rational belief revision*. Oxford, UK: Oxford University Press. 3, 25

van Fraassen, Bas C. (1980). *The Scientific Image*. Oxford: Clarendon Press. 10, 27

van Heijenoort, Jean (1967). *From Frege to Gödel: A Sourcebook of Mathematical Logic*. Cambridge, Massachusetts: Harvard University Press. 10, 18, 20, 25, 39, 43, 46

Veblen, Oswald (1914). The foundations of geometry. In J. W. A. Young (ed.), *Monographs on Topics of Modern Mathematics Relevant to the Elementary Field*. Longmans, Green. Reprinted 1955 by Dover, New York. 23

von Neumann, John (1925). Eine Axiomatisierung der Mengenlehre. *Journal für die reine und angewandte Mathematik*, 154:219–240. Reprinted in von Neumann [1961], pp. 35–56. English translation *An axiomatization of set theory* in van Heijenoort [1967], pp. 393–413. 20

—— (1927). Zur Hilbertschen Beweistheorie. *Mathematische Zeitschrift*, 26:1–46. 25

—— (1961). *Collected Works*, volume 1. Pergamon Press, New York. Edited by A.H. Taub. 46

Weinberg, Steven (2002). Afterword. how great equations survive. In Graham Farmelo (ed.), *It Must Be Beautiful. Great Equations of Modern Science*, pp. 253–257. London: Granta Books. 35

Weyl, Hermann (1918). *Das Kontinuum – Kritische Untersuchung über die Grundlagen der Analysis*. Veit, Leipzig. 47

—— (1919). Der circulus vitiosus in der heutigen Begründung der Mathematik. *Jahresbericht der Deutschen Mathematiker Vereinigung (JDMV)*, 28:85–92. English translation by Stephen Pollard and Thomas Bole in *The Continuum*, Dover, New York, 1994. 47

—— (1930). Axiomatik. Notes of lectures held at the University of Göttingen, Winter semester 1930/1, held at the library of the Institute for Advanced Study. 11

—— (1935). Emmy Noether. *Scripta Mathematica*, 3:1–20. 2

—— (1994 (1918)). *The Continuum. A Critical Examination of the Foundation of Analysis*. New York: Dover. English translations by Stephen Pollard and Thomas Bole of Weyl [1918] and Weyl [1919]. 22

Wilder, R. L. (1959). Axiomatics and the development of creative talent. In Leon Henkin, Patrick Suppes, and Alfred Tarski (eds.), *The Axiomatic Method, with Special Reference to Geometry and Physics*, pp. 474–488. 2

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