The bideterministic concatenation product

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Abstract

This paper is devoted to the study of the bideterministic concatenation product, a variant of the concatenation product. We give an algebraic characterization of the varieties of languages closed under this product. More precisely, let $\mathbf{V}$ be a variety of monoids, $\mathcal{V}$ the corresponding variety of languages and $\hat{\mathcal{V}}$ the smallest variety containing $\mathcal{V}$ and the bideterministic products of two languages of $\mathcal{V}$. We give an algebraic description of the variety of monoids $\widehat{\mathbf{V}}$ corresponding to $\hat{\mathcal{V}}$. For instance, we compute $\widehat{\mathbf{V}}$ when $\mathbf{V}$ is one of the following varieties : the variety of idempotent and commutative monoids, the variety of monoids which are semilattices of groups of a given variety of groups, the variety of $\mathcal{R}$-trivial and idempotent monoids. In particular, we show that the smallest variety of languages closed under bideterministic product and containing the language $\{1\}$, corresponds to the variety of $\mathcal{J}$-trivial monoids with commuting idempotents. Similar results were known for the other variants of the concatenation product, but the corresponding algebraic operations on varieties of monoids were based on variants of the semidirect product and of the Malcev product. Here the operation $\mathbf{V} \to \widehat{\mathbf{V}}$ makes use of a construction which associates to any finite monoid M an expansion $\widehat{M}$, with the following properties:

(1) M is a quotient of $\widehat{M}$,

(2) the morphism $\pi : \widehat{M} \to M$ induces an isomorphism between the submonoids of $\widehat{M}$ and of M generated by the regular elements and

(3) the inverse image under π of an idempotent of M is a 2-nilpotent semigroup.

This paper assumes some familiarity with Eilenberg's theory of varieties and especially the notion of syntactic monoid of a recognizable language. References for this theory are [6, 7, 10]. The main result of this theory states that there exists a one-to-one correspondence between certain families of recognizable sets (the varieties of languages) and certain families of finite semigroups (the varieties of finite semigroups).

A fundamental result of Schützenberger [13] states that the smallest variety of languages closed under concatenation product corresponds to the variety of aperiodic monoids. Since then, an important part of the existing literature on varieties has been devoted to the study of the concatenation product and its variants. These variants include the weak forms of the concatenation product introduced by Schützenberger [14] (the unambiguous product and the left and right deterministic products) and the counter product introduced by Straubing [15]. This paper is devoted to the study of the two-sided version of the deterministic products, called the *bideterministic product*.

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The general setting for this type of result can be summarized as follows. Let $\circ$ be a binary operation on languages — in our case the concatenation product or one of its variants — and let $\mathcal{V}$ be the variety of languages corresponding to a variety of monoids $\mathbf{V}$. Denote by $\mathcal{V}'$ the smallest variety containing $\mathcal{V}$ and closed under $\circ$. The question is to describe the varieties of monoids $\mathbf{V}'$ corresponding to $\mathcal{V}'$. For all the variants of concatenation mentionned previously, the variety $\mathbf{V}'$ is equal to a Malcev product of the form $\mathbf{W} \circledM \mathbf{V}$, where $\mathbf{W}$ is a certain variety of semigroups [16, 8, 9, 12, 17, 18]. This variety $\mathbf{W}$ is given in the following table:

Product type	Variety $\mathbf{W}$ such that $\mathbf{V}' = \mathbf{W} \circledM \mathbf{V}$
concatenation	aperiodic semigroups
unambiguous	semigroups S such that $eSe = e$ for each idempotent $e \in S$
right deterministic	semigroups S such that $eS = e$ for each idempotent $e \in S$
left deterministic	semigroups S such that $Se = e$ for each idempotent $e \in S$
counter	semigroups which are locally solvable groups

This is no longer true for the bideterministic product: in this case, the variety $\mathbf{V}'$ cannot be written as a Malcev product of some variety with $\mathbf{V}$ and a new algebraic operation is required. This new operation relies on a construction of independent interest, which associates to any monoid M a certain expansion $\widehat{M}$, with the following properties: M is a quotient of $\widehat{M}$ and the morphism $\pi : \widehat{M} \to M$ induces an isomorphism from $\langle \mathrm{Reg}(\widehat{M}) \rangle$, the submonoid of $\widehat{M}$ generated by the regular elements of $\widehat{M}$, onto $\langle \mathrm{Reg}(M) \rangle$. Furthermore,the inverse image under π of an idempotent of M is a 2-nilpotent semigroup. Our construction is somewhat reminiscent of the expansion proposed by Birget, Margolis and Rhodes in [4, 5], but turns out to be different, as we shall see on an example.

Now the key result states that a variety of languages is closed under bideterministic product if and only if the corresponding variety of monoids is closed under this expansion. We also give a more precise version of this result. Let $\mathcal{V}$ be a variety of languages and let $\mathbf{V}$ be the corresponding variety of monoids. Let $\hat{\mathcal{V}}$ be the smallest variety containing $\mathcal{V}$ and the bideterministic products of two languages of $\mathcal{V}$. Then the variety of monoids corresponding to $\hat{\mathcal{V}}$ is the variety of monoids generated by the monoids of the form $\hat{M}$ for some $M \in \mathbf{V}$. Similar results are known for the other variants of products, but again, there are based on totally different algebraic constructions (essentially variants of the semidirect product).

We compute $\widehat{\mathbf{V}}$ for various varities $\mathbf{V}$, including the variety of idempotent and commutative monoids, the variety of monoids which are semilattices of groups of a given variety of groups and the variety of $\mathcal{R}$-trivial and idempotent monoids.

As a byproduct, we characterize the smallest non trivial variety of languages containing the language $\{1\}$ and closed under bideterministic product : the corresponding variety of monoids is the variety of $\mathcal{J}$-trivial monoids whose idempotents commute.

1 Some preliminaries.

In this section, we recall some basic definitions or facts about finite semigroups and languages. All semigroups and monoids considered in this paper are either finite or free, although some results could be easily extended to periodic semigroups.

Let S be a semigroup. We denote by S^1 the semigroup equal to S if S has an identity and to $S \cup \{1\}$, where 1 is a new identity, otherwise. We denote by $\mathrm{E}(S)$ the set of idempotents of S. For each element s of S, the subsemigroup of S generated by s contains a unique idempotent, denoted s^ω. If P is a subset of M, $\langle P \rangle$ denotes the submonoid generated by P.

Given $s, t \in S$, we say that s is $\mathcal{R}$*-below* t (denoted $s \leq_{\mathcal{R}} t$) if there exists $x \in S^1$ such that $s = tx$. The elements s and t are $\mathcal{R}$*-equivalent* (denoted $s \,\mathcal{R}\, t$) if $s \leq_{\mathcal{R}} t$ and $t \leq_{\mathcal{R}} s$. Finally, we denote $s <_{\mathcal{R}} t$ if s is $\mathcal{R}$-below t but is not $\mathcal{R}$-equivalent with t. The relations $\leq_{\mathcal{L}}$, $\mathcal{L}$ and $<_{\mathcal{L}}$ are defined dually. For instance, $s \leq_{\mathcal{L}} t$ if there exists $x \in S^1$ such that $s = xt$.

Let s be a semigroup and let s be an element of S. An element $\bar{s}$ of S is called a *weak inverse* of s if $\bar{s}s\bar{s} = \bar{s}$. It is an *inverse* of s if $\bar{s}s\bar{s} = \bar{s}$ and $s\bar{s}s = s$. In this case, s is an inverse of $\bar{s}$. An element which has an inverse is called *regular*. We denote by $\mathrm{Reg}(S)$ the set of regular elements of a semigroup S. The following propositions state some elementary properties of weak inverses.

Proposition 1.1 *Let $\bar{s}$ be a weak inverse of s. Then $s\bar{s}$ and $\bar{s}s$ are idempotent and $\bar{s}$ is an inverse of $s\bar{s}s$.*

Proof. If $\bar{s}$ is a weak inverse of s, we have $\bar{s}s\bar{s} = \bar{s}$. This implies in particular $s\bar{s}s\bar{s} = s\bar{s}$ and $\bar{s}s\bar{s}s = \bar{s}s$ and thus $s\bar{s}$ and $\bar{s}s$ are idempotent. We also have

$$(s\bar{s}s)\bar{s}(s\bar{s}s) = (s\bar{s})(s\bar{s})(s\bar{s})s = s\bar{s}s \quad \text{and} \quad \bar{s}(s\bar{s}s)\bar{s} = \bar{s}$$

Thus $\bar{s}$ is an inverse of $s\bar{s}s$. $\square$

Proposition 1.2 *Let s and t be elements of a semigroup S such that $s \,\mathcal{R}\, st$ (resp. $ts \,\mathcal{L}\, s$). Then there exists a weak inverse $\bar{t}$ of t such that $st\bar{t} = s$ (resp. $\bar{t}ts = s$).*

Proof. Since $s \,\mathcal{R}\, st$, there exists an element $t' \in S^1$ such that $stt' = s$. Let ω be an integer such that $(tt')^\omega$ is idempotent, and set $\bar{t} = t'(tt')^{2\omega-1}$. Then $st\bar{t} = stt'(tt')^{2\omega-1} = s$. Furthermore $\bar{t}t\bar{t} = t'(tt')^{2\omega-1}tt'(tt')^{2\omega-1} = t'(tt')^{4\omega-1} = t'(tt')^{2\omega-1} = \bar{t}$. Thus $\bar{t}$ is a weak inverse of t. The proof for the $\mathcal{L}$ relation is dual. $\square$

A monoid M *divides* a monoid N if M is a quotient of a submonoid of N. A *variety of finite monoids* is a class of finite monoids closed under taking submonoids, quotients and finite direct products.

Recall that a relational morphism between monoids M and N is a relation $\tau : M \to N$ such that:

(1) $(m\tau)(n\tau) \subset (mn)\tau$ for all $m, n \in M$,

(2) $(m\tau)$ is non-empty for all $m \in M$,

(3) $1 \in 1\tau$

Equivalently, τ is a relation whose graph

$$\mathrm{graph}(\tau) = \{\ (m, n) \mid n \in m\tau\ \}$$

is a submonoid of $M \times N$ that projects onto M.

Let $\mathbf{V}$ and $\mathbf{W}$ be varieties. The *Malcev product* of $\mathbf{V}$ and $\mathbf{W}$ is the variety $\mathbf{V} \circledM \mathbf{W}$ defined as follows

$$\mathbf{V} \circledM \mathbf{W} = \{\ M \mid \text{There is a relational morphism } \tau : M \to N \text{ with } N \in \mathbf{W} \text{ and such that } e\tau^{-1} \in \mathbf{V} \text{ for all idempotents } e \in N\ \}$$

Let A be a finite set, called the *alphabet*, whose elements are *letters*. We denote by A^* the free monoid over A. Elements of A^* are words. In particular, the empty word, denoted by 1, is the identity of A^*. A *language* is a subset of A^*.

Let M be a monoid and L be a language of A^*. A monoid morphism $\varphi : A^* \to M$ *recognizes* a language L if there exists a subset P of M such that $L = \varphi^{-1}(P)$. The *syntactic congruence* of L is the equivalence $\sim_L$ on A^* defined by

$$u \sim_L v \text{ if and only if, for every } x, y \in A^* \quad (xuy \in L \Longleftrightarrow xvy \in L).$$

The quotient $A^*/{\sim_L}$ is the *syntactic monoid* of L and the natural morphism $\eta : A^* \to M(L)$ is called the *syntactic morphism*: it recognizes L and every surjective morphism $\varphi : A^* \to M$ that recognizes L can be factorized through it, that is, there is a surjective morphism $\alpha : M \to M(L)$ such that $\eta = \alpha \circ \varphi$.

For technical reasons, it is more appropriate to use a variant of the concatenation product called the marked product. The results stated in the introduction refer to this product. Given a finite alphabet A and a letter a of A, the *marked product* of two subsets (also called *languages*) L_0 and L_1 of the free monoid A^* is the language

$$L_0aL_1 = \{u \in A^* \mid u = u_0au_1 \text{ for some } u_0 \in L_0 \text{ and } u_1 \in L_1\}$$

Unambiguous, left and right deterministic products were introduced by Schützenberger. A product $L = L_0aL_1$ is *unambiguous* if every word u of L has a unique decomposition of the form $u = u_0au_1$ with $u_0 \in L_0$ and $u_1 \in L_1$. It is *left deterministic* if every word of L has exactly one prefix in L_0a. This means that in order to find the decomposition $u = u_0au_1$ of a word of L, it suffices to read u from left to right: the first prefix of u in L_0a will give u_0a, and thus the decomposition. Dually, a product $L = L_0aL_1$ is *right deterministic* if every word of L has exactly one suffix in aL_1. A product is called *bideterministic* if it is both deterministic and antideterministic. Schützenberger [14] characterized the smallest variety of languages containing the language $\{1\}$ and closed under unambiguous (resp. deterministic, antideterministic) products. Later on, it was shown in [8, 9, 12] that the closure of a variety of languages under unambiguous (resp. left deterministic, right deterministic) product correspond to the Malcev product $\mathbf{V} \to \mathbf{LI} \textcircled{M} \mathbf{V}$ (resp. $\mathbf{V} \to \mathbf{K} \textcircled{M} \mathbf{V}$, $\mathbf{V} \to \mathbf{K^r} \textcircled{M} \mathbf{V}$), where $\mathbf{LI}$, $\mathbf{K}$ and $\mathbf{K}^r$ are respectively the varieties of semigroups S such that, for every idempotent $e \in S$, $eSe = e$, (resp. $eS = e$, $Se = e$).

2 An expansion.

In this section, we give the formal definition of our new expansion, which is related to certain special factorizations of words.

Let M be a monoid, and let $\varphi : A^* \to M$ be a surjective (monoid) morphism. A *good factorization* (with respect to φ) is a triple $(x_0, a, x_1) \in A^* \times A \times A^*$ such that $\varphi(x_0a) <_{\mathcal{R}} \varphi(x_0)$ and $\varphi(ax_1) <_{\mathcal{L}} \varphi(x_1)$. A *good factorization of a word* $x \in A^*$ is a good factorization (x_0, a, x_1) such that $x = x_0ax_1$. Two good factorizations (x_0, a, x_1) and (y_0, b, y_1) are *equivalent* if $\varphi(x_0) = \varphi(y_0)$, $\varphi(x_1) = \varphi(y_1)$ and $a = b$. In particular, this implies $\varphi(x_0ax_1) = \varphi(y_0by_1)$. Here is a first useful lemma.

Lemma 2.1 *Let (x_0, a, x_1) be a good factorization, let x_0' be a right factor of x_0 and let x_1' be a left factor of x_1. Then (x_0', a, x_1') is a good factorization.*

Proof. Set $x_0 = x_0''x_0'$ and $x_1 = x_1'x_1''$. If (x_0', a, x_1') is not a good factorization, then $\varphi(x_0'a) \mathrel{\mathcal{R}} \varphi(x_0')$ or $\varphi(ax_1') \mathrel{\mathcal{L}} \varphi(x_1')$. We treat the first case, but the other case is dual. Since $\mathcal{R}$ is stable on the left, we have $\varphi(x_0'')\varphi(x_0'a) \mathrel{\mathcal{R}} \varphi(x_0'')\varphi(x_0')$, whence $\varphi(x_0) \mathrel{\mathcal{R}} \varphi(x_0a)$, a contradiction, since (x_0, a, x_1) is a good factorization. □

Define a relation $\sim_\varphi$ on A^*, by setting, for each $x, y \in A^*$, $x \sim_\varphi y$ if and only if the following three conditions are satisfied:

(1) $\varphi(x) = \varphi(y)$,

(2) each good factorization of x is equivalent to some good factorization of y,

(3) each good factorization of y is equivalent to some good factorization of x.

Then one can state.

Proposition 2.2 *The relation $\sim_\varphi$ is a congruence on A^*.*

Proof. By symmetry, it suffices to prove that $x \sim_\varphi y$ implies $xb \sim_\varphi yb$ for each $b \in A$. Suppose that $x \sim_\varphi y$. Then $\varphi(xb) = \varphi(yb)$. By symmetry again, it suffices to show that each good factorization of xb is equivalent to some good factorization of yb. Let (x_0, a, x_1) be a good factorization of xb. If $x_1 = 1$, then $(x_0, a, x_1) = (x, b, 1)$, and $(y, a, 1)$ is a good factorization of yb equivalent to (x_0, a, x_1). Otherwise, $(x_0, a, x_1) = (x_0, a, x'_1 b)$ for some $x'_1 \in A^*$, and by Lemma 2.1, (x_0, a, x'_1) is a good factorization of x. Since $x \sim_\varphi y$, there exists an equivalent good factorization (y_0, a, y'_1) of y. We claim that $(y_0, a, y'_1 b)$ is a good factorization of yb equivalent to (x_0, a, x_1). The only thing to prove is that $\varphi(ay'_1 b) <_{\mathcal{L}} \varphi(y'_1 b)$. But this is clear, since

$$\begin{aligned}\varphi(ay'_1 b) &= \varphi(a)\varphi(y'_1)\varphi(b) = \varphi(a)\varphi(x'_1)\varphi(b)\\ &= \varphi(a)\varphi(x_1) <_{\mathcal{L}} \varphi(x_1) = \varphi(x'_1)\varphi(b) = \varphi(y'_1 b). \quad \square\end{aligned}$$

Put $\widehat{M} = A^*/{\sim_\varphi}$ and let $\widehat{\varphi} : A^* \to \widehat{M}$ be the natural morphism induced by $\sim_\varphi$. By definition, $\widehat{\varphi}$ will be called the *expansion* of φ. Since $x \sim_\varphi y$ implies $\varphi(x) = \varphi(y)$, there exists a surjective morphism $\pi : \widehat{M} \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \hat{M} \arrow[dd, "\pi"] \\
A^* \arrow[ur, "\hat{\varphi}"] \arrow[dr, "\varphi"'] & \\
 & M
\end{tikzcd}$$

More generally, a monoid $\widehat{M}$ is said to be *an expansion* of a monoid M if there exists a free monoid A^* and a surjective morphism $\varphi : A^* \to M$ such that $\widehat{\varphi} : A^* \to \widehat{M}$ is the expansion of φ.

Example 2.1 Let $\varphi : \{a, b\}^* \to U_1 = \{1, 0\}$ be the surjective monoid morphism defined by $\varphi(a) = 1$ and $\varphi(b) = 0$. Then $(aa, b, 1)$ is a good factorization. The congruence $\sim_\varphi$ has three classes $C_1 = \{u \mid |u|_b = 0\}$, $C_2 = \{u \mid |u|_b = 1\}$, and $C_3 = \{u \mid |u|_b > 1\}$. Therefore $\widehat{U}_1 = \{1, s, 0\}$, where $s^2 = 0$, and $\widehat{\varphi} : A^* \to \widehat{U}_1$ is defined by $\widehat{\varphi}(a) = 1$ and $\widehat{\varphi}(b) = s$. The morphism $\pi : \widehat{U}_1 \to U_1$ is defined by $\pi(1) = 1$ and $\pi(s) = \pi(0) = 0$.

This example shows that our expansion is different from the expansion $\tilde{M}$ considered by Birget, Margolis and Rhodes in [4, 5]. Indeed, one has $\tilde{U}_1 = U_1$, but $\widehat{U}_1 \neq U_1$.

Example 2.2 Let G be a group and let $\varphi : A^* \to G$ be a surjective monoid morphism. Then there is no good factorization, and thus $\widehat{G} = G$.

We now verify the universal properties of our construction.

Proposition 2.3 *Let $\varphi : A^* \to M$, $\varphi' : A^* \to M'$ and $\sigma : M' \to M$ be surjective morphisms such that $\sigma \circ \varphi' = \varphi$. Then there exists a surjective morphism $\widehat{\sigma} : \widehat{M'} \to \widehat{M}$ such that the following diagram commutes*

$$\begin{tikzcd}
\hat{M}' \arrow[rr, "\pi"] \arrow[dd, "\hat{\sigma}"'] & & M' \arrow[dd, "\sigma"] \\
& A^* \arrow[ul, "\hat{\varphi}'"'] \arrow[ur, "\varphi'"] \arrow[dl, "\hat{\varphi}"] \arrow[dr, "\varphi"'] & \\
\hat{M} \arrow[rr, "\pi"'] & & M
\end{tikzcd}$$

Proof. We have to show that $x \sim_{\varphi'} y$ implies $x \sim_{\varphi} y$. First, $x \sim_{\varphi'} y$ implies $\varphi'(x) = \varphi'(y)$, whence $\varphi(x) = \varphi(y)$. Let (x_0, a, x_1) be a good factorization of x with respect to φ. Then (x_0, a, x_1) is also a good factorization of x with respect to φ', because if $\varphi'(x_0a) \mathrel{\mathcal{R}} \varphi'(x_0)$, then $\varphi(x_0a) = \sigma(\varphi'(x_0a)) \mathrel{\mathcal{R}} \sigma(\varphi'(x_0)) = \varphi(x_0)$, and, dually, $\varphi'(ax_1) \mathrel{\mathcal{L}} \varphi'(x_1)$ implies $\varphi(ax_1) \mathrel{\mathcal{L}} \varphi(x_1)$. Since $x \sim_{\varphi'} y$, there exists a good factorization of y with respect to φ' equivalent to (x_0, a, x_1), say (y_0, a, y_1). Then $\varphi'(x_0) = \varphi'(y_0)$ whence $\varphi(x_0) = \varphi(y_0)$, and similarly, $\varphi(x_1) = \varphi(y_1)$. Therefore, (y_0, a, y_1) is a good factorization of y with respect to φ equivalent to (x_0, a, x_1). Dually, each good factorization of y is equivalent to some good factorization of x, and $x \sim_{\varphi} y$. This proves the existence of a surjective morphism $\widehat{\sigma} : \widehat{M'} \to \widehat{M}$ such that $\widehat{\sigma} \circ \widehat{\varphi'} = \widehat{\varphi}$. It follows that

$$\pi \circ \widehat{\sigma} \circ \widehat{\varphi'} = \pi \circ \widehat{\varphi} = \varphi = \sigma \circ \varphi' = \sigma \circ \pi' \circ \widehat{\varphi'}$$

whence $\pi \circ \widehat{\sigma} = \sigma \circ \pi'$ since $\widehat{\varphi'}$ is surjective. □

We now study the algebraic properties of our expansion. Recall that a semigroup is n-nilpotent if it satisfies the identity $s_1 s_2 \cdots s_n = t_1 t_2 \cdots t_n$, or equivalently, if S has a zero, and if the set $S^n = \{s_1 s_2 \cdots s_n \mid s_1, \ldots, s_n \in S\}$ is equal to $\{0\}$.

Proposition 2.4 *For every idempotent e of M, the semigroup $\pi^{-1}(e)$ is a 2-nilpotent semigroup.*

Proof. Let $s, t \in \pi^{-1}(e)$ and let $x, y \in A^*$ be words such that $\widehat{\varphi}(x) = s$ and $\widehat{\varphi}(y) = t$. Then $\varphi(x) = \varphi(y) = e$. We claim that xy has no good factorization. Indeed, let (u_0, a, u_1) be a factorization of xy. By symmetry, we may assume that x is a left factor of u_0. Set $u_0 = xu'_0$. Then

$$\varphi(x) = \varphi(x)\varphi(x) = \varphi(xy) = \varphi(xu'_0au_1) \leq_{\mathcal{R}} \varphi(xu'_0a) \leq_{\mathcal{R}} \varphi(xu'_0) \leq_{\mathcal{R}} \varphi(x)$$

whence $\varphi(xu'_0) \mathrel{\mathcal{R}} \varphi(xu'_0a)$ and thus (u_0, a, u_1) is not a good factorization of xy. Therefore the words of $\varphi^{-1}(e)$ which have no good factorization are $\sim_{\varphi}$-equivalent. It follows that $\pi^{-1}(e)$ satisfies the equation $s_1 s_2 = t_1 t_2$ and thus is a 2-nilpotent semigroup. □

Proposition 2.5 *Let s, s_0 and s_1 be elements of $\widehat{M}$ such that $\pi(s_0 s) = \pi(s_0)$ and $\pi(s s_1) = \pi(s_1)$. Then $s_0 s s_1 = s_0 s_1$.*

Proof. Let $x, x_0, x_1 \in A^*$ be such that $\widehat{\varphi}(x) = s$, $\widehat{\varphi}(x_0) = s_0$ and $\widehat{\varphi}(x_1) = s_1$. By assumption, $\varphi(x_0x) = \varphi(x_0)$ and $\varphi(xx_1) = \varphi(x_1)$. We claim that $x_0xx_1 \sim_{\varphi} x_0x_1$. Let (u_0, a, u_1) be a good factorization of x_0xx_1. If $x = x'_0ax'_1$ for some words x'_0, x'_1 such that $u_0 = x_0x'_0$ and $u_1 = x'_1x_1$, then $\varphi(x_0) = \varphi(x_0x) <_{\mathcal{R}} \varphi(x_0x'_0) \leq_{\mathcal{R}} \varphi(x_0)$, a contradiction. Thus either x_0x is a prefix or u_0 or xx_1 is a suffix of u_1. Suppose for instance $u_0 = x_0xu'_0$. Then since $\varphi(x_0x) = \varphi(x_0)$, $(x_0u'_0, a, u_1)$ is a good factorization of x_0x_1 equivalent with (u_0, a, u_1). Conversely, let (u_0, a, u_1) be a good factorization of x_0x_1. Then either x_0 is a prefix of u_0 or x_1 is a suffix of u_1. Suppose for instance $u_0 = x_0u'_0$. Then since $\varphi(x_0x) = \varphi(x_0)$, $(x_0xu'_0, a, u_1)$ is a good factorization of x_0xx_1 equivalent with (u_0, a, u_1). This proves the claim and thus $s_0ss_1 = \widehat{\varphi}(x_0xx_1) = \widehat{\varphi}(x_0x_1) = s_0s_1$. $\square$

We arrive at the main property of our expansion.

Theorem 2.6 *The morphism $\pi : \widehat{M} \to M$ induces a bijection from $\mathrm{E}(\widehat{M})$ onto $\mathrm{E}(M)$, a bijection from $\mathrm{Reg}(\widehat{M})$ onto $\mathrm{Reg}(M)$ and an isomorphism from $\langle \mathrm{Reg}(\widehat{M}) \rangle$ onto $\langle \mathrm{Reg}(M) \rangle$.*

Proof. Since the image of a regular element is regular, $\pi(\langle \mathrm{Reg}(\widehat{M}) \rangle)$ is certainly contained in $\langle \mathrm{Reg}(M) \rangle$. We first show that π induces a surjection from $\mathrm{Reg}(\widehat{M})$ onto $\mathrm{Reg}(M)$. Let $r \in \mathrm{Reg}(M)$ and let $\bar{r}$ be an inverse of r. Let $s, t \in \widehat{M}$ be such that $\pi(s) = r$ and $\pi(t) = \bar{r}$. Then $\pi(st) = r\bar{r}$ is idempotent, and by Proposition 2.4, $(st)^3$ is idempotent. Therefore $x = (st)^3 s$ is a regular element such that $\pi(x) = r$. Similarly, π induces a surjection from $\mathrm{E}(\widehat{M})$ onto $\mathrm{E}(M)$.

We now show that the restriction of π to $\langle \mathrm{Reg}(\widehat{M}) \rangle$ is injective. This will show that π induces a bijection from $\mathrm{E}(\widehat{M})$ onto $\mathrm{E}(M)$, a bijection from $\mathrm{Reg}(\widehat{M})$ onto $\mathrm{Reg}(M)$ and an isomorphism from $\langle \mathrm{Reg}(\widehat{M}) \rangle$ onto $\langle \mathrm{Reg}(M) \rangle$. Let R be the set of all words which have no good factorizations. We claim that R is a submonoid of A^*. Indeed, if $x, y \in R$, and if (u_0, a, u_1) is a good factorization of xy, then either x is a left factor of u_0, or y is a right factor of u_1. Suppose, for instance, that $u_0 = xu_0'$ for some $u_0' \in A^*$ (the other case is symmetrical). Then, by Lemma 2.1, (u_0', a, u_1) is a good factorization of y, a contradiction. Thus $xy \in R$, proving the claim. Furthermore, it follows from the definition that R is saturated by $\sim_\varphi$, and that two elements r and r' of R are congruent modulo $\sim_\varphi$ if and only if $\varphi(r) = \varphi(r')$. In particular, π induces an isomorphism from $\widehat{\varphi}(R)$ onto $\varphi(R)$. Since $\widehat{\varphi}(R)$ is a submonoid of $\widehat{M}$, it remains to show that $\widehat{\varphi}(R)$ contains $\mathrm{Reg}(\widehat{M})$. This is the object of the next lemma.

Lemma 2.7 *Let x be a word such that $\widehat{\varphi}(x)$ is regular. Then x has no good factorization.*

Proof. . Let $s = \widehat{\varphi}(x)$. Since s is regular, there exists an element $\bar{s} \in M$ such that $s\bar{s}s = s$. Let $\bar{x}$ be a word of A^* such that $\widehat{\varphi}(\bar{x}) = \bar{s}$. In particular, $\widehat{\varphi}(x\bar{x}x) = \widehat{\varphi}(x)$. Suppose that x has a good factorization. Then since $x\bar{x}x \sim_\varphi x$, $x\bar{x}x$ also has a good factorization. There are three possible types for this factorization, depending on the place of the middle letter

(1) $(x_0, a, x_1\bar{x}x)$, where $x_0ax_1 = x$,

(2) $(x\bar{x}_0, a, \bar{x}_1x)$, where $\bar{x}_0a\bar{x}_1 = \bar{x}$,

(3) $(x\bar{x}x_0, a, x_1)$, where $x_0ax_1 = x$.

But (1) is not a good factorization, since $\varphi(x_1\bar{x}x) \mathrel{\mathcal{L}} \varphi(ax_1\bar{x}x)$, (2) is not a good factorization, since $\varphi(x\bar{x}_0) \mathrel{\mathcal{R}} \varphi(x\bar{x}_0a)$, and finally (3) is not a good factorization, since $\varphi(x\bar{x}x_0) \mathrel{\mathcal{R}} \varphi(x\bar{x}x_0a)$. Therefore x cannot have any good factorization. This concludes the proofs of Lemma 2.7 and Theorem 2.6. $\square$

3 Monoids in which the regular elements form a submonoid

The results of the previous section can be made precise if the regular elements of M form a submonoid of M. This condition is satisfied in particular if the product of two idempotents is an idempotent.

Proposition 3.1 *Let S be a semigroup in which the idempotents form a subsemigroup. Then the regular elements of S form a subsemigroup of S.*

Proof. Let r_1 and r_2 be regular elements of S. Then there exist idempotents e and f such that $e \mathcal{L} r_1$ and $r_2 \mathcal{R} f$. Therefore, there exist $x, y \in S^1$ such that

$$xr_1 = e \qquad r_1e = r_1 \qquad r_2y = f \qquad fr_2 = r_2$$

whence $r_1r_2 = r_1efr_2$. It follows that $ef \mathcal{J} r_1r_2$ since

$$x(r_1efr_2)y = (xr_1)(ef)(r_2y) = e(ef)f = ef.$$

Thus r_1r_2 is regular. $\square$

We need the following property of the semigroups in which the regular elements form a subsemigroup.

Lemma 3.2 *Let S be a semigroup in which the regular elements form a subsemigroup. If $\bar{s}_1$ (resp. $\bar{s}_2$) is a weak inverse of s_1 (resp. s_2), then there exist a weak inverse $\bar{s}$ of s_1s_2 such that $(s_1\bar{s}_1s_1)(s_2\bar{s}_2s_2) = (s_1\bar{s}_1)(s_1s_2)\bar{s}(s_1s_2)(\bar{s}_2s_2)$.*

Proof. By Proposition 1.1, $s_1\bar{s}_1s_1$ and $s_2\bar{s}_2s_2$ are regular, and thus

$$s = (s_1\bar{s}_1s_1)(s_2\bar{s}_2s_2)$$

is also regular and has an inverse s'. Set $\bar{s} = \bar{s}_2s_2s's_1\bar{s}_1$. We get $s_1\bar{s}_1s_1s_2\bar{s}s_1s_2\bar{s}_2s_2 = ss's = s$. Furthermore $\bar{s}$ is a weak inverse of s_1s_2 since

$$\bar{s}s_1s_2\bar{s} = \bar{s}_2s_2s's_1\bar{s}_1s_1s_2\bar{s}_2s_2s's_1\bar{s}_1 = \bar{s}_2s_2s'ss's_1\bar{s}_1 = \bar{s}_2s_2s's_1\bar{s}_1 = \bar{s}. \quad \square$$

Theorem 3.3 *Let M be a monoid in which the regular elements form a submonoid. Let $s_1, \ldots, s_n$ be elements of M such that, for $1 \le i \le n$, either $s_1 \cdots s_{i-1} \mathcal{R} s_1 \cdots s_i$ or $s_i \cdots s_n \mathcal{L} s_{i+1} \cdots s_n$. Then $s_1s_2 \cdots s_n$ is regular.*

Proof. We prove the result by induction on n. If $n = 1$, we have either $1 \mathcal{R} s_1$ or $s_1 \mathcal{L} 1$ and thus s_1 is regular. Assume that for some $i < n$ we have $s_1 \cdots s_{i-1} \mathcal{R} s_1 \cdots s_i$ and $s_1 \cdots s_i \mathcal{R} s_1 \cdots s_{i+1}$. Then $s_1 \cdots s_{i-1} \mathcal{R} s_1 \cdots s_{i-1}(s_is_{i+1})$ and one may apply the induction hypothesis to the sequence $s_1, \ldots, s_{i-1}, (s_is_{i+1}), s_{i+2}, \ldots, s_n$ to show that $s_1s_2 \cdots s_n$ is regular.

Next assume that $1 \mathcal{R} s_1$. Then s_1 is a unit in M and thus every relation of the form $s_1 \cdots s_{i-1} \mathcal{R} s_1 \cdots s_i$ is equivalent with $s_2 \cdots s_{i-1} \mathcal{R} s_2 \cdots s_i$. Therefore, induction applied to the sequence $s_2, \ldots, s_n$ shows that $s_2 \cdots s_n$ is regular. Since s_1 is regular, $s_1s_2 \cdots s_n$ is also regular.

A dual argument can be applied to the $\mathcal{L}$ relation, and thus we are left with the case where the following relations hold:

$$\begin{array}{rcll}
s_1s_2 \cdots s_n & \mathcal{L} & s_2 \cdots s_n & \quad (i = 1) \\
s_1 & \mathcal{R} & s_1s_2 & \quad (i = 2) \\
s_3s_4 \cdots s_n & \mathcal{L} & s_4 \cdots s_n & \quad (i = 3) \\
s_1s_2s_3 & \mathcal{R} & s_1s_2s_3s_4 & \quad (i = 4) \\
 & \vdots & & \\
s_1 \cdots s_{n-1} & \mathcal{R} & s_1s_2 \cdots s_n & \quad (i = n)
\end{array}$$

This implies in particular that n is even. By Proposition 1.2, there exist for $1 \le i \le n$ a weak inverse $\bar{s}_i$ of s_i such that

$$\left.\begin{array}{ll}
s_{i+1} \cdots s_n = \bar{s}_is_i(s_{i+1} \cdots s_n) & \text{if } i \text{ is odd} \\
s_1 \cdots s_{i-1} = (s_1 \cdots s_{i-1})s_i\bar{s}_i & \text{if } i \text{ is even}
\end{array}\right\} \tag{3.1}$$

Set, for $1 \le j \le n$, $s_{1,j} = s_1 \ \cdots \ s_j$. We show by induction on j that for $1 \le j \le n$, there exist weak inverses $\bar{s}_{1,j}$ of $s_{1,j}$ such that

$$s_1 \ \cdots \ s_n = s_{1,1}\bar{s}_{1,1}s_{1,2}\bar{s}_{1,2}s_{1,3}\bar{s}_{1,3} \ \cdots \ s_{1,j}\bar{s}_{1,j}s_{1,j}s_{j+1} \ \cdots \ s_n \tag{3.2}$$

First apply (3.1) with $i = 1$. We get $s_2 \ \cdots \ s_n = \bar{s}_1 s_1 s_2 \ \cdots \ s_n$ and thus $s_1 s_2 \ \cdots \ s_n = s_1 \bar{s}_1 s_1 s_2 \ \cdots \ s_n$, proving (3.2) for $j = 1$. By induction, assume that the result holds for j. If $j + 1$ is odd, we have by (3.1)

$$(s_{j+1}\bar{s}_{j+1}s_{j+1})(s_{j+2} \ \cdots \ s_n) = s_{j+1}(s_{j+2} \ \cdots \ s_n) \tag{3.3}$$

Therefore

$$s_{1,j}\bar{s}_{1,j}s_{1,j}s_{j+1} \ \cdots \ s_n = (s_{1,j}\bar{s}_{1,j}s_{1,j})(s_{j+1}\bar{s}_{j+1}s_{j+1})(s_{j+2} \ \cdots \ s_n)$$

Now by Lemma 3.2, there exists a weak inverse $\bar{s}_{1,j+1}$ of $s_{1,j}s_{j+1} = s_{1,j+1}$ such that

$$\begin{aligned}(s_{1,j}\bar{s}_{1,j}s_{1,j})(s_{j+1}\bar{s}_{j+1}s_{j+1}) &= (s_{1,j}\bar{s}_{1,j})(s_{1,j}s_{j+1})\bar{s}_{1,j+1}(s_{1,j}s_{j+1})(\bar{s}_{j+1}s_{j+1}) \\ &= (s_{1,j}\bar{s}_{1,j})(s_{1,j+1}\bar{s}_{1,j+1}s_{1,j+1})(\bar{s}_{j+1}s_{j+1})\end{aligned}$$

It follows that

$$\begin{aligned}s_{1,j}\bar{s}_{1,j}s_{1,j}s_{j+1} \ \cdots \ s_n &= (s_{1,j}\bar{s}_{1,j})(s_{1,j+1}\bar{s}_{1,j+1}s_{1,j+1})(\bar{s}_{j+1}s_{j+1})(s_{j+2} \ \cdots \ s_n) \\ &= (s_{1,j}\bar{s}_{1,j})(s_{1,j+1}\bar{s}_{1,j+1})(s_{1,j+1}s_{j+2} \ \cdots \ s_n) \qquad \text{by (3.3)}\end{aligned}$$

and thus (3.2) holds for $j + 1$.
If $j + 1$ is even, we have by (3.1)

$$s_{1,j}s_{j+1} = s_1 \ \cdots \ s_j s_{j+1} = s_1 \ \cdots \ s_j s_{j+1}\bar{s}_{j+1}s_{j+1} = s_{1,j}s_{j+1}\bar{s}_{j+1}s_{j+1} \tag{3.4}$$

Therefore by the induction hypothesis,

$$\begin{aligned}s_1 \ \cdots \ s_n &= s_{1,1}\bar{s}_{1,1}s_{1,2}\bar{s}_{1,2} \ \cdots \ s_{1,j}\bar{s}_{1,j}s_{1,j}s_{j+1} \ \cdots \ s_n \\ &= s_{1,1}\bar{s}_{1,1}s_{1,2}\bar{s}_{1,2} \ \cdots \ s_{1,j}\bar{s}_{1,j}s_{1,j}s_{j+1}\bar{s}_{j+1}s_{j+1} \ \cdots \ s_n\end{aligned}$$

Now by Lemma 3.2, there exists a weak inverse $\bar{s}_{1,j+1}$ of $s_{1,j}s_{j+1} = s_{1,j+1}$ such that

$$\begin{aligned}(s_{1,j}\bar{s}_{1,j}s_{1,j})(s_{j+1}\bar{s}_{j+1}s_{j+1}) &= (s_{1,j}\bar{s}_{1,j})(s_{1,j}s_{j+1})\bar{s}_{1,j+1}(s_{1,j}s_{j+1})(\bar{s}_{j+1}s_{j+1}) \\ &= (s_{1,j}\bar{s}_{1,j})(s_{1,j+1}\bar{s}_{1,j+1}s_{1,j+1})(\bar{s}_{j+1}s_{j+1})\end{aligned}$$

It follows that

$$\begin{aligned}s_1 \ \cdots \ s_n &= s_{1,1}\bar{s}_{1,1}s_{1,2}\bar{s}_{1,2} \ \cdots \ (s_{1,j}\bar{s}_{1,j})(s_{1,j+1}\bar{s}_{1,j+1}s_{1,j+1})(\bar{s}_{j+1}s_{j+1})s_{j+2} \ \cdots \ s_n \\ &= s_{1,1}\bar{s}_{1,1}s_{1,2}\bar{s}_{1,2} \ \cdots \ (s_{1,j}\bar{s}_{1,j})(s_{1,j+1}\bar{s}_{1,j+1}s_{1,j+1})s_{j+2} \ \cdots \ s_n \qquad \text{by (3.4)}\end{aligned}$$

and thus (3.2) also holds for $j + 1$. In particular, (3.2) gives for $j = n$,

$$s_1 \ \cdots \ s_n = (s_{1,1}\bar{s}_{1,1})(s_{1,2}\bar{s}_{1,2}) \ \cdots \ (s_{1,n}\bar{s}_{1,n})s_{1,n}$$

and thus $s_1 \ \cdots \ s_n$ is regular, as a product of the regular elements $s_{1,1}\bar{s}_{1,1}, \ \ldots \ ,$ $\bar{s}_{1,n-1}s_{1,n-1}$ and $s_{1,n}\bar{s}_{1,n}s_{1,n}$. □

If M is a monoid in which the regular elements form a submonoid, Lemma 2.7 can be completed as follows.

Proposition 3.4 *Let $\varphi : A^* \to M$ be a surjective monoid morphism, where M is a monoid in which the regular elements form a submonoid. A word $x \in A^*$ has a good factorization if and only if $\widehat{\varphi}(x)$ is non-regular.*

Proof. If x has a good factorization, then $\widehat{\varphi}(x)$ is non-regular by Lemma 2.7. Conversely, suppose that $\widehat{\varphi}(x)$ is non-regular. If $\varphi(x)$ is regular, then there exists by Theorem 2.6 a word y such that $\widehat{\varphi}(y)$ is regular and $\varphi(x) = \varphi(y)$. On the other hand, $\widehat{\varphi}(x) \neq \widehat{\varphi}(y)$ since $\widehat{\varphi}(x)$ is non-regular. Therefore, since y has no good factorization by Lemma 2.7, x must have a good factorization, otherwise $x \sim_\varphi y$ and $\widehat{\varphi}(x) = \widehat{\varphi}(y)$.

Now suppose $\varphi(x)$ non-regular. Let $x = a_1 \cdots a_k$, with $a_1, \ldots, a_k \in A$, and set for $1 \leq i \leq k$, $s_i = \varphi(a_i)$. If x has no good factorization, then in particular $(a_1 \cdots a_{i-1}, a_i, a_{i+1} \cdots a_k)$ is not a good factorization of x, whence

$$\text{either } \varphi(a_1) \cdots \varphi(a_{i-1}) \ \mathcal{R}\ \varphi(a_1) \cdots \varphi(a_i) \text{ or } \varphi(a_{i+1}) \cdots \varphi(a_k) \ \mathcal{L}\ \varphi(a_i) \cdots \varphi(a_k).$$

Now by Theorem 3.3, $\varphi(x) = \varphi(a_1 \cdots a_k)$ is regular, a contradiction. □

This last result has an important consequence. Let M be a monoid in which the regular elements form a submonoid and let $\alpha : A^* \to M$ be a surjective morphism. Suppose that M has a zero and a 0-minimal null ideal J and let $\beta : M \to M/J$ be the natural morphism. Finally, set $\varphi = \beta \circ \alpha : A^* \to M/J$ and let $\widehat{M/J}$ be the expansion of M/J relative to φ. These notations are summarized in the following diagram.

$$\begin{tikzcd}
& & A^* \arrow[dl, "\alpha"'] \arrow[dr, "\hat{\varphi}"] \arrow[ddll, bend right, "\varphi"'] & \\
& M \arrow[dl, "\beta"'] & & \widehat{M/J} \\
M/J & & &
\end{tikzcd}$$

Theorem 3.5 *With the previous notations, M is a quotient of $\widehat{M/J}$.*

Proof. Set $K = J \setminus \{0\}$. It suffices to show that, for every $u, v \in A^*$, $u \sim_\varphi v$ implies $\alpha(u) = \alpha(v)$. Since $u \sim_\varphi v$ implies $\varphi(u) = \varphi(v)$, it suffices to consider the case $\alpha(u), \alpha(v) \in J$.

Lemma 3.6 *Let x be a word such that $\alpha(x) \in K$. Then x has a good factorization.*

Proof. Let $x = a_1 \cdots a_k$, with $a_1, \ldots, a_k \in A$, and put $s_i = \alpha(a_i)$ for $1 \leq i \leq k$. First assume that $s_i \in K$ for some i. We claim that $f_i = (a_1 \cdots a_{i-1}, a_i, a_{i+1} \cdots a_k)$ is a good factorization of x. Indeed, $\alpha(a_1 \cdots a_{i-1}) \notin K$, otherwise

$$\alpha(x) = \alpha(a_1 \cdots a_{i-1})\alpha(a_i)\alpha(a_{i+1} \cdots a_k) = 0$$

since J is null. Similarly, $\alpha(a_{i+1} \cdots a_k) \notin K$, and thus $\varphi(a_1 \cdots a_{i-1}) \neq 0$, $\varphi(a_i) = 0$, $\varphi(a_{i+1} \cdots a_k) \neq 0$, proving the claim.

Suppose now $s_i \notin K$ for $1 \leq i \leq k$. Then $\varphi(a_i) = \alpha(a_i) = s_i$. If x has no good factorization, then none of the factorizations f_i is good. It follows that for $1 \leq i \leq k$, either $s_1 \cdots s_{i-1} \ \mathcal{R}\ s_1 \cdots s_i$ or $s_{i+1} \cdots s_k \ \mathcal{L}\ s_i \cdots s_k$. Therefore by Theorem 3.3, $\alpha(x) = s_1 \cdots s_k$ is regular, a contradiction. □

We now complete the proof of Theorem 3.5. If $\alpha(u) \in K$, (or if $\alpha(v) \in K$), u (or v) has a good factorization by Lemma 3.6. Since $u \sim_\varphi v$, u and v have equivalent factorizations, (u_0, a, u_1) and (v_0, a, v_1) respectively. Since $\varphi(u) <_\mathcal{R} \varphi(u_0)$, $\varphi(u_0) \neq 0$, whence $\varphi(u_0) = \alpha(u_0)$. Similarly $\varphi(u_1) = \alpha(u_1)$, $\varphi(v_0) = \alpha(v_0)$ and $\varphi(v_1) =$

$\alpha(v_1)$. Now since (u_0, a, u_1) and (v_0, a, v_1) are equivalent factorizations, $\varphi(u_0) = \varphi(v_0)$ and $\varphi(u_1) = \varphi(v_1)$. It follows that

$$\begin{aligned}\alpha(u) = \alpha(u_0 a u_1) = \alpha(u_0)\alpha(a)\alpha(u_1) &= \varphi(u_0)\alpha(a)\varphi(u_1) \\ &= \varphi(v_0)\alpha(a)\varphi(v_1) = \alpha(v_0)\alpha(a)\alpha(v_1) = \alpha(v_0 a v_1) = \alpha(v).\end{aligned}$$

Finally if $\alpha(u) \notin K$ and $\alpha(v) \notin K$, then $\alpha(u) = \alpha(v) = 0$. □

4 Bideterministic concatenation product

We now apply the results of the previous sections to the study of the bideterministic concatenation product.

Theorem 4.1 *Let $\varphi : A^* \to M$ be a surjective morphism. Every language recognized by $\widehat{\varphi}$ is a finite boolean combination of languages recognized by M or of bideterministic products of the form $L_0 a L_1$, where L_0 and L_1 are recognized by M.*

Proof. Since any language recognized by $\widehat{\varphi}$ is a finite union of sets of the form $\widehat{\varphi}^{-1}(s)$, it suffices to establish the result for $L = \widehat{\varphi}^{-1}(s)$, where $s \in \widehat{M}$. Set $m = \pi(s)$, and fix an element x of L. Put

$$T = \{(m_0, a, m_1) \in M \times A \times M \mid m_0\varphi(a)m_1 = m, m_0\varphi(a) <_{\mathcal{R}} m_0 \text{ and } \varphi(a)m_1 <_{\mathcal{L}} m_1\}.$$

This set is the disjoint union of two sets E and F, defined as follows

$$\begin{aligned}E = \{(m_0, a, m_1) \in T \mid \ &\text{there exists a good factorization } (x_0, a, x_1) \text{ of } x \\ &\text{such that } \varphi(x_0) = m_0 \text{ and } \varphi(x_1) = m_1\},\\ F = \{(m_0, a, m_1) \in T \mid \ &\text{there exists no good factorization } (x_0, a, x_1) \text{ of } x \\ &\text{such that } \varphi(x_0) = m_0 \text{ and } \varphi(x_1) = m_1\}.\end{aligned}$$

Note that the definition of E and F depends on s, but does not depend on the choice of x in L, since all the elements of L are congruent modulo $\sim_\varphi$. We claim that

$$L = \varphi^{-1}(m) \cap \Big(\Big[\bigcap_{(m_0,a,m_1)\in E} \varphi^{-1}(m_0)a\varphi^{-1}(m_1) \Big] \setminus \Big[\bigcup_{(m_0,a,m_1)\in F} \varphi^{-1}(m_0)a\varphi^{-1}(m_1) \Big] \Big).$$

First $\varphi^{-1}(m) = \widehat{\varphi}^{-1}(\pi^{-1}(m))$ contains $\widehat{\varphi}^{-1}(s) = L$. Let $u \in \varphi^{-1}(m)$ and let $(m_0, a, m_1) \in T$. Then $u \in \varphi^{-1}(m_0)a\varphi^{-1}(m_1)$ if and only if u has a good factorization (u_0, a, u_1) such that $\varphi(u_0) = m_0$ and $\varphi(u_1) = m_1$. Therefore,

$$u \in \bigcap_{(m_0,a,m_1)\in E} \varphi^{-1}(m_0)a\varphi^{-1}(m_1) \tag{4.1}$$

if and only if each good factorization of x is equivalent to a good factorization of u, and

$$u \in A^* \setminus \bigcup_{(m_0,a,m_1)\in F} \varphi^{-1}(m_0)a\varphi^{-1}(m_1) \tag{4.2}$$

if and only if each good factorization of u is equivalent to a good factorization of x. Therefore u satisfies (1), (2) and $\varphi(u) = m$, if and only if $u \sim_\varphi x$, or equivalently, if $u \in L$. This proves the claim and the theorem. □

Let $\varphi_0 : A^* \to M_0$ and $\varphi_1 : A^* \to M_1$ be two surjective morphisms recognizing the languages L_0 and L_1, respectively, and let a be a letter of A. Let $\varphi : A^* \to M_0 \times M_1$ be the morphism defined by $\varphi(u) = (\varphi_0(u), \varphi_1(u))$, and let $M = \varphi(A^*)$. Then we can state

Theorem 4.2 *If the product $L = L_0 a L_1$ is bideterministic, then L is recognized by $\widehat{\varphi}$.*

Proof. Put $L = L_0 a L_1$. It suffices to show that $x \sim_\varphi y$ implies $x \sim_L y$. First, $x \sim_\varphi y$ implies $\varphi_0(x) = \varphi_0(y)$ and $\varphi_1(x) = \varphi_1(y)$. Suppose that $uxv \in L$. Three cases may occur

(1) $uxv = u_0 a u_1 x v$, with $u_0 a u_1 = u$, $u_0 \in L_0$ and $u_1 x v \in L_1$. Then $x \sim_{L_1} y$ since $\varphi_1(x) = \varphi_1(y)$, and thus $u_1 y v \in L_1$ and $uyv = u_0 a u_1 y v \in L$.

(2) $uxv = uxv_0 a v_1$, with $v_0 a v_1 = v$, $uxv_0 \in L_0$ and $v_1 \in L_1$. This case is symmetrical to (1) and thus $uyv \in L$.

(3) $uxv = ux_0 a x_1 v$, with $x_0 a x_1 = x$, $ux_0 \in L_0$ and $x_1 v \in L_1$. If $\varphi_0(ux_0 a)\ \mathcal{R}\ \varphi_0(ux_0)$, there exists a word t such that $\varphi_0(ux_0 a t) = \varphi_0(ux_0)$, whence $ux_0 a t \in L_0$ and $L_0 a A^* \cap L_0 \neq \emptyset$. Therefore, $\varphi_0(ux_0 a) <_\mathcal{R} \varphi_0(ux_0)$, whence $\varphi(ux_0 a) <_\mathcal{R} \varphi(ux_0)$ and dually $\varphi(a x_1 v) <_\mathcal{L} \varphi(x_1 v)$. It follows that $(ux_0, a, x_1 v)$ is a good factorization of uxv. Now $x \sim_\varphi y$ implies $uxv \sim_\varphi uyv$, and there exists a good factorization (w_0, a, w_1) of uyv equivalent to $(ux_0, a, x_1 v)$. In particular, $\varphi(w_0) = \varphi(ux_0)$, whence $\varphi_0(w_0) = \varphi_0(ux_0)$, and $w_0 \in L_0$. Similarly, $w_1 \in L_1$ and $uyv = w_0 a w_1 \in L$. Thus $uxv \in L$ implies $uyv \in L$. Now, $uyv \in L$ implies $uxv \in L$ by a dual argument and $x \sim_L y$. $\square$

The previous results can be converted into results on varieties. Let $\mathbf{V}$ be a variety of monoids and let $\mathcal{V}$ be the corresponding variety of languages. We denote by $\widehat{\mathbf{V}}$ the variety of monoids generated by the monoids $\widehat{M}$ for $M \in \mathbf{V}$. Finally, we denote by $\widehat{\mathcal{V}}$ the variety of languages associated with $\widehat{\mathbf{V}}$.

Theorem 4.3 *For each alphabet A, $\widehat{\mathcal{V}}(A^*)$ is the boolean algebra generated by languages of the form L or $L_0 a L_1$, where L, L_0 and L_1 are in $\mathcal{V}(A^*)$ and $L_0 a L_1$ is a bideterministic product.*

Proof. Let $\mathcal{C}(A^*)$ be the boolean algebra generated by languages of the form L or $L_0 a L_1$, where L, L_0 and L_1 are in $\mathcal{V}(A^*)$ and $L_0 a L_1$ is a bideterministic product. We claim that $\mathcal{C}(A^*)$ is contained in $\widehat{\mathcal{V}}(A^*)$. By definition of a variety of languages, $\widehat{\mathcal{V}}(A^*)$ is a boolean algebra of languages and since $\mathbf{V}$ is a subvariety of $\widehat{\mathbf{V}}$, $\widehat{\mathcal{V}}(A^*)$ contains the languages of $\mathcal{V}(A^*)$. Let $L_0 a L_1$ be a bideterministic product with L_0 and L_1 in $\mathcal{V}(A^*)$. Let $\varphi_0 : A^* \to M_0$ and $\varphi_1 : A^* \to M_1$ be two surjective morphisms recognizing the languages L_0 and L_1, respectively, with $M_0, M_1 \in \mathbf{V}$. By Theorem 4.2, $L_0 a L_1$ is recognized by a monoid $\widehat{M}$, where M is a submonoid of $M_0 \times M_1$. Since $\mathbf{V}$ is a variety, it follows that $M \in \mathbf{V}$ and $\widehat{M} \in \widehat{\mathbf{V}}$, and thus $L_0 a L_1 \in \widehat{\mathcal{V}}(A^*)$, proving the claim.

We now show that $\widehat{\mathcal{V}}(A^*)$ is contained in $\mathcal{C}(A^*)$. Let $L \in \widehat{\mathcal{V}}(A^*)$. By definition, L is recognized by a monoid $\widehat{M}$, where $M \in \mathbf{V}$. Therefore, there exist a morphism $\alpha : A^* \to \widehat{M}$ such that $L = \alpha^{-1}(P)$, where $P = \alpha(L)$, and a surjective morphism $\varphi : B^* \to M$ which defines the expansion $\widehat{\varphi} : B^* \to \widehat{M}$. Then, by the universal property of the free monoid A^*, there exists a morphism $\beta : A^* \to B^*$ such that $\alpha = \widehat{\varphi} \circ \beta$. In particular $L = \beta^{-1}\big(\widehat{\varphi}^{-1}(P)\big)$. Now, $\widehat{\varphi}^{-1}(P)$ is recognized by $\widehat{\varphi}$ and thus, by Theorem 4.1, is in $\mathcal{C}(B^*)$. Therefore, it suffices to show that $\beta^{-1}(\mathcal{C}(B^*)) \subset \mathcal{C}(A^*)$. Since β^{-1} commutes with boolean operations, we need only consider the generators of $\mathcal{C}(B^*)$. If K is a language of $\mathcal{V}(B^*)$, then $\beta^{-1}(K) \in \mathcal{V}(A^*)$ by definition of a variety of languages. Therefore, it only remains to verify that if $K_0 b K_1$ is a bideterministic product, where $b \in B$ and $K_0, K_1 \in \mathcal{V}(B^*)$, then $\beta^{-1}(K_0 b K_1) \in \mathcal{C}(A^*)$. Given a subset S of B^* and a word $x \in B^*$, we define the *right* and *left quotients* of S by x as follows

$$Sx^{-1} = \{y \in B^* \mid yx \in S\} \quad \text{and} \quad x^{-1}S = \{y \in B^* \mid xy \in S\}$$

We claim that

$$\beta^{-1}(K_0bK_1) = \bigcup_{\{a \mid \beta(a)=x_0bx_1\}} \beta^{-1}(K_0x_0^{-1})a\beta^{-1}(x_1^{-1}K_1) \tag{4.3}$$

Indeed, if $\beta(a) = x_0bx_1$, then

$$\beta\big(\beta^{-1}(K_0x_0^{-1})a\beta^{-1}(x_1^{-1}K_1)\big) \subset (K_0x_0^{-1})x_0bx_1(x_1^{-1}K_1) \subset K_0bK_1$$

In the opposite direction, take $u \in \beta^{-1}(K_0bK_1)$. Then $\beta(u) \in K_0bK_1$ and there exists a factorization $\beta(u) = k_0bk_1$ with $k_0 \in K_0$ and $k_1 \in K_1$. Therefore, there is a factorization $u = u_0au_1$ such that $\beta(a) = x_0bx_1$, $\beta(u_0) = k_0x_0^{-1}$ and $\beta(u_1) = x_1^{-1}k_1$. It follows that $u \in \beta^{-1}(K_0x_0^{-1})a\beta^{-1}(x_1^{-1}K_1)$, proving the claim. Notice that formula (4.3) gives an expression of $\beta^{-1}(K_0bK_1)$ as a *finite* union of languages since any rational language has only a finite number of right (resp. left) quotients. Since a variety of languages is closed under left and right quotients, we have $K_0x_0^{-1}, x_1^{-1}K_1 \in \mathcal{V}(B^*)$ and thus $\beta^{-1}(K_0x_0^{-1}), \beta^{-1}(x_1^{-1}K_1) \in \mathcal{V}(A^*)$. Furthermore, each product $P = \beta^{-1}(K_0x_0^{-1})a\beta^{-1}(x_1^{-1}K_1)$ is bideterministic. Indeed, if for instance the product P was not deterministic, then there would be a word $u \in P$ with a factorization of the form $u = u_0au'au_1$ with $u_0a, u_0au'a \in \beta^{-1}(K_0x_0^{-1})a$ and $u'au_1, u_1 \in \beta^{-1}(x_1^{-1}K_1)$. Then $\beta(u)$ would be a word of K_0bK_1 with two distinct prefixes in K_0b, a contradiction. Therefore, $P \in \mathcal{C}(A^*)$ and $\beta^{-1}(K_0bK_1) \in \mathcal{C}(A^*)$, concluding the proof. □

With the previous notations, we denote by $\bar{\mathbf{V}}$ the smallest variety containing $\mathbf{V}$ and closed under the operation $M \to \widehat{M}$. This variety is in general larger than $\widehat{\mathbf{V}}$ since $\widehat{\mathbf{V}}$ is not necessarily closed under the operation $M \to \widehat{M}$. On the other hand, if $\mathbf{V} = \widehat{\mathbf{V}}$, then $\mathbf{V} = \bar{\mathbf{V}}$. Let $\bar{\mathcal{V}}$ be the corresponding variety of languages. A set $\mathcal{S}$ of languages of A^* is said to be *closed under bideterministic product* if, for every bideterministic product L_0aL_1, with $L_0, L_1 \in \mathcal{S}$, one has $L_0aL_1 \in \mathcal{S}$.

Theorem 4.4 *For each alphabet A, $\bar{\mathcal{V}}(A^*)$ is the smallest boolean algebra containing $\mathcal{V}(A^*)$ and closed under bideterministic product.*

Corollary 4.5 *Let $\mathbf{V}$ be a variety of monoids and let $\mathcal{V}$ be the corresponding variety of languages. Then $\mathcal{V}$ is closed under bideterministic product if and only if $\mathbf{V} = \widehat{\mathbf{V}}$.*

The next results give an explicit computation of $\bar{\mathbf{V}}$ for various varieties $\mathbf{V}$.

Proposition 4.6 *The following varieties of monoids $\mathbf{V}$ satisfy $\mathbf{V} = \bar{\mathbf{V}}$: any variety of groups, aperiodic monoids, $\mathcal{R}$-trivial monoids, $\mathcal{L}$-trivial monoids, $\mathcal{J}$-trivial monoids, monoids whose regular $\mathcal{D}$-classes are groups (respectively semigroups, idempotent semigroups, etc.), monoids whose idempotents form a submonoid, monoids with commuting idempotents and all the possible intersections of these varieties. In particular, the corresponding varieties of languages are closed under bideterministic product.*

Proof. Example 2.2 shows that any variety of groups $\mathbf{V}$ satisfies $\mathbf{V} = \bar{\mathbf{V}}$. By Theorem 2.6, the expansion $\widehat{M}$ preserves regular elements and idempotents. This proves that $\mathbf{V} = \bar{\mathbf{V}}$ for the variety of monoids whose regular $\mathcal{D}$-classes are groups (respectively semigroups, idempotent semigroups, etc.), for the variety of monoids whose idempotents form a submonoid and for the variety of monoids with commuting idempotents. Furthermore it is well known (see [10] for instance) that a monoid is aperiodic (resp. $\mathcal{R}$-trivial, $\mathcal{L}$-trivial, $\mathcal{J}$-trivial) if and only if its regular $\mathcal{H}$-classes (resp. $\mathcal{R}$-classes, $\mathcal{L}$-classes, $\mathcal{J}$-classes) are trivial. Thus $\mathbf{V} = \bar{\mathbf{V}}$ also holds for these varieties. □

The proofs of the next theorems share a few standard decomposition techniques (see [1] for instance) which we recall below.

Proposition 4.7 *Let M be a monoid and let I be its minimal ideal. Let M_{r} (resp. M_ℓ) be the transformation monoid defined by the right (resp. left) action of M on I. Then M is a submonoid of $(M/I) \times M_{\mathrm{r}} \times M_\ell$. Furthermore, if I is $\mathcal{R}$-trivial (resp. $\mathcal{L}$-trivial), then M is a submonoid of $(M/I) \times M_\ell$ (resp. $(M/I) \times M_{\mathrm{r}}$).*

Corollary 4.8 *If the minimal ideal G of a monoid M is a group, then M is a submonoid of $(M/G) \times G$.*

Proposition 4.9 *Suppose that a monoid M has a zero and a unique 0-minimal ideal J. If $J \setminus \{0\}$ is a subsemigroup of M, then $M \setminus \{0\}$ is a submonoid of M and M is a quotient of $(M \setminus \{0\}) \times U_1$.*

Proof. Let $I = J \setminus \{0\}$. Let $x, y \in M \setminus \{0\}$ and let $s \in I$. Since J is the unique 0-minimal ideal of M, I is contained in the ideal generated by x (resp. y). In particular, $s = x_1 x x_2 = y_1 y y_2$ for some $x_1, x_2, y_1, y_2 \in M$. Since I is a simple semigroup, it follows that $s^2 = s x_1 x x_2 \in I$ whence $s x_1 x \mathrel{\mathcal{R}} s$ and $s x_1 x \in I$. Similarly, $y y_2 s \in I$ and thus $s x_1 x y y_2 s \in I$. In particular, $xy \neq 0$. Therefore, $M \setminus \{0\}$ is a submonoid of M.

Let $\pi : (M \setminus \{0\}) \times U_1 \to M$ be the map defined, for every $m \in M \setminus \{0\}$, by $\pi(m, 1) = m$ and $\pi(m, 0) = 0$. Then π is a surjective morphism and thus M is a quotient $(M \setminus \{0\}) \times U_1$. □

Proposition 4.10 *Suppose that a monoid M has a zero and two distinct 0-minimal ideals J_1 and J_2. Then M is a submonoid of $(M/J_1) \times (M/J_2)$.*

Proof. Let $\pi_1 : M \to M/J_1$ and $\pi_2 : M \to M/J_2$ be the natural morphisms. Let $\pi : M \to (M/J_1) \times (M/J_2)$ be the morphism defined by $\pi(m) = \big(\pi_1(m), \pi_2(m)\big)$. We claim that π is injective. Indeed, suppose that $\pi(m) = \pi(m')$. Then $\pi_1(m) = \pi_1(m')$ and $\pi_2(m) = \pi_2(m')$ and thus m and m' are either simultaneously in J_1 (resp. J_2) or simultaneously in $M \setminus J_1$ (resp. $M \setminus J_2$). In the latter case, we have $m = m'$. Thus we may assume that m and m' belong to J_1 and to J_2. But the intersection of two distinct 0-minimal ideals is equal to $\{0\}$ and thus $m = m' = 0$. This proves the claim and the proposition. □

Theorem 4.11 *If $\mathbf{Sl}$ is the variety of idempotent and commutative monoids, then $\overline{\mathbf{Sl}}$ is the variety of $\mathcal{J}$-trivial monoids with commuting idempotents.*

Proof. By Proposition 4.6, the variety $\mathbf{V}$ of $\mathcal{J}$-trivial monoids with commuting idempotents satisfies $\mathbf{V} = \overline{\mathbf{V}}$. Since $\mathbf{Sl} \subset \mathbf{V}$, it follows that $\overline{\mathbf{Sl}} \subset \mathbf{V}$.

Conversely, let $M \in \mathbf{V}$. We show by induction on Card(M) that $M \in \overline{\mathbf{Sl}}$. The result is clear if Card$(M) = 1$, so we may assume Card$(M) > 1$. Since M is $\mathcal{J}$-trivial, M has a zero. If M has (at least) two 0-minimal ideals J_1 and J_2, then M divides $(M/J_1) \times (M/J_2)$ by Proposition 4.10. Now since Card$(M/J_1) <$ Card(M) and Card$(M/J_2) <$ Card(M), it follows that $M/J_1, M/J_2 \in \overline{\mathbf{Sl}}$ by the induction hypothesis, whence $M \in \overline{\mathbf{Sl}}$.

Assume now that M has a unique 0-minimal ideal J, and put $J = \{s, 0\}$. If s is regular, then by Proposition 4.9, M divides $(M/J) \times U_1$. Since $U_1 \in \mathbf{Sl}$ and Card$(M/J) <$ Card(M), it follows that $M/J \in \overline{\mathbf{Sl}}$ by the induction hypothesis, and thus $M \in \overline{\mathbf{Sl}}$. Finally suppose that $s^2 = 0$, so that $J^2 = 0$. Then by Proposition 3.1 and Theorem 3.5, M is a quotient of $\widehat{M/J}$. But $M/J \in \overline{\mathbf{Sl}}$ by the induction hypothesis, and thus $M \in \overline{\mathbf{Sl}}$. □

Theorem 4.11 can be extended as follows :

Theorem 4.12 *Let* $\mathbf{H}$ *be a variety of groups and let* $\mathbf{Sl}(\mathbf{H})$ *be the variety of monoids which are semilattices of groups of the variety* $\mathbf{H}$*. Then* $\overline{\mathbf{Sl}(\mathbf{H})}$ *is the variety of monoids with commuting idempotents whose regular* $\mathcal{J}$*-classes are groups of the variety* $\mathbf{H}$*.*

Proof. By prop 4.6, the variety $\mathbf{W}$ of all monoids with commuting idempotents whose regular $\mathcal{J}$-classes are groups of $\mathbf{H}$ satisfies $\mathbf{W} = \overline{\mathbf{W}}$. Since $\mathbf{Sl}(\mathbf{H}) \subset \mathbf{W}$, it follows that $\overline{\mathbf{Sl}(\mathbf{H})} \subset \mathbf{W}$.

Conversely, let M be a monoid in $\mathbf{W}$. We show by induction on $\mathrm{Card}(M)$ that $M \in \overline{\mathbf{Sl}(\mathbf{H})}$. The result is clear if $\mathrm{Card}(M) = 1$, so we may assume $\mathrm{Card}(M) > 1$. Since $M \in \mathbf{W}$, the minimal ideal of M is a group G, and by Corollary 4.8, M divides $(M/G) \times G$. But $G \in \mathbf{Sl}(\mathbf{H})$, and thus it suffices to show that $M/G \in \overline{\mathbf{Sl}(\mathbf{H})}$. In other words, we may assume that M has a zero. If M has (at least) two 0-minimal ideals J_1 and J_2, then M divides $(M/J_1) \times (M/J_2)$ by Proposition 4.10. Since $\mathrm{Card}(M/J_1) < \mathrm{Card}(M)$ and $\mathrm{Card}(M/J_2) < \mathrm{Card}(M)$, it follows that $M/J_1, M/J_2 \in \overline{\mathbf{Sl}(\mathbf{H})}$ by the induction hypothesis, whence $M \in \overline{\mathbf{Sl}(\mathbf{H})}$.

Assume now that M has a unique 0-minimal ideal J, and put $I = J \setminus \{0\}$. If I is regular (and thus, is a group), then M divides $(M/J) \times U_1$ by Proposition 4.9. Since $U_1 \in \mathbf{Sl}$ and $\mathrm{Card}(M/J) < \mathrm{Card}(M)$, it follows that $M/J \in \overline{\mathbf{Sl}(\mathbf{H})}$ by the induction hypothesis, and thus $M \in \overline{\mathbf{Sl}(\mathbf{H})}$. Finally suppose that I is not regular, that is, $J^2 = 0$. Then by Theorem 3.5, M is a quotient of $\widehat{M/J}$. But $M/J \in \overline{\mathbf{Sl}(\mathbf{H})}$ by the induction hypothesis, and thus $M \in \overline{\mathbf{Sl}(\mathbf{H})}$. □

Corollary 4.13 *Let* $\mathbf{V}$ *be the variety of* $\mathcal{J}$*-trivial monoids with commuting idempotents and let* $\mathcal{V}$ *be the corresponding variety of languages. Then, for every alphabet* A*,* $\mathcal{V}(A^*)$ *is the smallest boolean algebra containing the languages of the form* B^**, where* B *is a subset of* A *and closed under bideterministic product.*

Proof. It is well known (see [6, 7, 10] for instance) that, for every alphabet A, the class of languages of A^* recognized by an idempotent and commutative monoid is the boolean algebra generated by the languages of the form B^*, where B is a subset of A. The result now follows from Theorems 4.4 and 4.11. □

Corollary 4.14 *Let* $\mathbf{H}$ *be a variety of groups. Let* $\mathbf{V}$ *be the variety of monoids with commuting idempotents whose regular* $\mathcal{J}$*-classes are groups of the variety* $\mathbf{H}$*. Then, for every alphabet* A*,* $\mathcal{V}(A^*)$ *is the smallest boolean algebra closed under bideterministic product containing the languages* B^**, where* B *is a subset of* A *and the languages whose syntactic monoid is a group of the variety* $\mathbf{H}$*.*

Proof. The variety $\mathbf{V}$ of monoids which are semilattices of groups of the variety $\mathbf{H}$ is equal to $\mathbf{Sl} \vee \mathbf{H}$, the join of the varieties $\mathbf{H}$ and $\mathbf{Sl}$. Let $\mathcal{V}$ be the corresponding variety of languages. Then for every alphabet A, $\mathcal{V}(A^*)$ is the boolean algebra generated by the languages whose syntactic monoid is a group of the variety $\mathbf{H}$ and by the languages of the form B^*, where B is a subset of A. The result now follows from Theorems 4.4 and 4.11. □

Corollaries 4.13 and 4.14 should be compared with the following results of [3].

Theorem 4.15 *Let* $L \in A^*$ *be a recognizable language and let* M *be its syntactic monoid. The following conditions are equivalent:*

(1) *M is $\mathcal{J}$-trivial with commuting idempotents,*

(2) *L is a boolean combination of languages of the form $A_0^*a_1A_1^*a_2\cdots a_kA_k^*$, where $k\geq 0$, $a_1,\ldots,a_k\in A$, $A_0,\ldots,A_k\subset A$, and, for $1\leq i\leq k$, $a_i\notin A_{i-1}\cup A_i$.*

Theorem 4.16 *Let $\mathbf{H}$ be a variety of groups. Let $L\in A^*$ be a recognizable language and let M be its syntactic monoid. The following conditions are equivalent:*

(1) *M is a monoid with commuting idempotents whose regular $\mathcal{J}$-classes are groups of the variety $\mathbf{H}$,*

(2) *L is a boolean combination of languages of the form $L_0a_1L_1a_2\cdots a_kL_k$, where $k\geq 0$, $a_1,\ldots,a_k\in A$, and $L_i\subset A_i^*$, where A_i is a subset of A such that $a_i\notin A_{i-1}\cup A_i$ and the syntactic monoid of L_i in A_i^* is a group of the variety $\mathbf{H}$.*

Our next result gives a description of the variety $\bar{\mathbf{V}}$ when $\mathbf{V}$ is the variety of $\mathcal{R}$-trivial and idempotent monoids (also called left regular bands). The variety $\bar{\mathbf{V}}$ is given by three equations. Two of the three equations of this description, namely

$$(1)\quad (xy)^\omega=(xy)^\omega x \quad\text{and}\quad (2)\quad x^\omega y^\omega x=x^\omega y^\omega$$

characterize the $\mathcal{R}$-trivial monoids in which idempotents form a submonoid. Indeed, let M be an $\mathcal{R}$-trivial monoid in which idempotents form a submonoid, and let $x,y\in M$. Then $(xy)^\omega\ \mathcal{R}\ (xy)^\omega x$ and thus $(xy)^\omega=(xy)^\omega x$. Similarly, since $x^\omega y^\omega$ is idempotent, $x^\omega y^\omega x\ \mathcal{R}\ x^\omega y^\omega$ and thus $x^\omega y^\omega x=x^\omega y^\omega$. Conversely, if M satisfies the equations (1) and (2), M is $\mathcal{R}$-trivial (see [6, 10] for instance) and the product of two idempotents is idempotent since, by (2), $x^\omega y^\omega x^\omega y^\omega=x^\omega y^\omega y^\omega=x^\omega y^\omega$.

The third equation of the description is $x(xy)^\omega=(xy)^\omega$ and is not a consequence of the two first equations.

Theorem 4.17 *If $\mathbf{V}$ is the variety of $\mathcal{R}$-trivial and idempotent monoids, then $\bar{\mathbf{V}}$ is the variety defined by the equations $x(xy)^\omega=(xy)^\omega=(xy)^\omega x$ and $x^\omega y^\omega x=x^\omega y^\omega$.*

Proof. Let $\mathbf{W}$ be the variety defined by the equations $x(xy)^\omega=(xy)^\omega=(xy)^\omega x$ and $x^\omega y^\omega x=x^\omega y^\omega$. We first show that $\hat{\mathbf{W}}=\mathbf{W}$. Let $M\in\mathbf{W}$ and let $\pi:\hat{M}\to M$ be an expansion of M. Since $M\in\mathbf{W}$, M is $\mathcal{R}$-trivial and the idempotents of M form a submonoid and by Proposition 4.6, the same properties hold in $\hat{M}$. In particular, $\hat{M}$ satisfies the equations $(xy)^\omega=(xy)^\omega x$ and $x^\omega y^\omega x=x^\omega y^\omega$. It follows that, for every $x,y\in\hat{M}$,

$$(xy)^\omega=(xy)^\omega x=\big((xy)^\omega x\big)^2=(xy)^\omega x(xy)^\omega x=(xy)^\omega x(xy)^\omega$$

In particular, $x(xy)^\omega\ \mathcal{L}\ (xy)^\omega$ and thus $(xy)^\omega$ and $x(xy)^\omega$ are *regular* elements of $\hat{M}$. Now, by Theorem 2.6, π induces an isomorphism from $\langle\mathrm{Reg}(\widehat{M})\rangle$ onto $\langle\mathrm{Reg}(M)\rangle$. But since $M\in\mathbf{W}$, $\pi\big(x(xy)^\omega\big)=\pi\big((xy)^\omega\big)$ and thus $x(xy)^\omega=(xy)^\omega$. Thus $\hat{M}\in\mathbf{W}$ and $\hat{\mathbf{W}}=\mathbf{W}$. Since $\mathbf{V}\subset\mathbf{W}$, it follows that $\bar{\mathbf{V}}\subset\mathbf{W}$.

Conversely, let M be a monoid in $\mathbf{W}$. We show by induction on $\mathrm{Card}(M)$ that $M\in\bar{\mathbf{V}}$. The result is clear if $\mathrm{Card}(M)=1$, so we may assume $\mathrm{Card}(M)>1$. Let I be the minimal ideal of M. By Proposition 4.7, M is a submonoid of $(M/I)\times M_\ell$. We claim that $M_\ell\in\mathbf{V}$. Since M_ℓ divides M, M_ℓ is $\mathcal{R}$-trivial and it suffices to show that M_ℓ is idempotent. Let $x\in M$ and let $y\in I$. Then $xy\in I$ and thus xy is idempotent. It follows that $xy=(xy)^\omega=x(xy)^\omega=xxy=x^2y$. This shows that x and x^2 have the same left action on I, proving the claim. Therefore it suffices to show now that $M/I\in\bar{\mathbf{V}}$. If $\mathrm{Card}(I)\geq 2$, one can conclude by induction. Otherwise, we may suppose that $I=\{0\}$.

If M has (at least) two 0-minimal ideals J_1 and J_2, then M divides $(M/J_1)\times(M/J_2)$ by Proposition 4.10 and we conclude by induction. Assume now that M has a unique 0-minimal ideal J, and put $I=J\setminus\{0\}$. If I is regular, then M

divides $(M/J) \times U_1$ by Proposition 4.9. Since $U_1 \in \mathbf{Sl}$ and $\mathrm{Card}(M/J) < \mathrm{Card}(M)$, it follows that $M/J \in \bar{\mathbf{V}}$ by the induction hypothesis, and thus $M \in \bar{\mathbf{V}}$. Finally suppose that I is not regular, that is, $J^2 = 0$. Then by Theorem 3.5, M is a quotient of $\widehat{M/J}$. But $M/J \in \bar{\mathbf{V}}$ by the induction hypothesis, and thus $M \in \bar{\mathbf{V}}$. □

Corollary 4.18 *Let* $\mathbf{V}$ *be the variety defined by the equations* $x(xy)^\omega = (xy)^\omega = (xy)^\omega x$ *and* $x^\omega y^\omega x = x^\omega y^\omega$ *and let* $\mathcal{V}$ *be the corresponding variety of languages. Then for every alphabet* A*,* $\mathcal{V}(A^*)$ *is the smallest boolean algebra containing the languages of the form* B^*aA^**, where* B *is a subset of* A*, and closed under bideterministic product.*

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