Learning and understanding numeral systems: Semantic aspects of number representations from an educational perspective

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1 Introduction

1.1 Mathematical concepts and notation

In recent years philosophers of mathematics have begun to show greater interest in the activities involved in doing mathematics.[1] This turn to mathematical practice is motivated in part by the belief that an understanding of what mathematicians *do* will lead to a better understanding of what mathematics *is*. One obvious activity that mathematicians engage in is that of writing and manipulating meaningful symbols like numerals, formulas, and diagrams. These notational systems are used to represent abstract concepts and objects, and by operating with their symbolic representations we can learn about the their properties. Since such notational systems are crucial ingredients of mathematical practice, a better understanding of such systems and the way we handle them also contributes to a more encompassing understanding of mathematics.[2]

Natural numbers are among the most fundamental mathematical objects. In the history of mankind different linguistic systems for their representation have been invented, used, and forgotten. Most readers will have some familiarity with the system of Roman numerals, which was widely used throughout the Roman empire, but was replaced in the period between 1200 and 1500 CE by a decimal place-value system using what have come to be known as Hindu-Arabic numerals.[3] The exact reasons for this transition are still largely in the dark, although popular accounts of this development speculate frequently about certain deficiencies of the Roman numeral

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[1]See, e.g., the collections (Ferreirós and Gray, 2006) and (Van Kerkhove, 2009).

[2]The influence of numeral systems on the performance of mental numerical tasks has been studied in (Zhang and Norman, 1995); see also (Campbell and Epp, 2005, p. 350).

[3]See (De Cruz et al., 2010, fn. 14) for a brief discussion of this nomenclature.

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system.[4] However, as Schlimm and Neth (2008) have shown, the Roman system has no limitations in principle with regard to the basic arithmetic operations. In the present paper we discuss a different aspect of numeral systems other than that of their aptness for algorithmic computations, namely how different systems of numerals embody semantic information about the represented concepts.

In contrast to natural languages, notational systems are introduced for particular, often very specific purposes. Numerals, for example, are used to represent arbitrary numerosities (collections of one or more objects) and to compute efficiently. In general, being able to use a notational system effectively involves (a) understanding the relation between the symbolism and the represented concepts, and (b) knowing the correct rules for manipulating the notation. For an experienced user of a notational system these two go hand in hand, but for somebody who is starting to get acquainted with such a system, the situation can be very different. In the process of mastering a notation, being able to manipulate the notation correctly can lead to a better grasp of the represented concepts, while at the same time, a good understanding of the concepts can help to differentiate between correct and incorrect usage of the notation. Conversely, persistent incorrect usage of the notation reveals a fundamental lack of understanding of the relationship between the symbols and their meanings.

This latter discrepancy forms the basis of our analysis of two different numeral systems. In particular, we use systematic errors, as opposed to random errors due to carelessness or inattentiveness,[5] in computations by children who learn the decimal place-value system as indicators of a lack of full understanding of the relation between numerals and numbers. Such faulty procedures are also referred to as 'bugs' by Brown and Burton (1978). Here, the operations are carried out mainly as purely syntactic manipulations, without the student having a proper understanding of the semantics of the notation. In his developmental studies, Hughes also found 'serious limitations in children's understanding of arithmetical symbols' (Hughes, 1986, p. 111). By analyzing the kinds of mistakes that are made most frequently we show what semantic information is often not being taken into consideration by the student, and we conclude that the notational system does not straightforwardly convey this particular kind of information.

1.2 Semantic content

A few words on the *semantic content* of systems of numerals are in order at this point. The Arabic digit '5', the Roman numeral 'V', and the English

[4]See, e.g., (Menninger, 1992, p. 294) and (Ifrah, 1985, p. 431).

[5]We also count wrongly memorized basic addition and multiplication facts as such 'random errors.' Although they might be systematic for the individual they are not related to the structural features of the numeral systems.

word 'five' denote a particular natural number, namely 5. The first two symbols are constituents of a numerical notation system, i.e., a structured system of representation for numbers.[6] If these symbols are considered in isolation, the connection to their referents, the number 5, is by no means obvious. These symbols are *ciphers*: more or less arbitrarily chosen marks that are intended to represent a particular numerosity.[7] Numerals, however, do not usually come in isolation, but as parts of a system. The structure of such a system of numerals allows for the determination of the value of a numeral, i.e., the number it denotes, from knowledge about the basic symbols and the structural principles that are used to connect them. For example, in our decimal place-value system, the first occurrence of the symbol '5' in the numeral '505' denotes 500, while the second occurrence represents 5, and the number represented by '505' is obtained by adding 500 and 5. The situation is different, however, with the Roman numeral 'XX'.[8] Here, both occurrences of 'X' stand for the value 10, and the referent of the entire numeral is obtained by adding 10 and 10, resulting in 20.

In general, the symbols in a place-value system have a base value, but the value of a symbol within a specific numeral also depends on its position. We shall refer to the base value as the *explicit meaning* of the symbol, and to the value that it represents within a numeral as its *implicit meaning*.[9] Grasping the implicit meanings of the symbols is a difficult task for children, and educators have devised many *semantic tools* to make these meanings more explicit, and thus easier to understand. In an additive numeral system, like the Roman one, each symbol has a fixed meaning, regardless of where the symbol occurs in a numeral. In such a system less information is encoded implicitly, making it more concrete than our decimal place-value one. As we shall argue below, this difference has a great impact on the ease with which the place-value and the Roman numeral systems are learned.

1.3 Learning arithmetic in the decimal place-value system

In the development of children's arithmetic competency several phases can be distinguished, which have been studied extensively by developmental psychologists and cognitive scientists.[10] Usually at the age of five years[11] children learn a sequence of number words that are associated with small numerosities. While they might be able to recite the number words and

[6] This terminology is based on (Chrisomalis, 2004, p. 38).

[7] On the importance of cipherization in numeral systems, see (Boyer, 1944, p. 154).

[8] For more information about place-value systems and additive systems, like that of the Romans, see (De Cruz et al., 2010), this volume. For a brief introduction to the Roman numeral system, see Section 2.3, below.

[9] This implicit meaning can be understood as hidden information about the syntactic notation which enables us to interpret it correctly.

[10] See, e.g., (De Cruz et al., 2010), this volume, for an overview.

[11] See (Hasemann, 2007, p. 9).

write the corresponding numerals, they are initially unaware of the deeper recursive structure of the natural numbers and of their place-value representations. Instead, they develop different conceptions of numbers, which are connected with the symbolic representations of the numerals. At this stage, ‘21’ is read as if it were a single symbol that represents a collection of 21 elements, just as ‘2’ is a symbol for collections of two elements, but no structural relation between the occurrences of the ‘2’ is perceived. An understanding of the internal structure of the place-value representation is achieved gradually and goes hand in hand with the learning of strategies for computing with the basic arithmetic operations.

To illustrate the difficulties that a student encounters when learning the place-value system, let us briefly look at Padberg’s suggestion to teach the representation of two-digit numerals in five different ways (Padberg, 2005, p. 65). First, numbers like 23, are to be described as ‘2 tens and 3 units’. Second, using a table the place-values should be visualized as being distinct; e.g., $\begin{array}{c|c} \mathrm{T} & \mathrm{U} \\ \hline 2 & 3 \end{array}$. Third, the value of the tens is to be determined and the number written as a sum: $20 + 3$. Fourth, writing the corresponding number words (‘twenty-three’) is practiced. Finally, fifth, the numeral ‘23’ is introduced. Indeed, Padberg emphasizes that all of these representations are necessary for a successful introduction of the place-value notation and a full understanding of its workings. They are intended to separate the numerals, which are first conceived as single entities, into their constitutive parts. The steps also provide a careful explanation and visualization of how these parts contribute to the value of the numeral.

Also as part of the process, the number words are decomposed and the structural similarities between the number words and the numerals are highlighted. The more regular the number words are in a language, the easier the student can understand this relationship. Chinese and Japanese number words correspond exactly to the symbols in the place-value system (e.g., eleven and twelve being literally translated as ‘ten-one’ and ‘ten-two’), and research has shown that they are learned faster than the number words in English (Miller et al., 1995). German and Dutch number words introduce an additional level of difficulty, since they inverse the positions of the tens and units; e.g., the German word for 23 is ‘*dreiundzwanzig*’ (‘*drieëntwintig*’ in Dutch, literally: three-and-twenty). As a consequence, by following the order in the number word, German and Dutch students are initially misled into writing 23 as ‘32’, which is also the source of frequent errors in written calculations.[12]

[12]This source of error is so common, that there even is a term in German for getting two numbers the wrong way round (‘*Zahlendreher*’).

As we shall see, other common sources of mistakes made by learners concern the proper handling of zeros, empty places as well as places in general, in particular when operations like 'carrying' and 'borrowing' require the movement of digits from one place to another. Students often take refuge in totally syntactical execution of algorithms often resulting in notational problems and further mistakes.

1.4 Overview

In the next three sections we look at the basic arithmetic operations of addition, subtraction, and multiplication. For each of these we first describe the main difficulties that students reveal by making systematic errors in their computations in the decimal place-value system. Then, we discuss semantic tools that have been proposed in the didactical literature and are used in schoolbooks to develop a better semantic understanding of the basic written algorithms for these operations. Finally, we present how the computations could be carried out with Roman numerals, which is a purely additive system of numerals.

Section 2, on addition, will contain the most detailed presentation and discussion, while the other two will deal mainly with matters that haven't been covered before. For this reason, we also omit a discussion of division, since the description of even the most basic common algorithms would be quite cumbersome without bringing about any essentially new features.

2 Addition

2.1 Examples of difficulties with addition in the decimal place-value system

In the following we present eleven examples of common systematic errors that are made by students learning written addition with the decimal place-value system. Ten are taken from the book *Error patterns in computation: Using error patterns to improve instruction* by Robert Ashlock (1998),[13] and one example (A5) is from (Padberg, 2005, p. 99). These mistakes are the result of strategies that elementary school children have developed, often by abstracting from a small set of examples and over-generalizing. In other cases the students stumbled upon a situation in which they did not know how to continue and fixed the problem in an ad hoc manner, for example by reusing a symbol from the ones column when the tens column was empty. Thus, they have adopted a 'repair' strategy (VanLehn, 1983). Note that these strategies do not always lead to incorrect results, which explains how it is possible that the students have actually learned them: in those cases in

[13] We have used different labels for the exercises than Ashlock. We list here how our labels correspond to those of Ashlock: A8: AW1; A3: AW2; A6: AW3; A7: AW4; A1: P4; A4: P5; A10: P6; A2: P7; A9: P8; A11: P9.

which an incorrect strategy led to a correct result the student was reinforced in believing that the strategy was indeed correct.

We now analyze the types of frequent systematic errors made when performing addition and discuss some suggestions that have been put forward in the literature to help the students correct their incorrect grasp of the semantics of the operation (see Figures 1 and 2).

A1:	A2:	A3:	A4:	A5:
	9	^{32}		
43	8	618	48	27
+26	+ 7	+782	+37	+11
15	105	1112	75	36

FIGURE 1. Examples of common addition errors: Columns, direction, operation.

2.1.1 Difficulties with columns in general (A1, A2)

In example A1 the student determines the sum of all digits occurring in the problem, regardless of their place-value. The strategy pursued in A2 is more difficult to detect. One possibility is that the top two numbers are perceived as a single two-digit number to which the third number is added (i.e., $98 + 7$ is computed). An alternative strategy to obtain the same result would be first to add the lower two numbers and record the units digit in the result, and then add the carry to the top number.

Students who use the strategy A1 have clearly not understood the essence of the place-value notation. The single digits in a numeral like '43' are considered to be on par with each other. As a consequence, to add two numerals amounts to adding the values of each of their digits. We see clearly in this example that it is possible repeatedly to apply the single digit addition facts (possibly, by just 'counting on'), such that $4 + 3 + 2 + 6$ yields 15, and to know that 15 is represented by '15', without being aware of the internal structure of the place-value notation for numbers. The situation in A2 is similar, but almost in the opposite direction. Here two separate one-digit numerals are read as a single two-digit numeral, despite the fact that the digits are written one underneath the other, instead of one next to the other.

2.1.2 Difficulties with direction (A3, A4)

The next two examples illustrate difficulties with the direction in which the columns are dealt with. In A3 the digits are added column-wise, but from left to right, the tens are recorded, while the units are 'carried' to the next column on the right. Example A4 is similar, but the student simply ignores the tens digit.

Both strategies A3 and A4 indicate a good grasp of single-digit addition as well as some understanding of the importance of column-wise addition, but also betray a lack of understanding of the semantics of the numerals. The algorithms used are purely syntactic manipulations.

A related source of errors when two small numbers are added, according to Padberg (2005), is the inversion of two-digit numbers. For example, if the result of $54 + 4$ is given as '85'. In this case, the student has calculated the result of 58 correctly, but recorded the solution in the wrong way. This is an example of a '*Zahlendreher*' (see footnote 12), which is a quite frequent mistake made by German and Dutch students.

2.1.3 Difficulties with operation (A5)

In addition to the mistake just mentioned, a second typical mistake that Padberg points out reveals general difficulties with the operation of addition for multi-digit numerals. In A5 the tens are added correctly, but then the smaller unit is subtracted from, instead of added to, the larger unit.

$$
\begin{array}{cccccc}
\text{A6:} & \text{A7:} & \text{A8:} & \text{A9:} & \text{A10:} & \text{A11:} \\
 & & & & {\scriptstyle 11} & \\
 & {\scriptstyle 1} & & & 457 & {}^{\prime\prime} \\
26 & 98 & 8\ 8 & 3\ 5\ 9 & 368 & 775 \\
+\ 3 & +\ \ 3 & +3\ 9 & +\ \ 5\ 6 & +192 & +\ 483 \\
\hline
11 & 131 & 1117 & 81115 & 927 & 2158
\end{array}
$$

FIGURE 2. Examples of common addition errors: Empty places and carries.

2.1.4 Difficulties with empty places in columns (A6, A7)

The error shown in A6 is made by a student who can add two two-digit numbers correctly, but is baffled if one place in a column is empty. In this situation, all digits occurring in the problem are simply added (like in A1). A different way of coping when confronted with an empty place is to look for the 'next best' number and add that one. A calculation that results from this strategy is shown in A7, where the units are added correctly, but when the tens are added, the unit (3) — with an empty place in the tens column — is added again (together with a correct carry).

These two examples illustrate how empty places can be confusing; the students have learned how to manipulate the symbols correctly, but are puzzled about what to do if there is no digit that they can operate on. To resolve the impasse, they change the problem task or take the 'next best' digit, i.e., the one that is in a position close to the empty place. Although the students who employ these strategies would not produce errors in calculations that do not contain empty places, their behavior reveals that they have not yet fully understood the semantics of the numerals.

2.1.5 Difficulties with carries (A8, A9, A10, A11)

The most frequent errors deal with the handling of carries. According to Ashlock (1998, p. 101), these mistakes amount to 67% of all errors.[14]

In A8 the digits are added column-wise, but the carries are ignored and written in the row for the solution instead. In other words, the intermediate sums are written out directly in full. A similar behavior is the cause of the erroneous calculation in A9. Here, however, the carry of the units column is used for the tens column, but the empty place in the hundreds column causes the student to reuse the second value of the tens column (like in A7) and to disregard the carry. In the next two examples the carries are used at a later stage of the calculation, but incorrectly. In A10 the student always records the greater digit as the result and uses the lesser digit as the carry to the next column (except for the final, left-most column). In example A11 all carries are collected above the left-most column and are then recorded in the thousands place.

2.2 Semantic tools for addition algorithms

The mistakes discussed in the previous section, made by students who learn to compute with the decimal place-value system, are caused mainly by a lack of understanding of the semantics of numerals, and in particular by a lack of understanding of the implicit meanings of the digits. We now present some semantic tools that have been devised to help students to overcome these difficulties. These tools are intended to link the written, formal algorithms with a conceptual understanding of the numerals. In other words, they are intended to build bridges between the learners' conceptions of numbers and the mathematical content of numerical notation. We discuss how various schoolbooks and the relevant didactical literature propose to cope with this semantic gap. In particular we look at two schoolbooks for elementary schools (Becherer and Schulz, 2007; Böttinger et al., 2008) as well as the illustrations from the popular books on mathematics education by Padberg (2005) and Wittmann and Müller (2005).

The aim of the semantic tools discussed below is to make the implicit meanings of the symbols explicit. This is done by offering a different format for representing the values of the digits that occur in the numerals. In particular, this new format is chosen in such a way that it shows the grouping structure of the notation, i.e., the bundling of units into tens, of tens into hundreds, etc., in a visual and compelling fashion.[15]

[14] See also (Cox, 1975).

[15] See also the pictographic and iconic examples in (Hughes, 1986, p. 123).

2.2.1 Graphically representing the grouping of numbers

Figure 3 demonstrates various ways of displaying, in decreasing levels of concreteness, the semantic content of numerals in simple additions. In the left-hand diagram the values of the digits are represented by *number portraits*, which use boxes of different sizes to show their magnitudes. Here, the repeated occurrences of the basic elements ‘■’, ‘▌’, and ‘▪’ function as icons (Pierce, in Hartshorne and Weiss, 1932, p. 247) In addition, analogous three-dimensional representations have been developed to be used in the classroom. These ‘Dienes Multi-base Arithmetic Blocks,’ which mirror the decimal grouping structure of our numeral system, include small cubes, ten of which are grouped into sticks, and 100 of which fit into a box.[16]

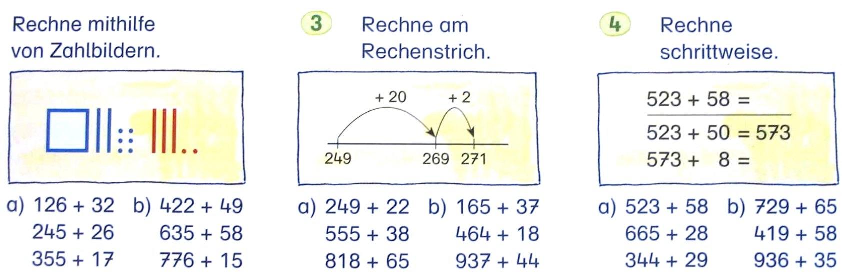

FIGURE 3. Number portraits to support understanding. The headings above the pictures mean from left to right: ‘Calculate with number portraits’, ‘Calculate with the number line’, and ‘Calculate stepwise’. Source: (Becherer and Schulz, 2007, p. 34).

Calculations using number portraits like in Figure 3 (left-hand side) are exactly the same procedures as in the Roman system, the only difference is the bundling. Roman numbers have a five and two bundling, our decimal place-value system has only a ten bundling. The middle diagram in Figure 3 appeals to an ordinal understanding of numbers represented as points on the number line. Finally, in the diagram on the right-hand side no additional semantic information is provided, but the values of the single digits are represented directly. The sequence of these three diagrams, from left to right, also shows how semantic information is reduced in the process of formalization.

2.2.2 Splitting up numerals

The splitting up of the numerals is also shown in the place-value table in Figure 4, where the place-values are indexed by their grouping (H =

[16]See (Davey, 1975).

'Hunderter' = hundreds, Z = 'Zehner' = tens, E = 'Einer' = ones). In addition, the student can also see the relation to the iconic number portraits. Again, this explication of the implicit meanings of the numerals is intended to help students understand the formal addition algorithm.

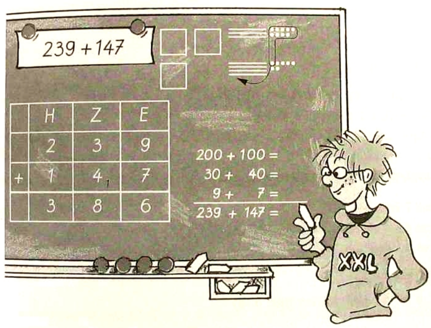

FIGURE 4. Visualization of the place values as a semantic tool. Source: (Padberg, 2005, p. 210).

2.2.3 Providing meaningful context

An alternative kind of semantic tool embeds the computations into a context with familiar objects that have a natural grouping structure, like money in different denominations or fruit that comes in units, crates, or containers. Figure 5 shows the combination of a place-value table with money-substitutes intended to provide a bridge between the student's everyday reasoning in a meaningful contexts and formal operations with numerals.

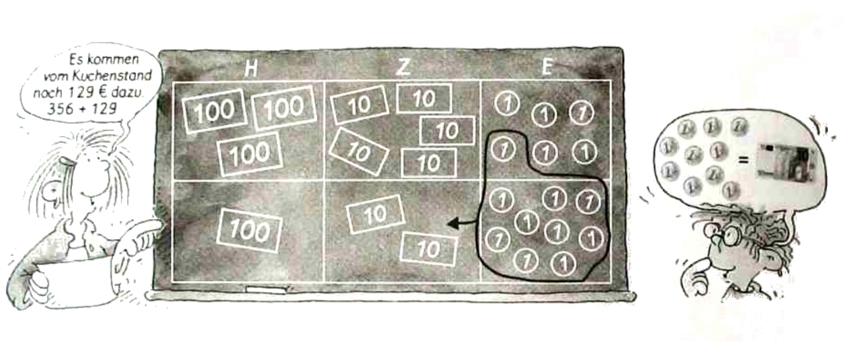

FIGURE 5. Money — a meaningful context for addition. Source: (Becherer and Schulz, 2007, p. 34).

2.3 Addition with Roman numerals

We now turn to an alternative notational system for numbers, namely the Roman numeral system, which is purely additive. Similar systems have been used in Egypt, Greece, and many other parts of the world.[17] Its basic constituents are the symbols I, V, X, L, C, D, and M, which represent the values 1, 5, 10, 50, 100, 500, and 1000, respectively. A numeral is written as a sequence of basic symbols and its value is obtained by adding the individual values of the symbols. For example, the numeral 'MDCCLXXI' stands for $1000 + 500 + 100 + 100 + 50 + 10 + 10 + 1 = 1771$.

The reader might be familiar with the contemporary conventions of writing the Roman numerals IV for 4, IX for 9, XL for 40, XC for 90, etc., but these were in fact only introduced during the Middle Ages, mainly to abbreviate the numerals on inscriptions. Since these subtractive conventions violate the semantics of a purely additive system, and since they were apparently not used by the Romans themselves, we do not consider them here to be part of the Roman numeral system under discussion.[18]

In an additive system, addition can be accomplished by simply writing all of the symbols together and then performing simplifications, i.e., replacing a group of symbols by a single symbol. For example, five 'I's are replaced by a 'V', two 'V's by an 'X', etc. Let us consider the problem A8 in Roman numerals, using a very crude strategy according to which all operations are executed explicitly:

$$\begin{array}{rrl} & \text{LXXXVIII} & \\ + & \text{XXXVIIII} & \\ \hline & \underbrace{\text{L}\underbrace{\text{XXXXX}}_{\text{L}}}_{\text{C}}\text{X}\underbrace{\text{VV}}_{\text{X}}\underbrace{\text{IIIII}}_{\text{V}}\text{II} & \begin{array}{l}\text{(copy addends)} \\ \text{(simplifications)}\end{array} \\ \hline & \text{CXXVII} & \end{array}$$

While the simplification process may look messy if intermediate steps are written out, it is accomplished by very simple rules of the form 'if you have five I's, replace them by a V,' etc., which reflect the grouping structure of the Roman numerals. As we have seen in Section 2.2, making such groupings explicit is in fact used in semantic tools to help students understand the relation between numerals and their values. Thus, they are regarded as somewhat 'intuitive' in the pedagogical literature, and we can assume that they are learned easily.

The addition algorithm for Roman numerals can be considerably simplified if the numerals are not written in a linear fashion, but in a more

[17]See (Ifrah, 1985).

[18]See (Cajori, 1928, pp. 30–37) for a discussion of the history of Roman numerals and the subtractive conventions.

structured way, as they would be organized on an abacus. The above problem would then be represented as:

$$
\begin{array}{cccccccl}
 & & \mathrm{L} & \mathrm{XXX} & \mathrm{V} & \mathrm{III} & & \\
+ & & & \mathrm{XXX} & \mathrm{V} & \mathrm{IIII} & & \\
\cline{2-6}
 & & \mathrm{L} & \mathrm{X} & \mathrm{V} & & & \text{(`carries')} \\
\cline{2-6}
 & \mathrm{C} & & \mathrm{XX} & \mathrm{V} & \mathrm{II} & & \text{(simplification)}
\end{array}
$$

Here the simplifications are done while the single occurrences of a symbol are processed and the intermediate results have been written down as 'carries.' Notice how the structure of an abacus, where symbols of the same numeric value are written in a column, resembles the place-value table shown in Figure 4, which is used as a tool for conveying the semantics of the Arabic numerals. We see here that the Roman numeral system reflects more closely the operations on an abacus than our decimal place-value system and thus seems to be more apt to convey the semantics of the numerals.

In our analysis of the common systematic mistakes that are made by students who learn the decimal place-value system (Section 2.1) we have identified five classes of difficulties: with the use of columns in general, with the direction of the algorithm, with the operation, with empty places in a column, and with carries. How would these kinds of difficulties, which can lead to mistakes in calculations in a place-value system, fare with an additive system? Since the Roman numeral system is purely additive, there is no need to keep track of columns; all numerals of the same shape have to be treated together, regardless of where they are positioned. A1's strategy to simply add the explicit values of all symbols occurring in the addends would work perfectly well with the Roman system.

Also the direction in which the single-digit additions are performed plays only a minor role in an additive system. Adding from left to right could possibly lead to a few forgotten simplifications, but they could be detected by inspecting the end result. Transferring carries from left to right, as done by A3, would also be an erroneous strategy in the Roman system. However, it seems unlikely that it would occur to a student who has learned the rule 'two Vs yield one X' to replace two or five 'X's by a 'V.' (This mistake would correspond to the use of a wrong addition fact.) In a place-value system the digits in each column and the carries have the same shape. Thus, one must understand the semantics of the numerals to judge whether a digit is being dealt with correctly. In an additive system, however, the shape alone of a symbol conveys its numerical value so that the semantic content of a symbol does not have to be extrapolated from the position of the symbol within a numeral.

In additive systems there is no symbol to mark an empty place. Thus difficulties that arise in the decimal place-value system in connection with the zero have no analog in the Roman system. A related difficulty discussed above results from an inability correctly to handle empty positions. This difficulty seems to arise because the students have learned to process each digit in a column and they are thrown off course if there is no digit in a certain row of the column they are working on. To solve this problem, they use a digit from a neighboring column. In the Roman system, even if the symbols are arranged in columns as in the previous algorithm, each column contains only symbols of the same shape. The following calculation is analogous to the one performed by A6:

$$\begin{array}{rccr} & \mathrm{XX} & \mathrm{V} & \mathrm{I} \\ + & & & \mathrm{III} \\ \hline & \mathrm{XX} & \mathrm{V} & \mathrm{IIII} \end{array}$$

In other words, which symbols are to be processed together is determined not only by the columns in which they appear, but also by the shapes of the symbols. In fact, the shape is the only relevant indicator; the columns are merely additional but dispensable aids. To 'borrow' a numeral from a different column would thus violate the rule that symbols of the same shape have to be processed together, and thus is an unlikely strategy for a student to adopt.

Failure to use the carries in the place-value addition algorithm would correspond in the Roman system to either (a) not applying the simplification rules, (b) applying them, but without recording the results of the simplification, or (c) applying them and using the results in an incorrect way. Since the simplification rules are most basic in dealing with Roman numerals, they are akin to the single-digit additions in the decimal place-value system. But the students who have learned the incorrect algorithms discussed above have managed to learn the single-digit additions, so we might assume that they would also have been able to learn the simplification rules, especially since there is only one rule to learn for each different symbol, while there are 10 single-digit addition facts to be mastered for each of the 10 digits in the decimal place-value system. Applying the simplification rule without recording the outcome (case b) can also result in an incorrect strategy for an additive system. However, since all symbols of the same shape are written in the same column, there is little room for ambiguities regarding where to write the result, i.e., even if the operations are understood purely syntactically, the additive system promises to be easier to learn and to apply.

2.4 Intermediate conclusion

Let us briefly summarize our findings so far. We have presented various kinds of systematic errors that students make when learning written additions in the decimal place-value system that reveal a lack of proper understanding of the semantic content of the numerals. As a consequence, the additions are performed as purely syntactic operations, which can easily lead to undetected mistakes. To overcome these problems, mathematics educators have developed semantic tools designed to convey the implicit meanings of the numerals. Our investigation of addition with Roman numerals has led to two observations. First, the Roman numeral system embodies already some of the principles that are used in the semantic tools discussed above, since the grouping structure is explicit in the symbolism (compare: ten 'I's yield an 'X', *versus* ten '1's yield a '1' in the column to the left, followed by a '0' in the current column). Because of this, the Roman numerals are often considered to be less abstract and thus easier to learn than our place-value numerals. Second, we have seen that the erroneous strategies that are mistakenly adopted by students using the decimal place-value system would either yield correct results when used with an additive numeral system, or would be less likely to be adopted.[19]

3 Subtraction

3.1 Examples of difficulties with subtraction in the decimal place-value system

The following subtraction example, given by Spiegel and Selter (2003, p. 24), shows how little semantic understanding can go into formal computations performed by children. When Malte is asked to calculate 701 − 698 he computes the result via the formal subtraction algorithm, which yields:

$$\begin{array}{rr} & 701 \\ - & 698 \\ \hline & 197 \end{array}$$

The interviewer asks: 'Do you know another possibility of computing?' Malte answers: 'From 698 to 700 are 2 and from 701 to 700 it is 1, therefore it's 3.' The interviewer inquires about the correct answer, and Malte decides to accept the result of the written computation, explaining: '197. This I have calculated and 3 was only hopp di hopp generated by thinking.'

This little episode shows how fundamental the lack of semantic understanding can be. The dissociation of written computations from an otherwise present understanding of numbers is described as 'a large gap between the children's concrete numerical understanding and their use of formal written symbolism' (Hughes, 1986, p. 95).[20] This is also supported by more

[19]It would be very interesting to have empirical studies on the use of additive numeral systems. Schlimm and Neth (2008) have shown differences in the complexity of the arithmetic operations and we imagine that these differences would also have some effect on the occurrences of computational errors. Unfortunately, however, we are not aware of any empirical studies of these issues.

[20]See in particular the discussion in (Hughes, 1986, pp. 95–133).

recent investigations concerning 'street mathematics,' which show that children are able to calculate correctly when the context is outside of school mathematics (Nunes et al., 1993).

As in the discussion of addition, let us now review briefly the systematic mistakes that are made frequently by elementary-school students, according to Ashlock (1998) (see Figures 6 and 7).[21]

S1:	S2:	S3:	S4:
$\begin{array}{r} 241 \\ -\ 96 \\ \hline 255 \end{array}$	$\begin{array}{r} 52 \\ -27 \\ \hline 30 \end{array}$	$\begin{array}{r} 47 \\ -\ 3 \\ \hline 14 \end{array}$	$\begin{array}{r} 446 \\ -302 \\ \hline 104 \end{array}$

FIGURE 6. Examples of common subtraction errors: Subtraction of smaller from larger, empty places, zero.

3.1.1 Difficulties with what to subtract from what (S1, S2)

In contrast to addition, which is commutative, the order in which subtractions are performed is crucial. In other words, switching the minuend and the subtrahend yields a different result, which is not the case when addends are interchanged. This peculiarity of subtraction comes with the fact that it is much easier to subtract a smaller number from a larger one than the other way around. After all, in the realm of concrete objects, we can take away two apples from seven apples, but not seven from two. This can lead students to adopt the rule 'always subtract the smaller digit from the larger one,' if they do column-wise written subtractions.[22] The boy in the initial example did exactly this in his calculation, and S1 presents another example.[23]

An alternative solution to the difficulty of subtracting a larger number from a smaller one is to adopt the result from the concrete example. Given two apples, taking away any number greater or equal than two leaves no apples at all, i.e., zero apples. This way of thinking seems to lie behind the reasoning exhibited in S2, which appears to follow the rule 'If a larger digit is subtracted from a smaller one, the result is zero.'

[21] We have renamed the labels that are used by Ashlock (1998). Here are the correspondences: S1: SW1; S6: SW2; S4: SW3; S7: SW4; S3: P10; S9: P11; S2: P12; S10: P13; S8: P14; S5: P15.

[22] See also (Hughes, 1986, p. 121).

[23] This is a mistake that could also occur within the Roman numeral system. But, it seems unlikely, since in this case it would be much more natural to adopt the rule 'subtract only equal symbols.'

3.1.2 Difficulties with empty places in columns (S3)

Like in the case of addition, the occurrence of an empty place can put the student into a situation where the most rudimentary algorithm fails and a repair strategy has to be invoked (see A7). In S3, when the place of the minuend in the tens column is empty, the student's way out is to reuse the minuend from the ones and subtract it again.

3.1.3 Difficulties with zero (S2, S4, S8)

The idea of adding zero appears to be more easily understood than subtracting zero.[24] In the case of S4, if a zero appears in the subtrahend then the result is also written as zero. Thus, it seems that the zero is not understood as representing the null quantity, or 'nothing,' in a purely formal fashion. Otherwise, taking away 'nothing' should not yield a change in the numerosity being denoted by the digit in the minuend.

$$
\begin{array}{r} \text{S5:} \\ {}^{4\,5} \\ \cancel{3}\cancel{6}3 \\ -341 \\ \hline 112 \end{array}
\qquad
\begin{array}{r} \text{S6:} \\ {}^{7}\phantom{{}^15} \\ 2\cancel{8}{}^{1}5 \\ -\ 6\,3 \\ \hline 2112 \end{array}
\qquad
\begin{array}{r} \text{S7:} \\ {}^{4}\phantom{{}^12{}^15} \\ \cancel{6}{}^{1}2{}^{1}5 \\ -3\,4\ \,8 \\ \hline 1\,8\ \,7 \end{array}
\qquad
\begin{array}{r} \text{S8:} \\ \\ 602 \\ -238 \\ \hline 274 \end{array}
\qquad
\begin{array}{r} \text{S9:} \\ \\ 437 \\ -\ 84 \\ \hline 453 \end{array}
\qquad
\begin{array}{r} \text{S10:} \\ {}^{2}\phantom{{}^16} \\ 43{}^{1}6 \\ -21\,8 \\ \hline 24\,8 \end{array}
$$

FIGURE 7. Examples of common subtraction errors: Direction, borrows.

3.1.4 Difficulties with direction (S5)

Also in written subtraction, the direction in which the columns are to be processed can be mistakenly reversed (see A3, A4). In S5 for example, the student proceeds from left to right. Moreover, whenever the same digit appears in both minuend and subtrahend, the student borrows a unit from the next column to be able to subtract without getting a result of zero. So, while the formal operation of borrowing is performed correctly (although in a case when it shouldn't be applied and in the wrong direction), the semantics of the numerals is clearly not understood at all.

3.1.5 Difficulties with borrows (S5, S6, S7, S8, S9, S10)

By far the greatest difficulties in written subtraction problems are caused by 'borrowing,' i.e., when the operation cannot be carried out within a single column. This situation is similar to that of addition, where the 'carries' present the greatest problems. In the example S6, the student has learned to take borrows, but has over-generalized this strategy, and now applies it when subtracting the units whether it is needed or not. The borrows are

[24] General difficulties with learning the digit zero are presented in (Wellman and Miller, 1986).

always taken from the left-most digit in S7. This kind of mistake might also be motivated by the presence of a zero. For example, in S8 the student regroups directly from the hundreds to the ones, i.e., all borrows are taken from the hundreds, since there is a zero in the position of the tens. The borrows are taken correctly in S9, but the student does not remember to subtract one ten (or one hundred) when she regroups. Finally, in S10 the student borrows one from the '3' in the tens and writes the result as a crutch on top. But, then he adds the '2' (i.e., the crutch) and the '3' before subtracting.

3.2 Semantic tools for subtraction algorithms

The problems described above show that applying the formal algorithms and understanding the semantics behind them do not always go together. This becomes very obvious if the algorithms involve steps that prima facie go against previously acquired knowledge. For example, from the first grades in elementary school it is learned that there is nothing less than zero, so subtracting a larger number from a smaller one does not make much sense. The strategies applied in S1 and S2 result from the students attempting to integrate the formal algorithm with their existing knowledge of numbers. Here the individual columns are seen in isolation and dealt with independently from each other, which reveals that the semantics of the place-value system hasn't been grasped yet.

The strategies found in the didactics literature for coping with these difficulties are often based on ideas similar to those discussed above in the context of addition. Figure 8 shows a detail from the schoolbook *Duden* in which children are encouraged to discuss their calculating strategies. Again the iconic representation of numerical quantities in terms of dots, sticks, and areas is used, as well as the pictographic representation on the number line.

Additional support for the semantical understanding of the number portrait, in particular of the debundling of the tens into the ones, could be provided by annotations, which are missing in the figure.

In this particular example, students are instructed first to subtract the tens and then add the units. For students who are still struggling with understanding the workings of the place-value system, the direction of the calculation, which starts from the left, could lead to further misunderstandings. The changes of number representations are sometimes also illustrated with an abacus, as can be seen in Figure 9.[25] Here the grouping and ungrouping is made explicit by the number of pebbles on the different lines: two pebbles on the 50-line can be replaced by one pebble on the 100-line,

[25] See also (Hughes, 1986), who advocates the introduction of different number representations to gain a better, semantically richer, understanding of the place-value system.

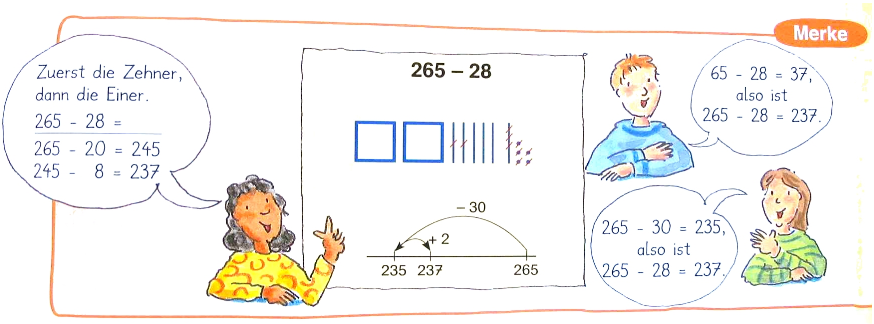

FIGURE 8. Subtraction strategies. Source: (Becherer and Schulz, 2007, p. 37).

and vice versa, without changing the numerical quantity that is being represented. A related approach is shown in (Lengnink, 2006), where students are encouraged to compare different number representations and discuss their benefits.

3.3 Subtraction with Roman numerals

As might be guessed from Figure 9, the grouping and ungrouping of numerals is more explicit in the additive numeral system of the Romans. Subtracting one numeral from another in such a system amounts to deleting all symbols that occur in the subtrahend from the minuend. For example, $38 - 12 = 26$ in the Roman system is computed as follows:

$$\begin{array}{rrrr} & \mathrm{XXX} & \mathrm{V} & \mathrm{III} \\ - & \mathrm{X} & & \mathrm{II} \\ \hline & \mathrm{XX} & \mathrm{V} & \mathrm{I} \end{array}$$

Here it is not essential to the algorithm that the symbols of the same value be written in the same column. In the decimal place-value system, if a digit of the subtrahend is greater than that of the minuend of the same magnitude, one has to 'borrow' a unit from the next magnitude. This process, as we have seen in Section 3.1, is the origin of many of the erroneous strategies developed by students. In the Roman system, if there are not as many different instances of a number symbol in the minuend as there are in the subtrahend, the simplification rules have to be applied in reverse. For example, to subtract II from V, the 'V' of the minuend has to be rewritten as 'IIIII'. Then, two of the 'I's can be deleted, leaving III. A slightly more complicated situation is shown in the following example, in which $16 - 7 = 9$ is calculated.

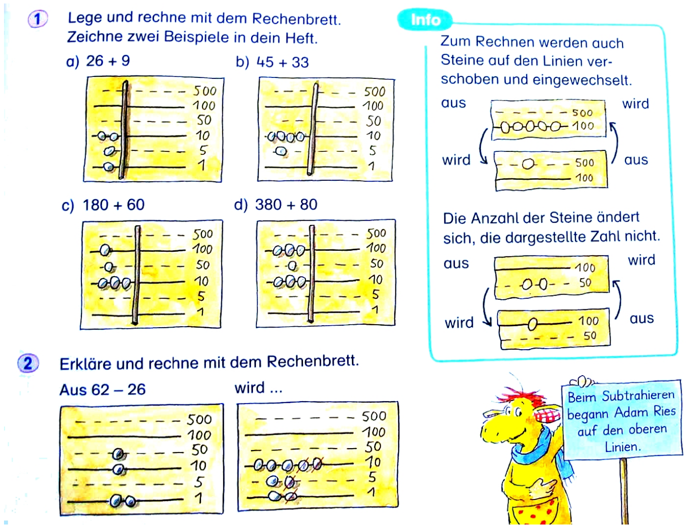

FIGURE 9. Historical exercise. The headings above the pictures mean: 'Place and calculate with the abacus. Draw two examples in your exercise book'. In the info box: 'To calculate, pebbles are also put on the lines and shifted: from ... you get .... The number of pebbles is changed, but not the represented number.' The sign in the bottom right corner reads: 'When subtracting, Adam Ries always began with the top-most lines.' Source: (Becherer and Schulz, 2007, p. 119).

$$\begin{array}{rccc} & \mathrm{X} & \mathrm{V} & \mathrm{I} \\ - & & \mathrm{V} & \mathrm{II} \\ \hline \end{array} \Longrightarrow \begin{array}{rccc} & \mathrm{X} & & \mathrm{IIIII} \\ - & & \mathrm{V} & \mathrm{II} \\ \hline \end{array} \Longrightarrow \begin{array}{rcc} & \mathrm{VV} & \mathrm{IIIII} \\ - & \mathrm{V} & \mathrm{II} \\ \hline & \mathrm{V} & \mathrm{IIII} \end{array}$$

Thus, the 'borrowing' in the additive system simply amounts to an application of the simplification rules in reverse. If a student has understood that an X stands for two V's, this should not create as many confusions as the borrowing in the place-value system does. However, the potential danger of subtracting the smaller value from the larger one within a column (i.e., subtracting 'I' from 'II' in the example above) remains.

Figure 10 shows a sample subtraction with Roman numerals that involves regrouping in which the single steps are presented explicitly. To avoid the impression that a representation of the numerals in columns is necessary, the numerals are written linearly. The single steps consist in either deleting the same amount of occurrences of the same symbol from the minuend and subtrahend, or in unbundling a symbol into symbols representing a smaller value. It should be noted that the order in which the symbols are being

	241	–	96	
1	$\mathrm{CCXXXX}\underline{\mathrm{I}}$	–	$\mathrm{LXXXXV}\underline{\mathrm{I}}$	Problem
2	CCXXXX	–	LXXXXV	Crossed out one I
3	$\mathrm{CCXXX}\overline{\mathrm{V}\underline{\mathrm{V}}}$	–	$\mathrm{LXXXX}\underline{\mathrm{V}}$	Unbundle one X into two Vs
4	$\mathrm{CC}\underline{\mathrm{XXX}}\mathrm{V}$	–	$\mathrm{LX}\underline{\mathrm{XXX}}$	Crossed out one V
5	CCV	–	LX	Crossed out three Xs, one still to be done
6	$\mathrm{C}\overline{\mathrm{L}\underline{\mathrm{L}}}\mathrm{V}$	–	$\underline{\mathrm{L}}\mathrm{X}$	Unbundle one C into two Ls
7	CLV	–	X	Crossed out one L
8	$\mathrm{C}\overline{\mathrm{XXXX}\underline{\mathrm{X}}}\mathrm{V}$	–	$\underline{\mathrm{X}}$	Unbundle one L into five Xs
9	CXXXXV			Crossed out one X, nothing left to subtract
	145			

FIGURE 10. Example of subtraction with Roman numerals (S1). The symbols with a line above (e.g., $\overline{\mathrm{VV}}$) are the result of unbundling; symbols that are underlined (e.g., $\underline{\mathrm{L}}$) are to be crossed out in the next step.

crossed out does not matter at all in this algorithm. This can be seen in the example, where an L-symbol (standing for 50) has been crossed out in line 7 before the final X-symbol (with value 10) is deleted. Alternatively, the X could have been deleted before the L, after the unbundling operation (line 8). In this particular problem, it might have been more efficient to delete four Xs in the first step, to obtain numerals that consisted of fewer letters.

The algorithm shown in Figure 10 is much closer to the intuitive understanding of subtraction as 'taking away.' The rules to be followed are fairly simple: (a) always delete the same symbols from the minuend and subtrahend, (b) if this is not possible, unbundle a symbol in the minuend to obtain the symbols that need to be deleted, according to the grouping rules for each symbol, which are just the reverse rules of those employed in addition (e.g., X→ VV and V→ IIIII). Since here no single-letter digits are subtracted from others, there is no danger that a student might adopt a rule 'always subtract the smaller from the larger number,' as was done in S1 and S2. Also, we have seen that the particular order in which the operations are carried out in the Roman subtraction algorithm does not alter the final result, so that no difficulties with direction, as those exhibited in S5, can arise. As a consequence of the absence of any symbol for zero or of empty places in columns, problems that lead to the erroneous strategies shown in S2, S3, S4, and S8 do not arise. Finally, the handling of 'borrows,' the main source of difficulties when learning subtraction in the decimal place-value system, is reduced to simple applications of the grouping rules.

It is difficult to know whether students who learn subtraction with Roman numerals would make different kinds of mistakes or would also show behavior that reveals gaps between the syntactic manipulation of symbols and the semantic understanding of the numerals. However, our discussion has shown that the common systematic mistakes that students make when learning subtraction with the decimal place-value system would most likely not be made with a purely additive numeral system, and, moreover, that the strategies that are used in the literature to help students understand the place-value system are already embodied in the Roman system of numerals. Thus, from a purely educational perspective, it seems very plausible that the Roman numeral system would be easier to learn and that it would not generate such a huge gap between formal, written computations and the numerical meanings of the notation.

4 Multiplication

4.1 Examples of difficulties with multiplication in the decimal place-value system

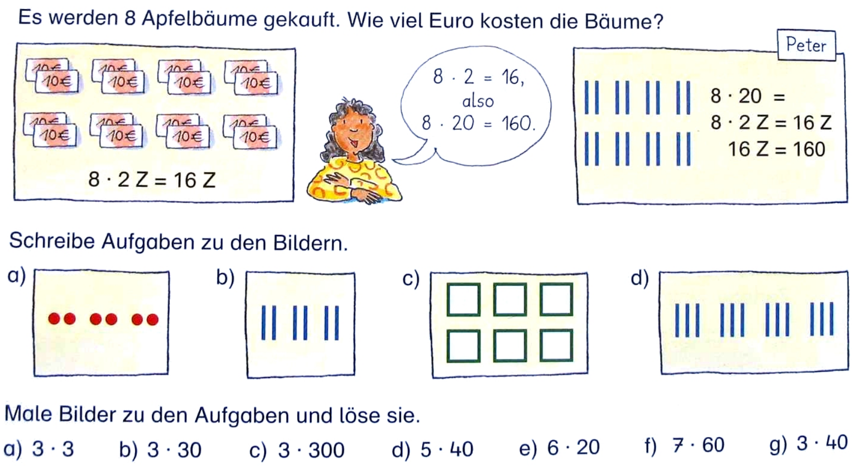

FIGURE 11. Pragmatic and semantic background of multiplication. The headings above the pictures mean: 'Eight apple-trees are bought. How many Euros do they cost?' and below 'Write down problems for the pictures' and 'Draw pictures for the following problems and solve them.' Source: (Becherer and Schulz, 2007, p. 56).

Different versions of the written multiplication algorithm are taught in different countries, sometimes even in different schools within a country, and the algorithms are also motivated in different ways. As an example, Figure 11 shows how multiplication in the special case with multiples of 10 is introduced in a German textbook for the third grade. The pictographic and iconic representations are intended to provide a bridge between the formal algorithm and reasoning with more concrete objects. However, such descriptions are often forgotten, even in the process of learning the algorithms. Consequently, students frequently learn to manipulate the numerals without gaining a deeper understanding of what they are doing. This leads to the typical mistakes discussed in (Ashlock, 1998), which are presented next (see Figures 12 and 13).[26]

4.1.1 Only column-wise operations (like for addition) (M1, M2)

One of the systematic mistakes that some students exhibit can be traced back to a generalization from the algorithms for addition and subtraction, which proceed by processing each column separately (with the occasional violations of this principle in the form of carries and borrows). In M1, for example, the student approaches each column as a single multiplication problem, with carries as commonly used in addition. In this particular case, a further complication arises because of the empty place in a column (see also A7), which is repaired by continuing to use the left-most digit of the second factor. Column-wise multiplication is also performed in M2, but here, if the resulting product has two digits, only the tens figure is recorded.

4.1.2 Difficulties with placement of intermediate results (M3)

In the usual multiplication algorithms the partial products that arise as intermediate results must be written in a fixed format to guarantee the correct result. If the student does not understand why this format is necessary, it will be easy for him to violate it. For example, in M3, the intermediate results are simply placed one underneath the other.

4.1.3 Difficulties with zero (M4, M5)

We have seen earlier that the digit that creates the most difficulties is zero. In M4, if a zero occurs in one of the factors it is inserted into the result, before regrouping is performed. Zero also creates problems if it occurs in an intermediate result. For example, in M5, the student correctly moves over one place, but incorrectly also writes down a zero.

4.1.4 Difficulties with operating with the crutch figure or carries (M9, M8, M10)

As we have seen in the discussion of addition and subtraction, if an operation requires the transgression of a column-boundary this can lead to major confusions for students who have not yet grasped the inner workings of the

[26]We have renamed the labels that are used by Ashlock in the following way: M9: MW1; M8: MW2; M1: MW3; M3: P16; M6: P17; M10: P18; M7: P19; M2: P20; M4: P21; M5: P22.

$$
\begin{array}{ccccc}
\text{M1:} & \text{M2:} & \text{M3:} & \text{M4:} & \text{M5:} \\
\begin{array}{rr} & {\scriptstyle 1} \\ & 524 \\ \times & 34 \\ \hline & 1576 \end{array} &
\begin{array}{rr} & 837 \\ \times & 294 \\ \hline & 122 \end{array} &
\begin{array}{rr} & 56 \\ \times & 32 \\ \hline & 112 \\ & 168 \\ \hline & 280 \end{array} &
\begin{array}{rr} & 5402 \\ \times & 6 \\ \hline & 32502 \end{array} &
\begin{array}{rr} & 57 \\ \times & 34 \\ \hline & 228 \\ & 1710 \\ \hline & 17328 \end{array}
\end{array}
$$

FIGURE 12. Examples of common multiplication errors: Column-wise operations, placement of intermediate results, zero.

place-value system. A few bugs that arise with multiplication out of such a situation are presented next.

The student in M6 has over-generalized the fact that often, and in particular in introductory examples, only the digit one has to be carried into the next column, such that she always adds one, regardless of the correct amount. Thus, after computing $6 \times 8 = 48$, the ‘8’ is recorded for the units, but a ‘1’ is carried into the tens, instead of a ‘4,’ and similarly for the hundreds. M7 also shows difficulties with keeping track of the numbers to be carried. The sequence of computations in this example is as follows: $6 \times 2 = 12$, write ‘2’ in the ones column and remember the 1; $6 \times 3 = 18$, plus the remembered 1 is 19, write ‘1’ and remember the 9; finally, $6 \times 1 = 6$, plus the remembered 9 is 15, which is written down, yielding the final result of ‘1512.’[27] The next students use ‘crutches’, or recorded intermediate results, as a memory-aid, but when performing the computations they use these crutches incorrectly. For example, M8 adds the number recorded as crutch before multiplying the tens figure, instead of adding it to the result after the multiplication. The next student shows how a symbol written down during the computation can be misleading at later stages. M9 correctly calculates the result of the multiplication with the ones, but then uses the crutch recorded when multiplying the ones again when multiplying the tens. A similar mistake is made by M10, who also computes the multiplication with the ones without problems, but uses both crutch figures when multiplying with the tens digit (i.e., she adds $1+3$ to the result of 3×6, though only the 1 should be added, the ‘3’ belonging to the multiplication with the ones).

4.2 Semantic tools for multiplication algorithms

As mentioned above, the different algorithms employed for multiplication may differ in different locations where the decimal place-value system is

[27] This is not a one-time miscalculation, but a systematic error that is made by this student.

M6:	M7:	M8:	M9:	M10:
$\begin{array}{r} 368 \\ \times\ \ 6 \\ \hline 1978 \end{array}$	$\begin{array}{r} 132 \\ \times\ \ 6 \\ \hline 1512 \end{array}$	$\begin{array}{r} {\scriptstyle 2}\ \ \\ 27 \\ \times\ \ 4 \\ \hline 168 \end{array}$	$\begin{array}{r} {\scriptstyle 2}\ \ \\ 46 \\ \times\ 24 \\ \hline 184 \\ 102\ \ \\ \hline 1204\ \ \end{array}$	$\begin{array}{r} {}^{1}{}^{3}65 \\ \times\ 37 \\ \hline 455 \\ 225\ \ \\ \hline 2705\ \ \end{array}$

FIGURE 13. Examples of common multiplication errors: Crutch figures, carries.

taught. However, the difficulties noted by Ashlock are fairly general. Consult, e.g., (Padberg, 2005) for a discussion of similar systematic errors made by German students. The common multiplication procedures are quite elaborate and depend crucially on writing the intermediate results in the correct positions. A student who does not know why the particular positions are used, i.e., who has not fully understood their meanings, can easily adopt erroneous strategies without being aware of it. Particular difficulties are posed by the zero, which is often not properly recognized as a place value or which is erroneously interpreted as one, yielding $a \times 0 = a$.

An alternative algorithm, developed with the intention of being closer to the semantics of the symbols but without sacrificing any computational efficiencies, is discussed in (Wittmann and Müller, 2005, Vol. 2, p. 135); empirical studies have shown that this algorithm helps to lower the error rate.[28] Their representation, shown in Figure 14, can also be supplemented by additional semantic information (the 'H,' 'Z,' and 'E', standing for hundreds, tens, and ones) during early learning stages, which can be discarded later. From the notation alone, the student can see how multiplying a digit in the tens place with one in the ones place gives a numeral that occupies the tens (and possibly hundreds) place in the result.[29] Moreover, the carries, or crutches, are recorded in determinate positions, so that misuses as those exhibited in M9, M8, and M10 are rendered impossible. Due to the rigid format, it seems that the notorious difficulties with empty places and zeros are also reduced.

4.3 Multiplication with Roman numerals

Although multiplication with Roman numerals has often been characterized as exceptionally difficult or even impossible,[30] an algorithm for multipli-

[28] See also (Padberg and Thiemann, 2002).
[29] Compare this representation with the examples in Figure 11.
[30] See, e.g., the very popular (Menninger, 1992, p. 294) and (Ifrah, 1985, p. 431).

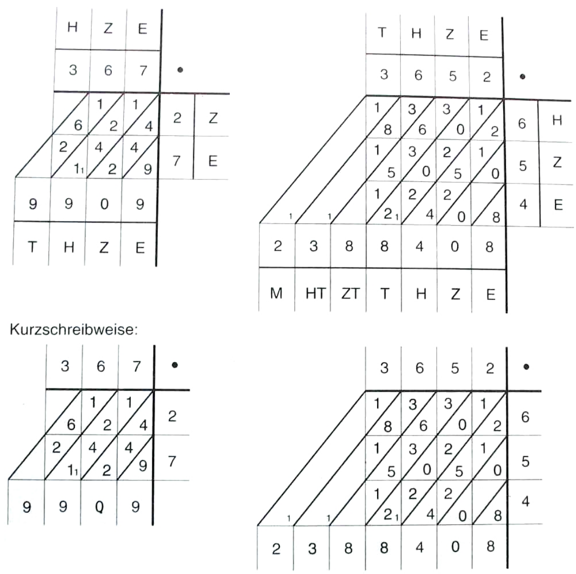

FIGURE 14. Linking formal multiplication algorithms with semantic information. Source: (Wittmann and Müller, 2005, p. 135).

cation in the Roman numeral system that is very similar to the familiar one used for the decimal place-value system is presented and discussed in (Schlimm and Neth, 2008). The general method is to multiply each digit of one numeral with each digit of the other and then to add up the intermediate results. This is exactly how the common multiplication algorithms for place-value systems proceed, too.[31] However, in the case of the place-value system one has to make sure to assign the correct magnitude to these intermediate results, i.e., to write them exactly in the right place. Indeed, the elaborate positioning-schemes of common algorithms as well as that of (Wittmann and Müller, 2005) are designed to achieve exactly this. As we have seen above, the resulting formal complexities can be perceived as being dissociated from any underlying semantics, and this is the origin of many systematic mistakes. With the Roman system, however, the magnitude of the value is inherent in the symbols (e.g., X×V = L and C×V = D), so that no special attention has to be devoted to this matter. Also single-letter operations that result in numerals ranging over more than one magnitude are handled in this way (e.g., V×V = XXV). In other words, the intermediate results can be gathered in any way and then simply added together. The

[31]See the examples in Figures 12, 13, and 14.

student only has to learn the multiplication table to be able to multiply.[32]

Here is an example of the computation of $106 \times 18 = 1908$ in Roman numerals in a format similar to that discussed in (Wittmann and Müller, 2005):

1. Compute intermediate results by single-letter multiplications:

×	X	V	I	I	I
C	M	D	C	C	C
V	L	XXV	V	V	V
I	X	V	I	I	I

2. Gather all intermediate results: MDCCCLXXXVVVVVIII.

3. Simplification yields the final result: MDCCCCVIII.

The table in item 1. above is filled using the single-letter multiplication facts that have to be learned by heart from a multiplication table. Since Roman numerals are on average longer than their decimal place-value counterparts, more intermediate results occur in general. Gathering these and simplifying the resulting numeral is done according to the standard procedure of addition with Roman numerals (see Section 2.3).

We see immediately from this procedure that no difficulties like those of column-wise multiplication arise, since there are no columns to be processed and the format of the representation differs substantially enough from that of addition to avoid any confusions. Nor should the positioning of the intermediate results pose any particular difficulties. Moreover, even the order in which the numerals are written on top and on the side of the table does not effect the result of the computation at all. In other words, switching columns or rows does not influence the final result. Further, since there are no zeros or empty places, confusions relating to these cannot arise. Finally, there are no crutches or carries to be dealt with separately, so that this source of common systematic errors with the decimal place-value system is not present in multiplication with Roman numerals.

In fact, the procedure for multiplication with Roman numerals is very similar to the semantic tool employed by (Wittmann and Müller, 2005) to help students learn multiplication with the decimal place-value system. The difficulties with carries, which do not arise in the Roman multiplication, cannot be completely avoided even in the representation of Wittmann and Müller. When the values in the diagonals in Figure 14 are added, it is possible that carries have to be processed to transgress column (or, in this case, diagonal) boundaries.

[32] See (Schlimm and Neth, 2008, p. 2010) for the multiplication table with Roman numerals.

5 Conclusion

In the decimal place-value system the meaning of a single symbol depends on the shape of the symbol *and* on the position of the symbol in the numeral. As long as the relation between these two components has not been properly understood by the student, the algorithms for the basic arithmetic operations appear to be arbitrary manipulations that, almost magically, lead to the correct results. In fact, as we have seen above, failure to grasp the semantic content of the notation leads many students to abstract erroneous procedures when learning the algorithms for addition, subtraction, and multiplication. The students do not realize that these algorithms are incorrect, because they haven't grasped the full semantics of place-value numerals.

By analyzing the algorithms for arithmetic operations for an additive numeral system, the Roman numerals, we have shown that many of the mistakes that students make when learning the decimal place-value system would not occur if they were to calculate with the additive system. The underlying reason for this observation is the distinctive feature of additive systems, namely that each symbol encodes a determinate value. Thus, the connection between a symbol and its semantic content is stronger, which facilitates learning the correct manipulation strategies for the notation. An interesting issue that is raised by our discussion concerns the relation between syntactic simplicity and semantic understanding. The stronger connection between syntax and semantics of numerals in an additive system goes hand in hand with algorithms that require simpler, although more frequent, manipulations. This syntactic simplicity might turn out to be an important factor for more error-free computations with the Roman system. After all, the complexity of the decimal place-value algorithms stems from the fact that they have to make sure the places are dealt with correctly.

The conclusion of our analysis is three-fold. On the one hand, we have suggested a new possible criterion for assessing different systems of numerals in terms of their semantic content, which is crucial for learning the symbolic manipulations for arithmetic operations and for a proper understanding of the numerals. In the discussions of different numeral systems, this pedagogical dimension of numeral systems has been largely overlooked. On the other hand, our discussion of the systematic errors that are frequently made by students who learn the decimal place-value system and the comparison of how such erroneous strategies would play out in an additive numeral system can be used to augment the repertoire of semantic tools used by mathematics teachers. This further supports the views of various educators who have emphasized the importance of teaching children different representation systems for numerals.[33] Finally, we have argued that notation can influence the learning of mathematics, and thus, indirectly, also mathematical practice. Consequently, philosophers interested in the latter should also be concerned about the nature of the language of mathematics, i.e., notational systems like numerals, formulas, and diagrams.

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[33] E.g., (Hughes, 1986; Lengnink, 2006).

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