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# Babbage's Guidelines for the Design of Mathematical Notations

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**Abstract**

The design of good notation is a cause that was dear to Charles Babbage's heart throughout his career. He was convinced of the "immense power of signs" (1864, 364), both to rigorously express complex ideas and to facilitate the discovery of new ones. As a young man, he promoted the Leibnizian notation for the calculus in England, and later he developed a Mechanical Notation for designing his computational engines. In addition, he reflected on the principles that underlie the design of good mathematical notations. In this paper, we discuss these reflections, which can be found somewhat scattered in Babbage's writings, for the first time in a systematic way. Babbage's desiderata for mathematical notations are presented as ten guidelines pertinent to notational design and its application to both individual symbols and complex expressions. To illustrate the applicability of these guidelines in non-mathematical domains, some aspects of his Mechanical Notation are also discussed.

**Keywords:** Babbage, design principles, discovery, mathematical notation, mechanical notation.

## 1 Introduction

Charles Babbage (December 26, 1791–October 18, 1871) was a British mathematician and inventor who is best known for his work on calculating machines, namely the Difference and Analytical Engines.[1] In addition, Babbage was deeply concerned about the use and development of good notations, attributing much of his career to them: "I believe my early perception of the immense power of signs in aiding the reasoning faculty contributed much to whatever success I may have had" (Babbage 1864, 364)[2]. He sought to make explicit guidelines that foster productive and efficient mathematical notations, presenting them together with more

[1] For background on Babbage's life and work, see Babbage (1864) and Hyman (1982).

[2] Page numbers to Babbage's papers refer to reprints in *The Works of Charles Babbage*, edited by Martin Campbell-Kelly (1989a, 1989b, 1989c).

general reflections in a series of papers: *Preface to the Memoirs of the Analytical Society* (1813), *Observations on the Notation employed in the Calculus of Functions* (1822), *On the Influence of Signs in Mathematical Reasoning* (1827), and an encyclopedia entry, *Notation* (1830).

Babbage's guidelines are based on his reflections on notations used in the history of mathematics, his mathematical work on the calculus of functions[3], his experiences with designing an efficient representation of complex machines, and his attempts at empirical research.

In an effort to present the remarkably diverse work of Babbage from a unified perspective, Grattan-Guinness characterizes him as following an "algorithmic/algebraic/semiotic" approach. Babbage is described as having introduced "some good notation" and also having considered "families of symbols, and symbolism in general" (Grattan-Guinness 1992, 38). Grattan-Guinness mentions "various desiderata for notations" that Babbage formulated, but discusses only one of them (Grattan-Guinness 1992, 39). It is the aim of the present paper to provide a comprehensive account of the semiotic thread in Babbage's work by presenting and discussing his reflections on the design of good notations. In addition to enriching our insights into the work and methods of Babbage, this provides a rich historical case study for further research on the role and effects of notations in mathematical practice, an area that has recently received considerable attention.

In the following, we begin by presenting Babbage's reflections on the importance of notation (Section 2). This is followed by a systematic discussion of Babbage's recommendations for mathematical notations, which we present as ten guidelines: *conciseness*, *simplicity*, *univocity*, *mnemonics*, *iconicity*, *analogy*, *modularity*, *generality*, *symmetry of symbols*, and *symmetry of structure*. These are grouped according to whether they are related to the general aims of notation (Section 3), to the meaning of individual symbols (Section 4), or to the formulation of complex expressions (Section 5). We complement the discussion with examples from his Mechanical Notation to illustrate the application of Babbage's guidelines to notational choices in non-mathematical domains.

# 2 On the importance of notation and of its study

## 2.1 Historical background

Before presenting Babbage's views on notations, let us take a brief look at the historical context in which they were formed. After Newton's retirement in the early 18th century, the collegiate standard in British mathematical training was to selectively study a portion of Newton's *Principia Mathematica* and to memorize it for examination purposes (Moseley 1964, 48). This contributed to the general preference in Britain of Newton's dot-notation over Leibniz's d-notation for differential calculus, and that of Newton's geometric methods over the analytic methods popular on the continent. The study of more recent discoveries made by French and German mathematicians was thereby effectively discouraged, and

[3] A field Babbage pioneered, which involves algebraically proving characteristics of general functions; see *An essay towards the calculus of functions* (1815) and *An essay towards the calculus of functions, part II* (1816).

British mathematics experienced a period of stagnation. Highlighting this lack of support for original mathematical inquiry, Babbage reports the response he received from his tutor when he raised some difficulties he had encountered while independently studying Lacroix's *Traité Élémentaire de Calcul Différentiel et de Calcul Intégral*: "these questions w[ill] not be asked in the Senate House, and [are] therefore of no consequence" (Moseley 1964, 47–48).

After learning the notation of Leibniz and the analytic methods through self-directed studies, Babbage came to the conclusion that the lack of progress in British mathematics was, at least in part, a consequence of adhering to an ineffective notation. For instance, in his view, the dots of Newton have a "want of analogy with other established notations, such as those relating to the symbols '$\Delta$' and '$\delta$', and present a "great difficulty, if not the impossibility, of representing, by their means, theorems relating to the separation of operations from quantities" (Babbage 1830, 424). That is, they fail to satisfy the guidelines of analogy and modularity (presented in Sections 4.4 and 5.1).

To counter this situation in British mathematics, Babbage founded the Analytical Society with friends John Herschel and George Peacock and engaged in a movement to revive the field by introducing, through translations, the powerful continental methods and notations (Koppelman 1971, 176). This movement for notational and methodological reform proved remarkably successful. As Elaine Koppelman describes in her historical overview of the period, *The Calculus of Operations and the Rise of Abstract Algebra* (1971), not only did it catalyze a renaissance in British mathematics, but it also played a causal role in some of its later advances by introducing notions that prompted the re-conceptualization of algebra as an abstract science of its own.

## 2.2 Toward a rational approach to notation

Babbage characterizes notation as "the art of adapting arbitrary symbols to the representation of quantities, and the operations to be performed on them" (Babbage 1830, 409). He is keenly aware that differences in notation may seem "apparently trivial", but maintains that "the convenience or inconvenience of notation frequently depends on differences as trifling" (Babbage 1827, 398). Moreover, while some contend that preferences of notation are merely a matter of convention or "in a sense aesthetic" (Koppelman 1971, 177), Babbage holds that our inclinations toward one notational system over another are ultimately based on some underlying rational principles:

> How frequently does it happen, even to the best informed, that they prefer one thing and reject another, from some latent sense of their propriety or impropriety, without being immediately able to state the reasons on which such a choice is founded; yet it cannot be doubted, when the selection appears to be the result of correct taste, that it is guided by unwritten rules, themselves the valued offspring of long experience. (Babbage 1830, 418)

Only once these unwritten rules are made explicit can we address questions regarding the design and use of notation in a rational way.

In discussing the difficulties of learning new mathematical concepts, Babbage writes that a poorly-adapted notation is not "by any means the sole obstacle" to understanding, but it is "one, which appears to me of some weight, and which might, without much difficulty, be removed" (Babbage 1827, 407). Indeed, Babbage considered the judicious choice of notation to be of utmost importance not only for students, but also for those "who may have occasion to express new relations" (Babbage 1830, 412) and for the progress of science as a whole:

> The subject of the principles and laws of notation is so important that it is desirable, before it is too late, that the scientific academies of the world should each contribute the results of their own examination and conclusions, and that some congress should assemble to discuss them. (Babbage 1864, 106)

Such careful deliberations would foster well-adapted notations and promote a uniform and consistent usage of symbols. In fact, Babbage himself made efforts to kindle these sorts of discussions at the Great Exhibition of 1851, held in London, where he distributed a short paper consisting of 20 'Laws of Mechanical Notation' "in considerable numbers, to foreigners as well as to his countrymen", asking for criticisms and additions (Babbage 1889, 242).

## 2.3 Understanding, reasoning, and discovery

In general, a good notation represents a complex situation in a way that is intelligible for a human agent and allows for efficient reasoning about the situation in question. That a notation is to be used by human agents implies that it must be suited to the perceptual apparatus and cognitive limitations of human beings, e. g., memory and attention, as well as to their reasoning process. Alternatively, a notation that is intended to be processed by machines, for example, would be subject to different constraints.

The situations that Babbage considers in his writings on notation are mainly mathematical (e. g., certain relations between quantities and operations on them) and mechanical (e. g., the workings of a calculating machine). In assessing different methods of representing these situations, Babbage frequently emphasized those that help overcome our cognitive limitations. He praised representations that alleviate "fatigue", "assist the memory", and "facilitate the processes by which [a] final arrangement [is] accomplished" (Babbage 1827, 403, 407–408). As will be seen in the next sections, Babbage recommends many of his guidelines for notation on the grounds that they unburden the memory and thereby free up mental energy for other purposes.

Babbage even thought it worthwhile to explore the effects of various material features of representations. His *Specimen of logarithmic tables printed with different coloured inks on variously coloured papers* (1831)[4] is a highly unusual and rare book that considers a specific question regarding ease of reading: Which combination of text and paper colour is least fatiguing to the eye? The publication consists of various samples of different combinations and encourages readers to use them to find out their preferences in the hope of reaching some empirically driven conclusion. Babbage himself ended up printing his tables in black ink on a "rather bright yellow paper", which Campbell-Kelly describes as "slightly dazzling" (Campbell-Kelly 1988, 163).

[4]See Campbell-Kelly (1988) for background on Babbage's work on logarithmic tables, including Babbage's considerations for the choice of font (type). Grattan-Guinness also remarks on the spacing of the arrays of digits in these tables "for ease of reading" and the use of different colours for the four basic arithmetic operations of the Analytical Engine, as described in (Babbage 1837, 42 and 52) (Grattan-Guinness 1992, 40).

Although Babbage discusses notations mainly as means for efficiently representing a given domain, he also sees a profound interplay between notation and mathematical reasoning. When a notation is well-suited to a particular problem, it can have a "remarkable influence [...] in the successful termination of [the] reasoning process" (Babbage 1827, 384). Thus, Babbage maintains that "any work devoted to the philosophical explanation of analytical language" must include "an examination of the various stages, by which, from certain data, we arrive at the solution of the questions to which they belong" (Babbage 1827, 386). For Babbage, these stages, each of which imposes different constraints on good notations, are: translating a situation into a notation, manipulating the notation to solve the problem in question, and re-translating the solution back into ordinary language. The last one, Babbage laments, "has been more neglected than any other" (Babbage 1827, 388).

Babbage maintains that, in addition to clarifying our understanding of a subject and supporting mathematical reasoning, a well-chosen notation has the potential to guide the researcher to new insights and discoveries:

> [W]e cannot employ a new symbol or make a new definition, without at once introducing a whole train of consequences, and in defiance of ourselves, the very sign we have created, and on which we have bestowed a meaning, itself almost prescribes the path our future investigations are to follow. (Babbage 1822, 344)

Because of this autonomy, Babbage contends, notations themselves can advance mathematical thought:

> [T]he symbols which have thus been invented in many instances from a partial view, or for very limited purposes, have themselves given rise to questions far beyond the expectations of their authors, and [...] have materially contributed to the progress of the science. (Babbage 1822, 343)

## 2.4 Babbage's writings on notations

Babbage's discussions of notation are presented mainly in the context of mathematics. While some of his remarks appear merely as comments in his writings, he also dedicated some publications exclusively to the discussion of notations. The four most important ones are the following: In the *Preface to the Memoirs of the Analytical Society* (1813) Babbage offers a history of mathematical analysis, paying special attention to its symbolic developments and touching on the benefits of good notations. His own research on the calculus of functions prompted the *Observations on the Notation employed in the Calculus of Functions* (1822), which discusses in particular the importance of conciseness and the use of analogy for the design of notations, using many examples from the calculus of functions. This discussion is broadened to various other considerations that contribute to the power of symbolic representations in *On the Influence of Signs in Mathematical Reasoning* (1827). Finally, Babbage's encyclopedia entry on notation (1830) presents his most comprehensive account of notational principles.

In the encyclopedia entry, Babbage introduced explicit "principles" and "rules" for the design of notations, which range from general advice (e. g., that notations should be concise) to specific recommendations (e. g., that the inverse of an operation should be denoted by the superscript '$-1$' (Babbage 1830, 411–413)). However, this classification appears rather unsystematic and does not cover all of the precepts that he discussed somewhat scattered in earlier publications. We have thus decided to organize Babbage's ideas according to whether they pertain to notation in general, to individual symbols and their meanings, or to the formulation of complex expressions, and to present them as ten different guidelines: *Conciseness*, *simplicity*, *univocity*, *mnemonics*, *iconicity*, *analogy*, *modularity*, *generality*, *symmetry of symbols*, and *symmetry of structure*. Some of these correspond directly to principles and rules that Babbage proposes, but others are extracted from his other discussions. The labels are our own, since Babbage himself stated the guidelines without giving them specific names. We hope that this structure lends a coherence to Babbage's reflections that is missing in his own presentations.

Although Babbage primarily wrote about guidelines that underlie good mathematical notation, he also spent considerable time developing a so-called Mechanical Notation to aid in the construction of his calculating machines. When Babbage started to work on these machines around 1819, he found, to his dismay, that the known methods of representing the workings of machines were inadequate. Thus, he developed the Mechanical Notation in order to "devise a more rapid means of understanding and recalling the interpretation of [his] own drawings" (Babbage 1864, 107). This system of representations shares some of the fundamental aims of mathematical notations, namely that it

> ought if possible to be at once simple and expressive, easily understood at the commencement, and capable of being readily retained in the memory from the proper adaptation of the signs to the circumstances they were intended to represent. (Babbage 1826, 209–210)

Thus, as we introduce Babbage's guidelines below, we will also use some examples from the Mechanical Notation to illustrate them, thereby demonstrating that the utility of Babbage's guidelines extends beyond the domain of mathematics.

## 3 General guidelines: Conciseness and simplicity

Babbage considers conciseness to be an essential property of a good notation, because it enables meaning to be communicated quickly. He writes:

> The great object of all notation is to convey to the mind, as speedily as possible, a complete idea of operations which are to be, or have been, executed; since every thing is to be exhibited to the eye, the more compact and condensed the symbols are, the more readily they will be caught, as it were, at a glance. (Babbage 1830, 412)

Devising a concise notation was, Babbage maintains, the original motivation for the adoption of dedicated signs in algebra. Finding it cumbersome to constantly write out the words for operations, early mathematicians "contented themselves [...] by employing one or two of the initial, or, in some cases, of the final letters, to denote them" (Babbage 1830, 409). Further simplifications were adopted over time, in particular the use of seemingly arbitrarily chosen symbols, which also reduced the risk of ambiguity. Once the meaning of the symbols was learned, a simple set of markings could convey a conceptually complex subject matter "at a glance".

In addition to facilitating our understanding of expressions, Babbage notes that conciseness can also enhance our powers of reasoning. Because shorter expressions increase the speed with which they can be understood, successive ideas can be processed faster, and this, Babbage contends, increases the accuracy of the reasoning and the amount of knowledge that can be processed:

> The closer the succession between two ideas which the mind compares, provided those ideas are clearly perceived, the more accurate will be the judgement that results; and the rapidity of forming this judgement, which is a matter of great importance, inasmuch as the quantity of knowledge we can acquire in a great measure depends on it, will be proportionably encreased. (Babbage 1827, 376)

The central importance of conciseness as an overarching principle is due to the fact that it can apply at many levels, guiding both the design and the use of a notation. The conciseness of a notation can easily be achieved by adopting more primitive signs. For example, expressions in the decimal notation are shorter than corresponding ones in the binary notation, but at the cost of using ten basic symbols instead of two. To counteract this proliferation of symbols for the sake of reducing the length of expressions, Babbage echoes William of Ockham's dictum (Kneale and Kneale 1964, 243) that we "*ought not to multiply the number of signs without necessity*" (Babbage 1830, 414). Reasons for introducing additional signs can be based on practical or cognitive matters. For example, while the introduction of novel operations or properties obviously necessitates the introduction of new symbols, Babbage also mentions "unusual combinations" (Babbage 1830, 416) to warrant such an introduction. He explains:

> The natural tendency of the science is to develop new relations and new combinations of those already known. When these new relations involve complicated combinations of such as are already received, or when they are of frequent occurrence, it becomes necessary, if it were merely for the sake of brevity, that some new symbol should be employed. (Babbage 1830, 414)

A specific rule that Babbage mentions for making expressions more concise is: "*parentheses may be omitted, if it can be done without introducing ambiguity*" (Babbage 1830, 421). Having fewer symbols reduces the amount of information that needs to be processed. Nevertheless, the choice of when to omit parentheses can be more difficult than it might seem at first. For example, when Babbage discusses this rule in relation to various customary ways of representing higher powers of trigonometric functions, he concludes that both $(sin\,\theta)^2$ and $sin\,\theta^2$ are equally well-suited to the task in simple cases; however, he concedes that with $\theta$ as a compound quantity, for example $2\theta$, the former notation is superior, because the latter would introduce ambiguity (e. g., $sin\,2\theta^2$) (Babbage 1830, 422).[5] We see here that

[5] Another example mentioned by Babbage is the representation of multiplication by juxtaposition or by specific symbols ('$\cdot$' or '$\times$'): while the former leads to shorter expressions and is preferable in simple cases, it can also lead to ambiguities when used together with other conventions in more complex expressions (Babbage 1830, 421).

even the overarching guideline of conciseness can stand in tension with other considerations (e. g., avoiding ambiguities).

Conciseness can be seen as a special case of a more general desideratum for notations, namely simplicity: "*[A]ll notation should be as simple as the nature of the operations to be indicated will admit*" (Babbage 1830, 412). In addition to the brevity of expressions, 'simplicity' can be understood in various ways, for example with regard to the primitive concepts represented, the shape of the symbols chosen, or the structure of expressions. Babbage seems most concerned with the latter two, namely with designing symbols with simple sets of markings that can easily convey their meaning and arranging them in such a way that facilitates our understanding of complex expressions. Again, Babbage is aware of certain limitations with regard to the applicability of these guidelines:

> It must, however, be remarked, that it is, in many cases, absolutely impossible to express the complicated operations required in the highest departments of analysis by formulae that can be called simple. Still, however, they may be simple with reference to the multiplied relations they express. (Babbage 1830, 412)

In addition to the general aims of conciseness and simplicity, Babbage put forward a number of more concrete suggestions for devising good notations, which we have grouped according to whether they apply to individual symbols or to complex expressions, and which will be presented in the next two sections.

# 4 Guidelines for symbols and their meanings

## 4.1 Univocity

The guideline of univocity is formulated by Babbage as "*we must adhere to one notation for one thing*" (Babbage 1830, 412). He adds: "it is particularly unphilosophical, and completely contrary to the whole spirit of symbolic reasoning, to employ the same signs for the representation of different operations" (Babbage 1830, 412).

Babbage illustrates the advantage of univocal mathematical symbols by drawing a comparison with ordinary language. He considers the definitions for words such as 'beauty' or 'government' to be vague, suggesting a "multitude of significations", which sometimes make it difficult to keep in view the "real ground on which our reasoning depends" (Babbage 1827, 372–373). In contrast, in mathematics, Babbage considers "the definitions themselves [to be] exceedingly simple, comprising but few ideas" (Babbage 1827, 372). Thus, when represented unambiguously in a symbolic language, the meaning of each symbol can be grasped with relative ease, no matter how complex the expression. This frees up mental energy that can be devoted to understanding more complicated relations and to extending different lines of reasoning. As Babbage explains, in contrast to words, algebraic signs are such that:

> [The] quality on which the whole force of our reasoning turns shall be visible to the eye [. . . which] enables the mind to apply that attention, which must otherwise be exerted in keeping it in view, to the more immediate purpose of tracing

its connection with other properties that are the objects of our research. (Babbage 1827, 373)

## 4.2 Mnemonics

While Babbage notes that the choice of individual symbols for a notation is in principle arbitrary, he is also aware that we process symbols differently depending on their shapes. Notations can be designed to better express the important features of the operations or quantities they represent; this results in more efficient notations, which the reader can understand and reason about more easily.

A straightforward way of making a notation more efficient is to employ mnemonic aids in choosing names for variables and operations. As an example, Babbage discusses improvements to the formulation of a problem as given in Newton's *Arithmetica Universalis*:

> The velocities of two moving bodies $A$ and $B$ being given, and also their distance, and the difference of the times of the commencement of their motion, to determine the point in which they will meet.
>
> Let $A$ have such a velocity that it will pass over the space $c$ in time $f$; and let $B$ have such that it will pass over the space $d$ in time $g$, and let the interval between the two bodies be $e$, and that of the times when they begin to move be $h$. (Babbage 1827, 399; cf. Newton 1720, 72)

The exact steps in the solution not being important here, we take for granted that, should both bodies be moving in the same direction and $B$ begin moving first, the solution will amount to (Babbage 1827, 399):

$$x = \frac{ceg + cdh}{cg - df}.$$

To use a more efficient notation, Babbage proposes to denote the velocity per second of $A$ by $v$ (in place of $\frac{c}{f}$), the velocity per second of $B$ by $v'$ (in place of $\frac{d}{g}$), the space between $A$ and $B$ by $s$ (in place of $e$), and the time in seconds one starts before the other by $t$ (in place of $h$). Solving the problem in a similar way, he arrives at the following solution (Babbage 1827, 401):

$$x = v\frac{s + tv'}{v - v'}.$$

Babbage's choice to condense the representations for the velocities of $A$ and $B$ to a single sign, respectively, reduces the total number of variables and thus simplifies the expression. Moreover, denoting velocity, space, and time by '$v$', '$s$', and '$t$' makes the "signs recall the thing signified" (Babbage 1827, 402). By linking the variables' names directly to their meanings, this formulation renders the result more immediately intelligible and eliminates the need to refer back to the definitions in the text to check what each letter represents. Recalling these definitions may be a trivial task, but it still slows down the pace at which the reader can work through the problem and understand the solution. The avoidance of unnecessary cognitive burdens is emphasized by Babbage as one of the main advantages of a good notation:

> The advantage of selecting in our signs, those which have some resemblance to, or which from some circumstance are associated in the mind with the thing signified, has scarcely been stated with sufficient force: the fatigue, from which such an arrangement saves the reader, is very advantageous to the more complete devotion of his attention to the subject examined; and the more complicated the subject, the more numerous the symbols [ . . . ], the more indispensible will such a system be found. (Babbage 1827, 403)

## 4.3 Iconicity

The advantage of using mnemonics can also be obtained with arbitrary symbols if their shapes are chosen in such a way that they somehow suggest their meanings. The passage quoted above continues:

> This rule is by no means confined to the choice of letters which represent quantity, but is meant to extend, when it is possible, to cases where new arbitrary signs are invented to denote operations. (Babbage 1827, 403)

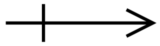
Examples of this generalization are the signs for the relations of greater than, less than, and equality: '$>$', '$<$', '$=$', where the intended relation between objects is, in a sense, embodied by the adopted sign. The signs for greater than and less than "are so contrived, that the largest end is always placed next to the largest quantity, and consequently, the smallest end next to the smallest quantity," which, once understood, makes these signs more "immediately recal [sic] the thing which they are intended to represent" (Babbage 1827, 404). Similarly, the sign for equality indicates by its balanced nature that "the same relation exist[s] between its two parts" (Babbage 1827, 403).

These representations are effective because they have, to some extent, an iconic character, i. e., a likeness or resemblance to the things they signify (Peirce 1894). Although the relations of greater than, less than, and equality are abstract and thus do not have a specific form or shape, the symbols used to represent them exemplify particular instances of the relations they denote. This makes it easier for readers to understand these symbols, which, in turn, lets them concentrate more energy on reasoning about the problem at hand.[6]

Babbage relied heavily on iconic representations when designing parts of his Mechanical Notation. To denote the logic of a machine — i. e., the "connection of each movable piece of the machine with every other on which it acts" (Babbage 1864, 107–108) — Babbage devised a method of using different arrows to represent the specific nature of the motion or attachment between pieces. For example, in *On a Method of Expressing by Signs the Action of Machinery* (1826), three kinds of arrows are introduced:

$\longrightarrow$ to indicate that "[o]ne piece may be driven by another in such a manner that when the driver moves, the other also always moves",

[6] Another example of iconicity that Babbage is very fond of is the algebraic representation of geometry contrived by Carnot, which uses an overbar, $\overline{AB}$, to denote a line between two points *A* and *B*, a curved overbar, $\widehat{AB}$, for an arc of a circle or curve, a point, $\widehat{AB}\,\dot{}\,\widehat{CD}$, for an intersection of two lines or curves, and a caret, $\overline{AB}\,^{\wedge}\,\overline{CD}$, for an angle between two lines or curves.

 for the case that “one piece may receive its motion from another by being permanently attached to it”, and

 to indicate that “one piece may be driven by another, and yet not always move when the latter moves” (Babbage 1826, 212).

Once introduced, these symbols can easily be remembered because their shapes aptly express their meanings: a continuous line suggests continuous motion, intersecting bars suggest a connection, and a dotted line suggests stasis.

Babbage valued this benefit highly, and thus developed similarly motivated symbols to represent the timing of a machine (i. e., the state of motion or rest of any given part of a machine). For instance, he represented a constant velocity with a straight line and a changing velocity with a curved line (Babbage 1826, 215–216).[7]

## 4.4 Analogy

The guideline of analogy also pertains to the design of individual symbols, but it guides the transfer of representations from one domain to another, maintaining that similar signs should be adopted for similar operations. As Babbage explains,

> *When it is required to express new relations that are analogous to others for which signs are already contrived, we should employ a notation as nearly allied to those signs as we conveniently can.* (Babbage 1830, 413)

In practical terms, this guideline urges us to draw from established notations when possible.

The field in which Babbage most often invoked the guideline of analogy himself was the calculus of functions. Since he pioneered it[8], he had both little to direct his investigations and free reign to devise new notations. Babbage drew on established conventions for representing known quantities with the first letters of the alphabet ($a$, $b$, $c$, etc.) and unknown quantities with the last letters of the alphabet ($x$, $y$, $z$, etc.)[9] and, analogously, denoted known functions with the first Greek letters ($\alpha$, $\beta$, $\gamma$, etc.) and unknown functions with some of the last Greek letters ($\phi$, $\chi$, $\psi$, etc.). As for the naming convention itself, Babbage remarks:

> This is in itself a matter perfectly arbitrary [. . . ] but whenever one of these [. . . ] is fixed upon for this purpose, if we wish to consider known and unknown functions, and to treat of their relations, it is no longer a matter of indifference how they are to be distinguished. (Babbage 1830, 414)

So, although it is simply a matter of custom, transgressing the guideline of analogy would create an unnecessary inconvenience — burdening readers with remembering extraneous, not to mention incongruous, information. Abiding by the tradition, on the other hand, can make expressions in the calculus of functions easier to process from the beginning, since one is already familiar with them.

[7]For a complex example of Babbage’s Mechanical Notation, see his table for the timing of an eight-day clock (Babbage 1826, 223).

[8]See footnote 3 in Section 1.

[9]This practice seems to have originated with Descartes (Cajori 1928, 381–383).

Babbage extends another analogy to the calculus of functions by designating repeated functions, for example $\psi(\psi(x))$, by $\psi^2(x)$. This notation is evidently taken from that of exponentiation, or repeated multiplication, in which the repetition of a quantity, $xx$, is denoted by $x^2$. To those acquainted with the sign for traditional exponentiation, the meaning of its extension in the realm of functions is intuitively grasped.[10] With this extension, Babbage perceives an even greater benefit to adopting similar signs for similar situations: it can foster discovery. His discussion of how re-purposing exponents to apply to functions naturally inspired several avenues for exploration is worth quoting at length:

> [I]t now followed [...] that
>
> $$f^{n+m}(x) = f^n f^m(x), \qquad (A)$$
>
> when $n$ and $m$ are whole numbers.[11]
>
> At this point of generalization, a question occurred as to the meaning of $f^n$ when $n$ is a fractional, surd, or negative number, and in order to determine it, recourse was had to a new convention not inconsistent with, but comprehending in it the former one. The index $n$ was now defined by means of the equation ($A$) and was said to indicate such a modification of the function to which it is attached that that equation shall be verified.
>
> From this extended view of the equation ($A$), several curious results follow; if $n = 0$, it becomes
>
> $$f^m(x) = f^0 f^m(x).$$
>
> This informs us that $f^0$ is such an operation that when performed on any quantity, it does not change it, or putting $f^m(x) = y$, it gives
>
> $$f^0(y) = y,$$
>
> a result which is analogous to $x^0 = 1$.
>
> Let $m = -1$, $n = 1$, we have
>
> $$f^0 x = f^1 f^{-1}(x), \text{ or } f(f^{-1}x) = x;$$
>
> $f^{-1}(x)$ must therefore signify such a function of $x$, that if we perform upon it the operation denoted by $f$, it shall be reduced to $x$. (Babbage 1822, 344–345)

This discussion demonstrates that drawing from established notations when tackling novel domains — that is, where a meaningful connection exists — can suggest analogous lines of inquiry that reveal characteristics about the new object of study. In this case, extending the analogy with exponentiation suggested how to conceive of $f^0(x)$ and $f^{-1}(x)$. Here, the

[10] However, confusion may arise if traditional exponentiation is understood strictly in terms of multiplication (raising the question: what is the product of two functions?). Once the operation is understood as the *repetition* or *iteration* of the multiplication procedure, the link becomes clear and the new notation is easily intelligible.

[11] It may come to the reader's attention that denoting a function by '$f$' contradicts the convention that Babbage established of using the Greek alphabet to denote functions. Maybe Babbage thought the mnemonic aid offered by this representation outweighed other considerations, or perhaps he simply failed to abide by his own guidelines.

guideline of analogy produces a powerful kind of understanding; the notation expands our knowledge of the mathematical entity it represents beyond the limits of what was held when the notation was first introduced.

Hence, Babbage believes it is important to leverage these analogies where they exist. In fact, a striking passage in Babbage's autobiography describes a sort of cross-pollination of good notation; he explains that, in his labeling scheme for mechanical parts (described in more detail in Section 5.3), a distinction between upright letters for pieces of framing and italicized letters for movable parts inspired the application of an analogous rule in analysis: "Let all letters indicating operations or modifications be expressed by *upright* letters; Whilst all letters representing quantity should be represented by *inclined* letters" (Babbage 1864, 106). More generally, Babbage writes: "Whenever I am thus perplexed it has often occurred to me that the very simple plan I have adopted in my *Mechanical Notation* for lettering drawings might be adopted in analysis" (Babbage 1864, 105–106). This demonstrates that Babbage was even open to drawing analogies between notations across different subject matters.

Yet another application of the guideline of analogy relates specifically to inverse operations. The idea is to avoid introducing a new sign for representing the inverse of an operation, and instead to modify the existing sign in some way. Babbage contemplated different formulations of this maxim, but ultimately decided on: "when the operation is an inverse one the sign implying it shall be the direct sign in an inverted position" (Babbage 1820, 150–151).[12] This is in accord with the use of common signs, such as those denoting the relations of greater than and less than, '$>$' and '$<$'. Though symmetric signs, for example addition, '$+$', do not lend to this rule, Babbage stresses that "in framing any sign which admits of an inverse one[,] some attention should be bestowed on its form if it is at all likely to have general use" (Babbage 1820, 150).

Adopting analogous symbols for inverses limits the proliferation of mathematical signs, and thus saves the reader from having to memorize additional ones. Concomitantly, it makes explicit the link between the operation and its inverse. As with the previous guidelines, denoting inverses in this way has the advantage of freeing the mind from trivial burdens, such that its full force can be applied to reasoning about an expression.

In summary, devising notations with the guideline of analogy in mind can serve several functions: reducing the total number of mathematical signs, making expressions easier to remember and process, and opening new avenues for exploration by suggesting characteristics of, and relations between, the objects represented.

[12] Although the encyclopedia article on notation, which includes a different rule (namely: "Whenever we wish to denote the inverse of any operation, we must use the same characteristic with the index $-1$"), was published in 1830, the article was originally included in an unpublished collection of *Essays on the Philosophy of Analysis*, written around 1820; these are stored in the British Museum Manuscripts Room as MS 37202. A small two-page discussion of notation in the manuscript indicates that the revision to the published rule for inverses was intended as a footnote. It is not clear why this edit did not make it into the final edition of the article.

# 5 Guidelines for complex expressions

## 5.1 Modularity

The guideline of modularity is formulated by Babbage in terms of the 'separability of parts' as follows: "*all notation should be so contrived as to have its parts capable of being employed separately*" (Babbage 1830, 418). Its aim is to maintain a one-to-one correspondence between the parts of an expression and the parts of the property or operation that is represented, thereby guaranteeing that sub-expressions are themselves meaningful and can be manipulated independently.[13] Babbage notes that this guideline is frequently adhered to because of the incremental nature of mathematical innovation. He explains:

> [With] this progress [i. e., mathematical invention] proceeding from the simple to the more complicated, [the inquirer's] notation would naturally increase by continued additions. Such being its origin, it will necessarily follow, that at any stage it might be used without reference to those additions with which subsequent considerations had obliged him to augment it. (Babbage 1830, 418)

Babbage still considers it worthwhile to state the guideline explicitly, so that it can be appealed to in future discussions.

The practical advantage of modularity is that it allows for the different parts of an expression to be understood and manipulated in isolation, which is generally easier because they are simpler. As Babbage notes:

> It is this power of separating the difficulties of a question which gives peculiar force to analytical investigations, and by which the most complicated expressions are reduced to laws and comparative simplicity. (Babbage 1827, 377)

Alternatively, in certain cases, it can provide the researcher with an additional element of freedom in their investigations. For example, Babbage remarks that:

> Arbogast [ . . . ] by a peculiarly elegant mode of separating the symbols of operation from those of quantity, and operating *upon them* as upon analytical symbols [ . . . derives] general theorems with unparalleled conciseness. (Babbage 1813, 48)

Indeed, the difficulty of separating Newton's dot-operators for differential calculus from their respective quantities, and reasoning with them independently is, in part, why Babbage favoured Leibniz's notation (as mentioned in Section 2.1).

## 5.2 Generality

Of the greatest benefits of mathematical representations, for Babbage, is that they enable one to reason with general expressions:

[13] This guideline is the only one quoted in (Grattan-Guinness 1992, 39). However, it is discussed, somewhat misleadingly, using the examples of different variations of the sine-squared function, which Babbage himself adduces to illustrate the need of avoiding ambiguities (see Section 3, above).

> The power which language gives us of generalizing our reasonings concerning individuals by the aid of general terms, is nowhere more eminent than in the mathematical sciences, nor is it carried to so great an extent in any other part of human knowledge. (Babbage 1827, 378)

We refer to this ability to express a number of distinct cases using a common formula as 'generality'. The guideline of generality emphasizes the utility of leaving certain quantities and operations indeterminate. This allows "one single investigation [to supersede] the necessity of a multitude", which simplifies the scope of the problem and reduces the mathematician's workload (Babbage 1827, 386).

For example, in calculating possible outcomes of a given betting scheme, Babbage denotes profit by $u(-1)^a$, in which $a$ represents an even whole number if the bet was successful and an odd one if it was unsuccessful (Babbage 1827, 382). By "rendering the events indeterminate," Babbage need not "consider separately all [ . . . ] cases, and [ . . . ] repeat the same or nearly the same reasoning for each individual case," but can formulate a simpler, general solution that can be adapted to a particular situation as necessary (Babbage 1827, 382). More generally, Babbage notes that:

> The utility of the unknown quantities in algebra [e. g., *x, y*], arises from their capability of being operated on without reference to the determined values for which they are placed, the advantage of employing letters for the known quantities [e. g., *a, b*], consists in [ . . . ] the consequent extension of their reasoning from an individual case to a numerous species. (Babbage 1827, 379)

Crucially, generality allows for this extension without introducing ambiguity. In fact, Babbage observes that general expressions may impart more definite meanings:

> When letters are used to the exclusion of number, the relations are not merely more apparent, but the results, although attained with difficulty, are more worthy of confidence: the reason of which, is to be found in this circumstance, that when letters only are employed, the functional characteristics convey no meaning except that on which the force of the reasoning depends; but, if numbers are used, they convey, besides this signification, a multitude of others, which distract the attention, although they are quite insignificant in producing the result. (Babbage 1827, 384)

On this description, generality also helps by removing unnecessary particulars and thereby emphasizing the most important elements of a mathematical problem.

Another sense in which an expression can be formulated in a general way is in leaving a trace of the operations to be performed instead of simply recording their result. Babbage writes that, for example:

> The indication of the extraction of roots by means of an appropriate sign, instead of actually performing the operation, is one of the circumstances which add generality to the conclusions of Algebra. (Babbage 1827, 381)

In this case, multiple distinct cases (e. g., the positive and negative roots of a square) can be represented by a general formula. Importantly, by revealing part of the mathematical process to the reader, this formulation can kindle a better understanding of the result. Babbage notes that knowing precisely how the different parts of a solution are derived can help illuminate the meaning of the solution as a whole:

> [This] principle of indicating operations, instead of executing them, when employed with judgement, contributes frequently in no small degree to the perspicuity of the result, and sometimes enables us to read in the conclusion every stage which has been passed through it in the progress towards it. (Babbage 1827, 381)

Although a tension arises here with the general guideline of conciseness discussed earlier (Section 3), Babbage warns us that whether an expression should be formulated to leave a trace of its operations "ought in a great measure to depend on the objects we have in view" (Babbage 1827, 381); for example, he writes that it would be improper to adhere to this suggestion "when by an opposite course any reduction or contraction can be made in the formula; for example, it would be better to write

$$y = \sqrt{(a-x)^2 + b^2}, \qquad \text{than} \qquad y = \sqrt{(a+x)^2 - 4ax + b^2}.$$

## 5.3 Symmetry of symbols

The guidelines of mnemonics and iconicity (Sections 4.2 and 4.3) encourage the use of individual symbols that evoke their meanings, but the underlying principle can be generalized further to the use of fonts, capitalization, etc., in such a way that similarities between symbols express similarities with regard to their meanings. Babbage introduces this guideline as one notion of symmetry (Babbage 1827, 395), which we shall refer to as 'symmetry of symbols'. (His other notion of symmetry will be discussed in the next section.)

According to the symmetry of symbols, we should incorporate a "resemblance between the systems of characters assumed to represent the data of a question" (Babbage 1827, 395). Using similar representations for similar objects makes the relation between them explicit. To illustrate this point, Babbage asks us to consider four notational variants for representing two straight lines (Babbage 1827, 398):

$$\begin{array}{cccc} y = ax + b & y = ax + \alpha & y = ax + b & y = ax + b \\ y = a'x + b' & y = bx + \beta & y = \alpha x + \beta & y = cx + d \\ (1) & (2) & (3) & (4) \end{array}$$

For Babbage, the first method of representation is well-adapted in that, "*a*, under all its forms, represents the tangent of an angle, and that *b*, in every form, always represents a particular ordinate"; and these two classes of things "hav[ing] no relation to each other [...] are therefore justly represented by dissimilar signs" (Babbage 1827, 398). Thus, the similarity between $a$ and $a'$ indicates that what they represent in both equations is similar, and the dissimilarity between $a$ and $b$ indicates that they represent different types of elements. The second case, in which the line and the angle are represented by the same letter (albeit in different alphabets), "will infallibly suggest some idea of a relation that does not exist" (Babbage 1827, 398). The third variant shares the benefit of the first, but is limited by the number of different alphabets we have at our disposal, because a new one must be used for each additional linear equation. The last method is poorly adapted because the names of the letters offer no indication of the relations that exist among the elements they represent.

As the expressions become more complex, these advantages and disadvantages are compounded. For example, when we seek ordinates for the points of intersection, we will have (Babbage 1827, 398):

$$y = \frac{ab' - a'b}{a - a'} \qquad y = \frac{a\beta - b\alpha}{a - b} \qquad y = \frac{a\beta - \alpha b}{a - \alpha} \qquad y = \frac{ad - cb}{a - c}$$

(1) (2) (3) (4)

With the first and third variants, "we can see at a glance, however numerous the lines introduced, to what property of them each individual letter refers" (Babbage 1827, 399). So, no matter how complex the system of equations becomes, these methods can shorten the time needed to process the information by avoiding the need to retain the meaning of each individual sign. In contrast, the disadvantage of the fourth variant is even more apparent, since "we must, in order to discover the meaning of any letter, refer back for each individual one, to the original translation into algebraic language" (Babbage 1827, 399). The equation in the second column displays some additional symmetry, but the use of the same letter (in different alphabets) with different meanings may suggest incorrect associations.

Babbage took care to incorporate a symmetry of symbols when devising a labeling scheme for his Mechanical Notation. He remarks that the letters chosen to designate the parts of a machine had "hitherto [...been] chosen without any principle, and in fact gave no indication of anything except the mere spot on the paper on which they were written" (Babbage 1864, 107). To rectify this poor practice, he introduced a labeling system to represent a machine's structure. In Babbage's time, machine components were categorized as *pieces of framing* (movable or fixed) or *movable parts* (axes, springs, etc.) — both of which contained *working points* (specific points either acting on or being acted on by other pieces). At the most basic level, his labeling system is based on the following rules (Babbage 1864, 107): "Upright letters (such as a, b, c, A, B, C) for pieces of framing, italicized letters (such as *a, b, c, A, B, C*) for movable parts, and lowercase letters for working points." Additional conventions are given to denote the relative order and level of parts by means of different alphabets and accented lettering (Babbage 1826, 210).

Babbage's labeling scheme uses systems of characters that reveal information about and relations among the data in question. These conventions "enable the attention to be more easily confined to the immediate object sought" and make it easier to understand mechanical diagrams at a glance (Babbage 1864, 107).

## 5.4 Symmetry of structure

The guideline of symmetry of structure "applies to the position, as well as the choice of letters, employed in an enquiry" (Babbage 1827, 407). It thus builds upon the guideline of the symmetry of symbols and involves reformulating an expression such that its structure more clearly suggests to the reader the order or meaning of the operations to be performed.

As an example, Babbage discusses formulas to express an angle of a triangle in terms of the radii of three circles. Given the radii $a, b, c$ and the angle $\theta$ opposite $a$, we have the equations (Babbage 1827, 408):

$$\cot\frac{\theta}{2} = \sqrt{\frac{b}{a} + \frac{c}{a} + \frac{bc}{a^2}} \qquad \text{and} \qquad \cot\frac{\theta}{2} = \frac{\sqrt{ab + ac + bc}}{a}.$$

In the first, the denominators in the fractions under the root sign are unequal, which Babbage deems unsymmetrical. Reformulating the equation as shown on the right has the effect that "the numerator is instantly perceived to be the square root of the sum of the products of the radii, two by two" (Babbage 1827, 408). Thus, the latter formulation discloses a similarity of situation which was concealed in the prior expression of the formula. Moreover, through this disclosure, the process by which the result is derived is made more intelligible. Although the calculations required by both compositions are of roughly the same complexity, the more symmetric composition affords a quicker understanding of the expression.[14]

Babbage aptly summarizes the compounding benefits of adhering to the symmetry of both symbols and structure:

> By employing the first species of symmetry, we assist the memory in remembering the ideas indicated by signs; by the use of the second, we enable it more easily to retain the form in which our investigation has arranged those signs, as well as facilitate the processes by which that final arrangement was accomplished. By the happy union of the two, our formulae acquire the wonderful property of conveying to the mind, almost at a glance, the most complicated relations of quantity, exciting a succession of ideas, with rapidity and accuracy, which would baffle the powers of the most copious language. (Babbage 1827, 407–408)

# 6 Conclusion

For Babbage, the foremost aim of mathematical notation is to succinctly condense meaning; good mathematical notation can convey meaning directly, efficiently, and unambiguously. This view about the power of notation arose from his work in mathematics and on computing machines. In this paper, we have collected, presented, and discussed Babbage's guidelines for notation: *conciseness*, *simplicity*, *univocity*, *mnemonics*, *iconicity*, *analogy*, *modularity*, *generality*, *symmetry of symbols*, and *symmetry of structure*. *Conciseness* and *simplicity* are very general aims for notation, ensuring that notations remain clear and intelligible and that new signs are only introduced when necessary. *Univocity* calls for unique and unambiguous meanings for individual symbols; while *mnemonics* and *iconicity* prioritize designing symbols that provide an indication of their meanings via their shape; *analogy* stresses the value of extending existing symbols to other, related domains. *Modularity* emphasizes the robust and simplifying power of contriving signs which can be considered separately within a complex expression; *generality* encourages the use of formulations that unite many distinct

[14]Babbage also illustrates the guideline of symmetry of structure with his use of different kinds of parentheses within an expression (Babbage 1830, 423).

cases and disclose steps in the solution process; finally, *symmetry of symbols* and *symmetry of structure* highlight the importance of leveraging common resemblances across both the features and structures of mathematical expressions.

There certainly exist tensions between these guidelines. Nevertheless, taken together, they spin a malleable web that informs and supports choices of notation. Good symbolism concretizes the way an expression should be interpreted and thus facilitates the immediate working of a problem. But a well-formed notation also exhibits creative power: it can foster discovery, open new avenues for exploration, and suggest novel properties of the objects represented. While one might find these guidelines too trivial to mention, Babbage was adamant that disregarding them could hinder progress in science and mathematics by leading to unnecessary confusion. He warns in his autobiography:

> Unless some philosophical principles are generally admitted as the basis of all notation, there appears a great probability of introducing the confusion of Babel into the most accurate of all languages. (Babbage 1864, 105)

With his reflections on notation, Babbage laid the groundwork for the discussion and adoption of such principles. These reflections and his notations were noted by a few. For example, De Morgan wrote: "With the exception of an article by Mr. Babbage, in the Edinburgh Encyclopædia, we do not know of anything written in modern times on notation in general" (De Morgan 1842, 443); Lardner quipped: "What algebra is to arithmetic, [Babbage's mechanical notation] is to mechanism" (Lardner 1834, 212); and Dodge commented in his eulogy of Babbage that his notation "is regarded by many eminent engineers as the most wonderful and useful discovery the great inventor ever made" (Dodge 1873, 28). Nevertheless, we must regrettably agree with Grattan-Guinness's assessment of the fate of Babbage's concern with notations, namely that it "failed to raise the interest it deserved" (Grattan-Guinness 1992, 39) — at least, so far.

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# Highlights

- First systematic presentation of Babbage's reflections on how to design good notations
- Ten guidelines for good notations
- Illustration of Babbage's design principles using examples from mathematics and his Mechanical Notation

# Contents