Lectures in Complex Analysis

Dana Berman

E-mail address: dana.berman@mail.mcgill.ca
URL: http://cs.mcgill.ca/~dberma3/

Last Updated: February 17, 2019.
MATH 366 LECTURE NOTES, MCGILL UNIVERSITY.
Instructor: Professor Sara Harrison.

Contents

Chapter 1. Introduction to Complex functions 1
§1. Lecture 1 1
§2. Lecture 2 4
§3. Lecture 3 6
§4. Lecture 4 9
Exercises and Solutions 12

Chapter 2. Cauchy's theorem and Applications 19
§1. Lecture 5 19
§2. Lecture 6 22
§3. Lecture 7 25
§4. Lecture 8 29
Exercises and solutions 31

Chapter 3. Meromorphic functions 37
§1. Lecture 9 37
§2. Lecture 10 40
§3. Lecture 11 43
§4. Lecture 12 46
Exercises and solutions 48

Chapter 4. Entire functions 53
§1. Lecture 13 53

§2. Lecture 14 56
§3. Lecture 15 60
§4. Exercises and Solutions 63
Chapter 5. The Gamma and Zeta functions 67
§1. Lecture 16 67
§2. Lecture 17 70
§3. Lecture 18 72
§4. Exercises and Solutions 75
Chapter 6. Conformal Maps 79
§1. Lecture 19 79
§2. Lecture 20 81
§3. Lecture 21 84
§4. Lecture 22 86
§5. Exercises and Solutions 88

Chapter 1

Introduction to Complex functions

Lecture 1

In this course we will, broadly speaking, study functions of complex variables. One central idea will be complex differentiability. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be "complex differentiable", or holomorphic if the following limit is well-defined:

$$\lim_{|h|\to 0} \frac{f(z+h) - f(z)}{h} \qquad \text{where } h \in \mathbb{C}.$$

Such functions have certain particular characteristics. For instance, if f is holomorphic in a domain Ω and γ is a closed curve whose interior is contained in Ω, then

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

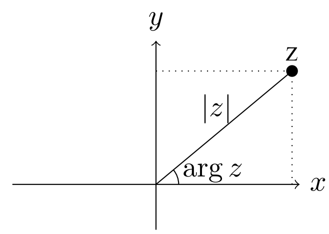
Furthermore, if f is holomorphic then it is infinitely differentiable. Another interesting fact is that if f and g are holomorphic and equal on some region, then $f = g$ on the entire complex plane.

All of the above facts will be proven during the course of the semester, which will end with Newman's proof of the Prime Number Theorem.

Certain results used in this course are known results from analysis or calculus. We therefore recommend that the reader has a solid foundation in both subjects before attacking these notes.

Complex Numbers. We now begin to present the course material. We first construct the complex numbers, denoted $\mathbb{C}$, from the reals $\mathbb{R}$. We define an element $\imath$ which is such that

$i^2 = -1$ and let the complex numbers be the set of numbers of the form $z = x + iy$ where $x, y \in \mathbb{R}$. If $y = 0$ then we say that z is simply a real number and if $x = 0$ we say that z is purely imaginary. In this notes, it is assumed that for any $z \in \mathbb{C}$, unless otherwise specified, x denotes the real part of z and y the imaginary part of z. We often visualize the complex numbers on a plane:



We will often make use of the **norm** of a complex number, which we define by $|z| = |x + iy| = \sqrt{x^2 + y^2}$. We say that a complex number is "small" if it is small in norm. The **argument** of a complex number $z = x + iy$ is the angle between z and the x-axis, i.e. $\arg z = \tan^{-1}(y/x)$. Furthermore, the **conjugate** of z is given by $\overline{z} = x - iy$ and note that $|z| = z \cdot \overline{z}$. Finally, the real and imaginary parts of z are

$$\Re(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \Im(z) = \frac{z - \overline{z}}{2}$$

Polar Coordinates. We first define the exponential function on the complex plane. Given a complex number $z = x + iy$

$$e^z = e^x e^{iy} = e^x \left(\cos(y) + i \sin(y)\right)$$

Later on, it will become evident why this definition is reasonable. Using, the above, any complex number z may be written in polar coordinates, i.e. $z = re^{i\theta}$ for some $r > 0$, $\theta \in [0, 2\pi)$. In this case, $|z| = r$ and θ is the argument of z. Note that for any $z, w \in \mathbb{C}$, we have $\arg(z \cdot w) = \arg z + \arg w$.

Useful equalities and inequalities. Proving the below identities is left as an easy exercise to the reader.

a) $|z + w|^2 = |z|^2 + |w|^2 + 2\Re(z \cdot w)$. In particular, if $zw = 0$ then $|z + w|^2 = |z|^2 + |w|^2$.
b) $-|z| \leq \Re(z) \leq |z|$
c) $-|z| \leq \Im(z) \leq |z|$
d) $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$ for any $n \in \mathbb{N}$
e) $||z| - |w|| \leq |z - w| \leq |z| + |w|$

Lemma 1.1 (Cauchy's inequality). *For any $n \in \mathbb{N}$ and complex numbers $z_1, w_1, z_2, w_2, \ldots, z_n, w_n$,*

$$\left| \sum_{i=1}^{n} z_i w_i \right|^2 \leq \left(\sum_{i=1}^{n} |z_i|^2 \right) \left(\sum_{i=1}^{n} |w_i|^2 \right)$$

Proof. For any $\lambda \in \mathbb{C}$, it holds that

$$\sum_{i=1}^{n} |z_i - \lambda \overline{w_i}|^2 = \sum_{i=1}^{n} |z_i|^2 + |\lambda|^2 \sum_{i=1}^{n} |w_i|^2 - 2\Re\left(\overline{\lambda} \sum_{i=1}^{n} z_i w_i\right) \geq 0 \tag{1}$$

Choosing $\lambda = \frac{1}{\sum_{i=1}^{n} |w_i|^2} \sum_{i=1}^{n} z_i w_i$, we have

$$\lambda \sum_{i=1}^{n} \overline{z_i w_i} = \frac{1}{\sum_{i=1}^{n} |w_i|^2} \left| \sum_{i=1}^{n} z_i w_i \right|^2$$

Combining the above with (1) we obtain

$$\sum_{i=1}^{n} |z_i|^2 + \frac{1}{\sum_{i=1}^{n} |w_i|^2} \left| \sum_{i=1}^{n} z_i \cdot w_i \right|^2 - \frac{2}{\sum_{i=1}^{n} |w_i|^2} \left| \sum_{i=1}^{n} z_i \cdot w_i \right|^2 \geq 0$$

Rearranging the terms we obtain the desired inequality. □

Remark 1.1. The above lemma is a special case of the Cauchy-Schwartz inequality. Indeed, the complex numbers form a *Hilbert space* with multiplication being the inner product.

Stereographic projection. It is often useful to consider the *extended* complex plane, i.e. the complex plane plus the point at infinity. The point at infinity is an accumulation point of the complex plane and may be algebraic manipulations may be performed under the following conventions; Let $z \in \mathbb{C}$, then

$$z + \infty = \infty, \quad \frac{z}{0} = \infty, \quad \frac{z}{\infty} = 0, \quad 0 \cdot \infty = 0$$

It is also possible to construct a bijection between any sphere and the extended complex plane. Thus, the extended complex plane can be represented by a sphere S, called the Riemann sphere. In order to construct a bijection, we consider a unit sphere (that is of diameter 1) centered at $(0, 0, 1/2)$. In other words, let the xy-plane represent $\mathbb{C}$ and place the sphere "on" the origin. Now, we identify each point in $z \in \mathbb{C}$, with the point on the sphere that intersects the line crossing z and $(0, 0, 1)$. We call the point $(0, 0, 1)$ the north pole and identify it with ∞. Explicitly, this bijection is given by

$$z = x + iy \mapsto \frac{1}{x^2 + y^2 + 1} \left(x, y, x^2 + y^2\right)$$

with inverse

$$(x, y, z) \mapsto \frac{x + iy}{1 - z}$$

Lecture 2

Definition 1.2. *A region $\Omega \subseteq \mathbb{C}$ is a non-empty connected open set. That is, Ω if open and if $A \cup B = \Omega$ for disjoint sets A and B, then one of A and B is empty.*

Definition 1.3. *Let Ω be an open subset of $\mathbb{C}$, then $f : \Omega \to \mathbb{C}$ is said to be continuous at $z_0 \in \Omega$ if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|z - z_0| < \delta$ implies that $|f(z) - f(z_0)| < \varepsilon$. We say that f is continuous on Ω if it is continuous at every point in Ω.*

Lemma 1.4. *A function f is continuous if and only if both the real and imaginary parts of f are continuous.*

Definition 1.5. *We say that f has a maximum at $z_0 \in \Omega$ if for all $z \in \Omega$, we have $|f(z_0)| \geq |f(z)|$.*

Theorem 1.6. *A continuous function on a compact set is bounded and acheives it's maximum on that set.*

Complex Differentiability.

Definition 1.7. *Let $\Omega \subseteq \mathbb{C}$ be an open set and $z_0 \in \Omega$. We say that a function $f : \Omega \to \mathbb{C}$ is complex differentiable (or holomorphic) at z_0 if there exists a complex number $f'(z_0)$ (the derivative of f at z_0) such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon.$$

Lemma 1.8. *A complex-valued function f is holomorphic at z_0 if and only if there exists some number $f'(z_0)$, a real number $\delta > 0$ and a function $\psi : \{h \in \mathbb{C} \mid 0 < |h| < \delta\} \to \mathbb{C}$ such that $\lim_{|h| \to 0} \psi(h) = 0$ and for all $h \in \mathbb{C}$ with $0 < |h| < \delta$*

$$f(z_0 + h) - f(z_0) - f'(z_0) \cdot h = \psi(h) \cdot h$$

Lemma 1.9. *If f, g are holomorphic, then so are $f \cdot g$, $f + g$ and $f \circ g$. Moreover,*

(1) $(f + g)' = f' + g'$
(2) $(f \cdot g)' = f' \cdot g + g' \cdot f$
(3) $(f \circ g)' = (f' \circ g) \cdot g'$

Example 1.2. One can easily show that the function $f(z) = z$ is holomorphic. Thence, by the above lemma any polynomial is holomorphic.

Example 1.3. The function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ defined by $f(z) = 1/z$ is holomorphic and $f'(z) = -1/z^2$.

Example 1.4. The function $f(z) = \overline{z}$ is not holomorphic since the limit of

$$\frac{f(z + h) - f(z)}{h} = \frac{\overline{h}}{h}.$$

does not exists. Indeed, if h is restricted to being real, then the above tends to 1 while if h is restricted to being complex the limit is -1.

Cauchy-Riemann equations. Given a complex functions f, we may write $f(z) = u(x, y) + iv(x, y)$ where x and y are respectively the real and imaginary parts of z and $u, v : \mathbb{R}^2 \to \mathbb{R}$. Alternatively, we may write $f(x + iy) = F(x, y) \cdot \langle 1, i\rangle$ where $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(x, y) = (u(x, y), v(x, y))$. Recall that F is said to be real differentiable with derivative (Jacobian) J_F if

$$\lim_{\|h\|\to 0} \frac{\|F(z+h) - F(z) - J_F(z) \cdot h\|}{\|h\|} = 0$$

If the function f is differentiable, then it's derivative may be computed along the real line and along the complex line. It follows that

$$f'(x+iy) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$f'(x+iy) = \frac{\partial f}{\partial (iy)} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

From the above equalities we obtain the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Corollary 1.10. *The laplacian of the real and imaginary parts of a holomorphic function is always zero.*

Proof. We will assume that the function f is twice complex differentiable.[1] Then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial y^2}$$

It follows that $\nabla^2 u = \Delta u = 0$. Similarly, $\Delta v = 0$. □

Definition 1.11. *We define the following operators;*

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \textit{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Remark 1.5. If f is a holomorphic function, then

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(f' + f'\right) = f'$$

and

$$2\frac{\partial f}{\partial \overline{z}} = \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = f' - f' = 0$$

Proposition 1.12. *If f is a holomorphic at $z = x + iy$, then the function F as we have defined it above is real differentiable at (x, y).*

[1] This will be shown later on in the course that this always holds true for holomorphic fcuntions.

Theorem 1.13. *If u, v are continuously differentiable at (x, y) and satisfy the Cauchy-Rieamann equations, then f is holomorphic.*

Proof. It a well-known result in calculus that if u, v are continuously differentiable then there exists functions $\psi_1(h), \psi_2(h)$ which tend to $|h| = |h_1 + ih_2|$ tends to zero and such that

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x} h_1 + \frac{\partial u}{\partial y} h_2 + |h| \psi_1(h)$$

and

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x} h_1 + \frac{\partial v}{\partial y} h_2 + |h| \psi_2(h).$$

It follows from the above and the Cauchy-Riemann equations that

$$\begin{aligned} f(z+h) - f(z) &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) h_1 + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) h_2 + |h| \left(\psi_1(h) + \psi_2(h)\right) \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) h_1 + \left(i\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right) ih_2 + |h| \left(\psi_1(h) + \psi_2(h)\right) \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) h + |h| \left(\psi_1(h) + \psi_2(h)\right) \end{aligned}$$

Thus

$$\frac{f(z+h) - f(z)}{h} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \frac{|h|}{h} \left(\psi_1(h) + \psi_2(h)\right)$$

Taking the limit as h tends to zero, we find that f is indeed holomorphic with derivative $\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. □

Lecture 3

Complex Power Series. Complex power series are a particular type of complex functions, specifically those of the form:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and } a_n \in \mathbb{C}$$

The first question we ask ourselves is *where does such a sequence converge?* As we have previously remarked, $\mathbb{C}$ is a Hilbert space and therefore also a Banach space. Ergo, every absolutely convergent series is also convergent. Furthermore, if $|w| \leq |z|$ and $f(z)$ converges (where f is as above), then by the comparison test $f(w)$ also converges. Recall the following definitions from analysis;

Definition 1.14. *A series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n z^n|$ converges.*

Definition 1.15. *A sequence of functions $f_n(z)$ defined on a set Ω converges uniformly to $f(z)$ if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $z \in \Omega$, $|f_n(z) - f(z)| < \varepsilon$.*[2]

[2] This varies from point-wise convergence as N may be chosen independently of z.

Example 1.6. Let $f_n(z) = z(1 + 1/n)$, then $\lim_{n\to\infty} f_n(z) = z$ for all z, i.e. $f_n(z) \to z$ pointwise. On the other hand, the value $|f_n(z) - f(z)| = |z|/n$ may not be bounded independently of z, implying that this sequence of functions does not converge uniformly.

Lemma 1.16. *The limit of a uniformly convergent sequence of continuous functions is continuous.*

Proof. Suppose that $\{f_n\}$ is a sequence of continuous functions converging to f uniformly on a neighbourhood V of x. Fix $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that for all $n \geq N$ and $y \in V$, $|f_n(y) - f(y)| < \varepsilon$. Since each f_n is continuous, we may fix $n \geq N$ and pick $\delta > 0$ such that $|f_n(x) - f_n(y)| < \varepsilon$ whenever $|x - y| < \delta$. Combining these facts, we have that whenever $|x - y| < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 3\varepsilon$$

This completes the proof. □

Proposition 1.17. *Given a series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, *there exists* $0 \leq R \leq \infty$ *called the radius of convergence such that*

(1) If $|z| < R$, *the series is absolutely convergent.*
(2) If $|z| > R$ *the series is divergent.*

Therefore, the disk of convergence of f *is* $D_R(0) = \{z \in \mathbb{C} \mid |z| < R\}$. *Furthermore,* R *is given by Hadamard's formula:*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$$

Proof. Suppose first that $|z| > R$, then

$$\limsup_{n\to\infty} |a_n| |z|^n = \limsup_{n\to\infty} \left(|a_n|^{1/n} |z| \right)^n$$

Let $\{a_{n_k}\}$ be a subsequence of a_n such that $|a_{n_k}|^{1/n_k} \to 1/R$. Then for all sufficiently large k, we must have that $|a_{n_k}|^{1/n} |z| > 1$. It follows that $\limsup_{n\to\infty} |a_n| |z|^n \geq 1$ which shows that the series must diverge.

We now consider the case $|z| < R$. Pick $\varepsilon > 0$ such that $\varepsilon < 1/|z| - L$ where $L = 1/R$. We may find $N \in \mathbb{N}$ such that $|a_n|^{1/n} \leq L + \varepsilon$ whenever $n \geq N$. Then for all $n \geq N$

$$|a_n|^{1/n} |z| \leq (L + \varepsilon) |z| < \left(L + \frac{1}{|z|} - L \right) |z| = 1$$

Finally, defining $r := (L + \varepsilon) |z| < 1$, we see that $f(z)$ converges absolutely since

$$\sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} |a_n z^n| \leq \sum_{n=0}^{N-1} |a_n z^n| + \sum_{n=N}^{\infty} r^n < \infty$$

□

Theorem 1.18. *A complex-valued series is holomorphic in it's disk of convergence and it's derivative is given by differentiating each individual term. In particular, any power series is infinitely complex differentiable in it's disk of convergence.*

Proof. Consider a function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence R and define

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

We first wish to show that g has the same radius of convergence as f. By proposition 1.17, this is indeed the case since

$$\limsup_{n\to\infty} |na_n|^{1/n} = \left(\lim_{n\to\infty} n^{1/n}\right)\left(\limsup_{n\to\infty} |a_n|^{1/n}\right) = \limsup_{n\to\infty} |a_n|^{1/n}$$

Furthermore, we claim that f is holomorphic and g is it's derivative. Define

$$S_N(z) = \sum_{n=0}^{N} a_n z^n \qquad \text{and} \qquad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

For fixed z in the disk of convergence, there exists r such that $|z| < r < R$ and $h \in \mathbb{C} \setminus 0$ sufficiently small so that $|z+h| < r$. Then for any $N \in \mathbb{N}$

$$\begin{aligned} &\frac{f(z+h) - f(z)}{h} - g(z) \\ &= \left(\frac{S_N(z+h) - S_N(z)}{h} - S_N'(z)\right) + \left(S_N'(z) - g(z)\right) + \left(\frac{E_N(z+h) - E_N(z)}{h}\right) \end{aligned} \tag{2}$$

Moreover, we have

$$\left|\frac{E_N(z+h) - E_N(z)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z+h)^n - z^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n|\, n r^{n-1}$$

where the last step follows from the identity $A^N - B^N = (A-B)(A^{N-1} + A^{N-2}B + \cdots + B^{N-1})$. Note that we were free to change the order of summation by absolute convergence. Finally, the right hand side tends to 0 as $N \to \infty$ since g converges absolutely in $D_R(0)$ and $r < R$.

Likewise, since g is absolutely convergent in the $D_R(0)$, $|S_N'(z) - g(z)| \xrightarrow{N\to\infty} 0$. Ergo, for any $\varepsilon > 0$, we may find $M \in \mathbb{N}$ such that whenever $N \geq M$,

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| = \frac{S_N(z+h) - S_N(z)}{h} - S_N'(z) + \varepsilon \xrightarrow{|h|\to 0} \varepsilon$$

since S_N is holomorphic for any $N \in \mathbb{N}$. ∎

Lecture 4

In the last lecture we developed some basic theory on power series. We now give some examples of power series which are relevant in complex analysis.

Example 1.7. The complex geometric series is as follows:

$$f(z) = \sum_{n=0}^{\infty} z^n$$

The above converges in the disk $|z| < 1$ and on that disk, $f(z) = \frac{1}{1-z}$.

Example 1.8 (The exponential function)**.** The power series for the exponential function on $\mathbb{R}$ is given by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

and converges for any $z \in \mathbb{R}$. We therefore define $\exp(z)$ for complex z by the above power series which converges for $z \in \mathbb{C}$, since

$$\sum_{n=0}^{\infty} \frac{|z|^n}{n!} < \infty$$

In conclusion, we have defined e^z for any $z \in \mathbb{C}$ and it is a holomorphic function since it is given by a convergent power series. By theorem 1.18, we may that the derivative of e^z in the complex plane is itself by differentiating each individual term. Later one in the course, we will show that the extension of the exponential function to a holomorphic function on the complex plane is in fact unique. Specifically, this is a corollary of the *Identity theorem*, also called the principle of *analytic continuation*.

Example 1.9 (Trigonometric functions)**.** As for the exponential function, we may extended sine and cosine (uniquely) to a holomorphic function on $\mathbb{C}$. For all $z \in \mathbb{C}$, we define

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad \sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Furthermore, by theorem 1.18, we find the derivative of $\cos(z)$ to be $-\sin(z)$ and the derivative of $\sin(z)$ to be $\cos(z)$.

Remark 1.10 (Euler's formula)**.** For any $x \in \mathbb{R}$, we have

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n \text{ even}}^{\infty} \frac{(-1)^{n/2} x^n}{n!} + i \sum_{n \text{ odd}}^{\infty} \frac{(-1)^{(n-1)/2} x^n}{n!} = \cos(x) + i\sin(x)$$

This identity is called Euler's formula and actually holds for any $z \in \mathbb{C}$. We leave the proof of this identity for $z \in \mathbb{C}$ as an exercise which can be done (tediously) by brute force, or more elegantly after proving the identity theorem. Algebraically manipulating Euler's formula also yield the following identities;

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \quad \text{and} \quad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2}$$

Line integrals. An important result in complex analysis which we mentioned in our introduction is Cauchy's theorem. Loosely, it states that if $f(z)$ is holomorphic in the interior of a closed curve γ then

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

We first introduce some notation and theorems from calculus. The next chapter is dedicated to proving the aforementioned theorem and studying it's applications.

Remark 1.11. Given a function $\phi : \mathbb{R} \to \mathbb{C}$,

$$\int \phi(x)\,\mathrm{d}x = \int \Re(\phi(x))\,\mathrm{d}x + i \int \Im(\phi(x))\,\mathrm{d}x$$

Definition 1.19. *We define a parameterized curve to be a function $z : [a,b] \subseteq \mathbb{R} \to \mathbb{C}$. We say that a curve is smooth if it is continuously differentiable on $[a,b]$ and it's derivative does not vanish on $[a,b]$. At $t = a, b$ we define the derivative by*

$$z'(a) = \lim_{h\to 0^+} \frac{z(a+h) - z(a)}{h} \quad \textit{and} \quad z'(b) = \lim_{h\to 0^-} \frac{z(b+h) - z(b)}{h}$$

The curve is called piece-wise smooth *if z is continuous on $[a,b]$ and there exists a finite set of points $a = t_1 < t_2 < \cdots < t_n = b$ such that for all $1 \le k < n$, z is continuously differentiable on $[t_k, t_{k+1}]$.*

Definition 1.20. *Two parameterization $z(t) : [a,b] \to \mathbb{C}$ and $\tilde{z}(s) : [c,d] \to \mathbb{C}$ are equivalent if there exists a bijection $t : [c,d] \to [a,b]$ such that $t'(s)$ exists and is positive and*

$$\tilde{z}(s) = z(t(s))$$

Note that z is smooth if and only if $\tilde{z}$ is smooth.

Definition 1.21. *A smooth curve $\gamma \subseteq \mathbb{C}$ is defined by a family of equivalent smooth parameterizations. We define γ^- to be the curve obtained by γ when the orientation is reversed. The curveγ is said to be simple if it is not self-intersecting and closed if $z(a) = z(b)$ where $z : [a,b] \to \mathbb{C}$ is a parameterization of γ*

A piecewise smooth curve is defined similarly.

Example 1.12. A circle $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$ with centre z_0 and radius r is a smooth curve. An example of a parameterization of the circle with positive orientation is $z(t) = z_0 + re^{it}$ where t ranges in $[0, 2\pi]$.

Definition 1.22. *Given a a piece-wise smooth curve with parameterization $z : [a, b] \to \mathbb{C}$*

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} f(z(t))z'(t)\,\mathrm{d}t$$

where $a = t_1 < t_2 < \cdots < t_n = b$ are such that z is continuously differentiable on each interval $[t_k, t_{k+1}]$. Moreover, the length of γ is given by

$$\ell(\gamma) = \int_a^b \left|z'(t)\right|\,\mathrm{d}t$$

Lemma 1.23. *The integral of a function over a curve is a well-defined concept, i.e. it is independent of equivalent parametrizations.*

Proof. It suffices to prove the lemma for smooth curves. Suppose that $z(t) : [a, b] \to \mathbb{C}$ and $\tilde{z}(s) : [c, d] \to \mathbb{C}$ are equivalent parametrizations. Then

$$\int_a^b f(z(t))z'(t)\,\mathrm{d}t = \int_a^b f(z(t(s)))z'(t(s))t'(s)\,\mathrm{d}s = \int_a^b f(\tilde{z}(s))\tilde{z}'(s)\,\mathrm{d}s$$

A similar proof shows that the length of a curve is also independent of parametrizations. $\square$

Proposition 1.24. *Given functions f, g and a curve γ,*

(1) For all $\alpha, \beta \in \mathbb{C}$,

$$\int_\gamma \alpha f(z) + \beta g(z)\,\mathrm{d}z = \alpha \int_\gamma f(z)\,\mathrm{d}z + \beta \int_\gamma g(z)\,\mathrm{d}z$$

(2) $\displaystyle\int_\gamma f(z)\,\mathrm{d}z = -\int_{\gamma^-} f(z)\,\mathrm{d}z$

(3) $\displaystyle\left|\int_\gamma f(z)\,\mathrm{d}z\right| \leq \sup_{z\in\gamma} |f(z)|\,\ell(\gamma)$

Definition 1.25. *A* primitive *of a function f on an open set Ω is a function F which is holomorphic on Ω and such that*

$$F'(z) = f(z) \quad \textit{for all } z \in \Omega$$

Theorem 1.26 (Fundamental theorem of calculus)**.** *If a continuous function f has a primitive F in Ω and γ is a piece-wise smooth curve in Ω which begins at ω_1 and ends at ω_2, then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(\omega_2) - F(\omega_1)$$

Proof. Suppose first that γ is smooth and let z is a parameterization of γ.

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b F'(z(t))z'(t)\,\mathrm{d}t = \int_a^b (F(z(t)))'\,\mathrm{d}t = F(z(b)) - F(z(a))$$

If γ is piece-wise smooth and z is a parameterization of γ then

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^{n-1} \int_{t_k}^{t_{k+1}} f(z)\,\mathrm{d}z = \sum_{k=1}^{n-1} \left[F\left(z\left(t_{k+1}\right)\right) - F\left(z\left(t_k\right)\right)\right] = F\left(z\left(t_n\right)\right) - F\left(z\left(t_1\right)\right)$$
$$= F\left(\omega_2\right) - F\left(\omega_1\right)$$

□

Corollary 1.27. *If the conditions in the theorem hold and γ is closed, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Remark 1.13. At the beginning of this section, we mentioned that an important result in complex analysis is

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for *almost any* closed curve γ and holomorphic function f. By the above corollary, it is sufficient to show that there exists a region containing γ and it's interior on which f has a primitive.

Corollary 1.28. *If a function f is holomorphic on Ω and $f' \equiv 0$, then f is constant.*

Proof. For any two point $a, b \in \mathbb{C}$, let γ be a simple curve from a to b, then

$$f(b) - f(a) = \int_\gamma f' = 0$$

□

Example 1.14. Consider the function $f(z) = 1/z$. This function is holomorphic on every point except 0. Let C denote the unit circle centered at the origin. We may parametrize C by $z(t) = e^{it}$ where $t \in [0, 2\pi]$. Then

$$\int_C f = \int_0^{2\pi} f(e^{it}) i e^{it}\,\mathrm{d}t = \int_0^{2\pi} i = 2\pi i$$

Exercises and Solutions

Problem 1. Prove that the complex numbers cannot be well ordered.

Solution. By way of contradiction, suppose that there exists a relation $\prec$ on $\mathbb{C}$. We consider the two only possibilities: $i \succ 0$, and $i \prec 0$.

(1) Suppose first that $i \succ 0$. Then we may multiply both sides of the equation by i to obtain that $i^2 = -1 \succ 0$. Multiplying $-1 \succ 0$ by -1 on both side, we also find that $1 \succ 0$. Thus,

$$0 = 1 + (-1) \succ 0 + 0 = 0$$

which is absurd.

(2) Likewise, if $i \prec 0$. we first add $(-i)$ on both sides to obtain $(-i) \succ 0$. Then we may multiply both sides of the equation by $(-i)$ to obtain $(-i)^2 = -1 \succ 0$. Multiplying again by -1, we also find that $1 \succ 0$ whence

$$0 = 1 + (-1) \succ 0 + 0 = 0$$

which is clearly a contradiction.

□

Problem 2. Define

$$\mathcal{S} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$

Prove that there exists an isomorphism between $\mathcal{S}$ and $\mathbb{C}$ with respect to both addition and multiplication.

Solution. We define the following map and prove that it is an isomorphism between $\mathcal{S}$ and the complex plane:

$$\phi\left(\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}\right) = \alpha + i\beta$$

It is clear that this map is bijective. To see that ϕ is a homomorphism with respect to addition, let $A, B \in \mathcal{S}$ have entries a, b and c, d respectively, then

$$\begin{aligned}\phi(A) + \phi(B) = \phi\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) + \phi\left(\begin{bmatrix} c & d \\ -d & c \end{bmatrix}\right) = a + ib + c + id &= (a + c) + i\,(b + d) \\ &= \phi(A + B)\end{aligned}$$

Likewise, ϕ is a homomorphism with respect to multiplication;

$$\begin{aligned}\phi(A) \cdot \phi(B) = (a + ib) \cdot (c + id) = (ac - bd) + i\,(ad + bc) &= \phi\left(\begin{bmatrix} ac - bd & ad + bc \\ -(ad + bc) & ac - bd \end{bmatrix}\right) \\ &= \phi(A \cdot B)\end{aligned}$$

□

Problem 3. For fixed $w \in \mathbb{C}$, the mapping

$$F : \mathbb{C} \to \mathbb{C} \qquad z \mapsto \frac{w - z}{1 - \overline{w}z}$$

is called a Blaschke factor. Prove that if $\overline{z}w \neq 1$, then

$$\frac{w - z}{1 - \overline{w}z} < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1$$

and

$$\frac{w - z}{1 - \overline{w}z} = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1$$

Moreover, prove that if $w \in \mathbb{D}$ then F is a bijective map from the unit disk to itself.

Solution. A series of algebraical manipulations show that

$$\frac{|w-z|}{|1-\overline{w}z|} \leq 1$$

if and only if

$$\begin{aligned} |w-z|^2 \leq |1-\overline{w}z|^2 &\iff |w|^2 - \overline{w}z - w\overline{z} + |z|^2 \leq 1 - \overline{w}z - w\overline{z} + |w|^2 |z|^2 \\ &\iff 0 \leq (1-|w|)(1-|z|) \end{aligned}$$

Indeed, the above inequality holds whenever $|w|, |z| \leq 1$. Furthermore, we have a strict inequality if $|w|, |z| < 1$ and equality if $|z| = 1$ or $|w| = 1$.

Fix $w \in \mathbb{D}$. To see that $F : \mathbb{D} \to \mathbb{D}$ is a bijection, it suffices to note that

$$F \circ F(z) = \frac{w(1-\overline{w}z) - w + z}{1 - \overline{w}z - \overline{w}(w-z)} = \frac{z - |w|^2 z}{1 - |w|^2} = z$$

That is, F is it's own inverse. □

Problem 4. Consider the bijection between the sphere $\mathcal{S}$ of radius 1 centered at the origin and the extended complex plane given by a stereographic projection. Prove that two points $z, w \neq \mathcal{N} = (0,0,1)$ correspond to opposite sides of the sphere if and only if $z\overline{w} = 1$.

Solution. The bijection between $\mathcal{S}$ and $\overline{\mathbb{C}}$ is given by

$$f : \mathcal{S} \to \overline{\mathbb{C}}, \quad (x,y,z) \mapsto \frac{x+iy}{1-z} \qquad \text{and} \qquad f^{-1}(z) : \overline{\mathbb{C}} \to \mathcal{S}, \quad z \mapsto \left(\frac{2a}{|z|^2+1}, \frac{2b}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

where $z = a + ib$. Suppose first that $z, w \in \mathbb{C}$ correspond to diametrically opposite points of $\mathcal{S} - \{\mathcal{N}\}$. Then, writing $z = a + ib$, the point z corresponds to

$$\left(\frac{2a}{|z|^2+1}, \frac{2b}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)$$

It follows that w maps to

$$w = f\left(\frac{-2a}{|z|^2+1}, \frac{-2b}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right) = \frac{\frac{-2a}{|z|^2+1} + i\frac{-2b}{|z|^2+1}}{1 - \frac{1-|z|^2}{|z|^2+1}} = -\frac{a+ib}{|z|^2} = \frac{-z}{|z|^2}$$

It follows that $w|z|^2 = w\overline{z}z = -z$ and hence $w\overline{z} = -1$.

For the converse direction, suppose now that $z\overline{w} = -1 = w\overline{z}$ where $z = a + ib$. Then

$$w = \frac{-z}{|z|^2} = \frac{-a}{|z|^2} + i\frac{-b}{|z|^2} \quad \text{and} \quad |w| = \left| \frac{-z}{|z|^2} \right| = \frac{1}{|z|}$$

Thence,

$$f^{-1}(w) = \left(\frac{-2a/|z|^2}{|w|^2+1}, \frac{-2b/|z|^2}{|w|^2+1}, \frac{|w|^2-1}{|w|^2+1} \right) = \left(\frac{-2a}{|z|^2+1}, \frac{-2b}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right) = -f^{-1}(z)$$

That is, z and w correspond to diametrically opposite sides of the sphere. □

Problem 5. Prove Abel's summation formula, that is show that for any finite sequences $\{a_n\}_{n=1}^N$, $\{b_n\}_{n=1}^N$ of complex numbers and $M \le N$,

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n$$

where $B_n = \sum_{k=1}^n b_k$ and $B_0 = 0$. Then use Abel's summation formula to prove that for any sequence of complex numbers $\{a_n\}_{n=1}^\infty$,such that $\sum_{n=1}^\infty$ converges, the limit of $\sum_{n=1}^\infty r^n a_n$ as $r \to 1^-$ evaluates to $\sum_{n=1}^\infty a_n$.

Solution.

$$\begin{aligned}
\sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n &= \sum_{n=M}^{N-1} a_{n+1} B_n - \sum_{n=M}^{N-1} a_n B_n = \sum_{n=M+1}^{N} a_n B_{n-1} - \sum_{n=M}^{N-1} a_n B_n \\
&= a_N B_{N-1} - a_M B_M + \sum_{n=M+1}^{N-1} a_n (B_{n-1} - B_n) \\
&= a_N B_{N-1} - a_M B_M - \sum_{n=M+1}^{N-1} a_n b_n \\
&= a_N B_{N-1} + a_N b_N - a_M B_M + a_M b_M - \sum_{n=M}^{N} a_n b_n \\
&= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N} a_n b_n
\end{aligned}$$

For the second part of the problem, fix $r < 1$, and denote $A_n = \sum_{k=1}^n a_n$. By Abel's summation formula,

$$\sum_{n=1}^{N} r^n a_n = r^N A_N - r A_0 - \sum_{n=1}^{N-1} \left(r^{n+1} - r^n \right) A_n = r^N A_N + (1-r) \sum_{n=1}^{N-1} r^n A_n$$

Letting N tend to infinity and denoting by A the sum $\sum_{n=1}^\infty a_n$, we have

$$\sum_{n=1}^{\infty} r^n a_n - \sum_{n=1}^{\infty} a_n = \left[(1-r) \sum_{n=1}^{\infty} r^n A_n \right] - A = (1-r) \sum_{n=1}^{\infty} r^n (A_n - A)$$

Since $A_n \to A$ as $n \to \infty$ we may fix $\varepsilon > 0$ and find N be such that for all $n \geq N$, we have $|A_n - A| < \varepsilon$. Then if M denotes the maximum of $|A_n - A|$ for $1 \leq n \leq N-1$, we have

$$\begin{aligned}\left|\sum_{n=1}^{\infty} r^n a_n - \sum_{n=1}^{\infty} a_n\right| &\leq (1-r)\sum_{n=1}^{N-1} r^n |A_n - A| + (1-r)\sum_{n=N}^{\infty} r^n |A_n - A| \\ &\leq (1-r)\sum_{n=1}^{N-1} r^n M + (1-r)\sum_{n=N}^{\infty} r^n \varepsilon \\ &\leq \left(r - r^N\right) M + \varepsilon \xrightarrow{r \to 1^-} \varepsilon\end{aligned}$$

Since ε was picked arbitrarily, this concludes the proof. □

Problem 6. Show that

(1) $\sum_{n=1}^{\infty} nz^n$ does not converges on any point of the unit circle (i.e. $\{z \in \mathbb{C} \mid |z| = 1\}$),
(2) $\sum_{n=1}^{\infty} z^n/n^2$ converges on every point of the unit circle,
(3) $\sum_{n=1}^{\infty} z^n/n$ converges on every point of the unit circle except $z = 1$.

Solution.

(1) Suppose that $|z| = 1$, then

$$\lim_{n\to\infty} |nz^n| = \lim_{n\to\infty} n = \infty$$

Since the above limit is not 0, the series diverges on the unit circle.

(2) The second sum converges absolutely on the unit circle since

$$\sum_{n=1}^{\infty} \left|\frac{z^n}{n^2}\right| = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < \infty$$

(3) From analysis/calculus, it is known that the sum does not converge if $z = 1$. For any $z \in \mathbb{C}, z \neq 1$ such that $|z| = 1$, since the terms of the series z^n/n tend to 0 as $n \to \infty$, it suffices to show that the series is bounded. Define $s_n = \sum_{k=1}^{n} z^k$, then

$$|s_n| = \frac{\left|z - z^{n+1}\right|}{|1-z|} \leq \frac{2}{|1-z|} \tag{2}$$

By Abel's summation formula,

$$\begin{aligned}\sum_{n=1}^{N} \frac{z^n}{n} = \frac{1}{N}s_N - \sum_{n=1}^{N-1} s_n = \frac{1}{N}s_N - \sum_{n=1}^{N-1} \frac{z - z^{n+1}}{1-z} &= \frac{1}{N}s_N - \frac{1}{1-z}\sum_{n=1}^{N-1} z + \frac{1}{1-z}\sum_{n=1}^{N-1} z^{n+1} \\ &= \frac{1}{N}s_N - \frac{1}{1-z}s_{N-1} + \frac{1}{1-z}\left(s_N - z\right)\end{aligned}$$

By equation (2),

$$\left|\sum_{n=1}^{N}\frac{z^n}{n}\right| \le \frac{1}{N}\frac{2}{|1-z|} + \frac{2}{|1-z|^2} + \frac{2}{|1-z|^2} + \frac{1}{|1-z|} \le \frac{4}{|1-z|^2} + \frac{3}{|1-z|}$$

□

Problem 7. Compute the following integrals

(1) $\int_\gamma z^n \, \mathrm{d}z$ where γ is a circle with positive orientation centered at the origin,

(2) $\int_\gamma z^n \, \mathrm{d}z$ where γ is a circle which does not contain the origin,

(3) $\int_\gamma (z-a)^{-1}(z-b)^{-1} \, \mathrm{d}z$ where γ is a circle with positive orientation centered at the origin of radius r and $|a| < r < |b|$.

We encourage the reader to memorize these integrals as they will be useful in the next chapter.

Solution.

(1) Letting $r > 0$ denote the radius of the circle, if $n \neq -1$ then

$$\int_\gamma z^n \, \mathrm{d}z = \int_0^{2\pi} \left(re^{it}\right)^n ire^{it} \, \mathrm{d}t = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, \mathrm{d}t = ir^{n+1} \left. \frac{e^{i(n+1)t}}{i(n+1)} \right|_0^{2\pi} = 0$$

If $n = -1$,

$$\int_\gamma z^{-1} \, \mathrm{d}z = \int_0^{2\pi} \left(re^{it}\right)^{-1} ire^{it} \, \mathrm{d}t = \int_0^{2\pi} i = 2\pi i$$

(2) If $n \neq -1$, z^n has primitive $z^{n+1}/(n+1)$ on γ and it's interior, thus by corollary 1.27, the integral evaluates to 0. We therefore know from class results that the integral evaluates to 0. If $n = -1$, then

$$F(z) = F(re^{i\theta}) = \log(r) + i\theta$$

where we restrict the domain of θ to some interval of length 2π is a primitive of z^{-1}, hence the integral still evaluates to 0. The above function is a generalization of the logarithmic function to the complex plane. We will cover the use of this function in detail in the next chapter.

(3) By the two previous parts, if γ_a is the circle centered at a and γ_b is the circle centered at b (with the same radius as γ) then

$$\begin{aligned}\int_\gamma \frac{1}{(z-a)(z-b)} \, \mathrm{d}z &= \int_\gamma \frac{1}{(a-b)(z-a)} + \frac{1}{(b-a)(z-b)} \, \mathrm{d}z \\ &= \frac{1}{a-b}\int_{\gamma_a} z^{-1} \, \mathrm{d}z + \frac{1}{b-a}\int_{\gamma_b} z^{-1} \, \mathrm{d}z = \frac{2\pi i}{a-b}\end{aligned}$$

□

Cauchy's theorem and Applications

Lecture 5

In the previous lecture, we mentioned that in order to prove Cauchy's theorem, it suffices to show that (under certain conditions), a holomorphic function has a primitive. The below theorem is a first important step in this direction.

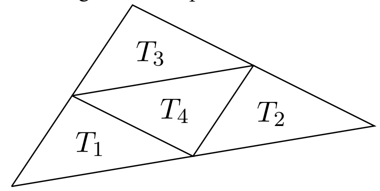
Theorem 2.1 (Goursat's Theorem)**.** *Given an open set $\Omega \subseteq \mathbb{C}$, a holomorphic function $f : \Omega \to \mathbb{C}$ and a triangle T contained in Ω,*

$$\int_T f = 0$$

Proof. For any triangle T, the notation $z \in T$ means that z is a point on the triangle T or in it's interior. This abuse of notation extends to set inclusions. Let T with diameter d and perimeter p. Separate T into four triangles T_1, T_2, T_3, T_4 so that

$$\int_T f = \int_{T_1} f + \int_{T_2} f + \int_{T_3} f + \int_{T_4} f \tag{3}$$

Figure 1. Separation of T



The diameter of each of the above triangles is at most $d/2$ and the perimeter is at most $p/2$. If $1 \leq n_1 \leq 4$ be the integer maximizing the magnitude of $\int_{T_k} f$ then

$$\left|\int_T f\right| \leq 4\left|\int_{T_{n_1}} f\right|$$

Repeating the above process for T_{n_1}, we find a triangle $T_{n_2} \supset T_{n_1}$ with diameter at most $d/2^2$ and perimeter at most $p/2^2$ such that

$$\left|\int_T f\right| \leq 4\left|\int_{T_{n_1}} f\right| \leq 4^2\left|\int_{T_{n_2}} f\right|$$

Repeating this process for every positive integer yields a sequence of triangles $\{T_{n_k}\}_{k=1}^{\infty}$ such that for every $k \in \mathbb{N}$, the diameter of T_{n_k} is at most $d/2^k$, the perimeter of T_{n_k} is at most $p/2^k$ and

$$\left|\int_T f\right| \leq 4^k\left|\int_{T_{n_k}} f\right| \tag{4}$$

Let z_0 to be any point in $\bigcap_{k\in\mathbb{N}} T_{n_k}$. By lemma 1.8, we may find a function ψ such that $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$ for all z in T and $\psi(z) \to 0$ as $z \to z_0$. By equation (4),

$$\begin{aligned}\left|\int_T f\right| \leq 4^k\left|\int_{T_{n_k}} f\right| &\leq 4^k\left(\underbrace{\left|\int_{T_{n_k}} f(z_0)\,\mathrm{d}z\right|}_{=0} + \underbrace{\left|\int_{T_{n_k}} f'(z_0)(z-z_0)\,\mathrm{d}z\right|}_{=0} + \left|\int_{T_{n_k}} \psi(z)(z-z_0)\,\mathrm{d}z\right|\right) \\ &\leq 4^k \sup_{z\in T_{n_k}} \psi(z)\frac{d}{2^k}\frac{p}{2^k} = d\cdot p \sup_{z\in T_{n_k}} \psi(z) \xrightarrow{k\to\infty} 0\end{aligned}$$

□

Corollary 2.2. *Goursat's theorem holds for any polygon since polygons may be sectioned into triangles.*

Cauchy's Theorem.

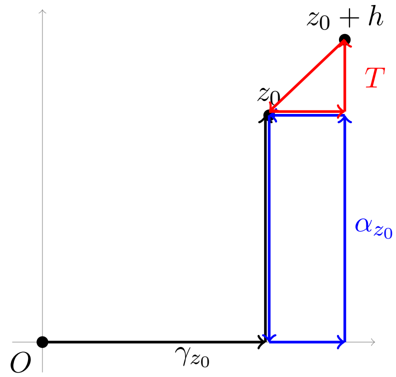
Theorem 2.3. *Let f be a holomorphic function in an open disk D, then f has a primitive.*

Proof. Suppose without loss of generality that D is centered at the origin and define

$$F : D \to \mathbb{C}, \qquad z_0 \mapsto \int_{\gamma_{z_0}} f(z)\,\mathrm{d}z$$

where γ_{z_0} is the piecewise smooth curve formed by taking the union of the line segments from 0 to $\Re(z_0)$ and from $\Re(z_0)$ to z_0.

Figure 2. Paths from z_0 to $z_0 + h$



Fix $z_0 \in \mathbb{C}$ and let $h \neq 0$ and a complex number h sufficiently small so that $z + h \in D$. Then

$$F(z_0 + h) - F(z_0) = \int_{\alpha_{z_0}} f(z)\, \mathrm{d}z + \int_T f(z)\, \mathrm{d}z + \int_z^{z_0+h} f(z)\, \mathrm{d}z$$

where α_{z_0} is the positively oriented rectangle with end-points $\Re(z_0)$, $\Re(z_0+h)$, $\Re(z_0+h)+\imath\Im(z_0)$ and z_0 and T is the positively oriented triangle with end-points z_0, $\Re(z_0+h)+\imath\Im(z_0)$ and $z_0 + h$.

By Goursat's Theorem 2.1 and it's corollary 2.2,

$$F(z_0 + h) - F(z_0) = \int_z^{z_0+h} f(z)\, \mathrm{d}z$$

By continuity of f, there exists a function ψ such that for all $z \in \mathbb{C}$ with $|z - z_0| < h$, $f(z) = f(z_0) + \psi(z)$ and $\psi(z) \to 0$ as $z \to z_0$.

$$\frac{F(z_0 + h) - F(z_0)}{h} = \frac{1}{h} \int_z^{z_0+h} f(z_0) + \psi(z)\, \mathrm{d}z = f(z_0) + \int_z^{z_0+h} \psi(z)\, \mathrm{d}z$$

Letting $h \to 0$, we conclude $F'(z_0) = f(z_0)$. □

Corollary 2.4 (Cauchy's theorem in a disk)**.** *It follows from the above theorem an corollary 1.27, that if f is holomorphic in a disk and γ is a curve contained in the disk, then*

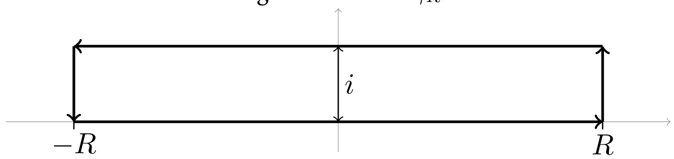
$$\int_\gamma f(z)\, \mathrm{d}z = 0$$

Example 2.1. We will use the above corollary to evaluate the improper Riemann integral

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x \xi}\, \mathrm{d}x = e^{-\pi \xi^2}$$

for any $\xi \in \mathbb{R}$. Let $f(z) = e^{-\pi z^2}$ (which is entire, i.e. holomorphic on $\mathbb{C}$) and define let γ_R to be the positively oriented rectangle with end point $-R$, R, $R + i\xi$, $-R + i\xi$.

Figure 3. Curve γ_R



By the corollary, the integral of f over γ_R is zero for any $R > 0$, thus

$$0 = \int_{-R}^{R} e^{-\pi z^2}\,\mathrm{d}z + \int_{0}^{\xi} ie^{-\pi(R+iz)^2}\,\mathrm{d}z + \int_{R}^{-R} e^{-\pi(z+\imath\xi)^2}\,\mathrm{d}z + \int_{\xi}^{0} ie^{-\pi(-R+iz)^2}\,\mathrm{d}z \tag{5}$$

As $R \to \infty$, the first integral tends to 1 (seen in calculus courses). Furthermore,

$$\left|\int_{0}^{\xi} ie^{-\pi(R+iz)^2}\,\mathrm{d}z\right| \le \int_{0}^{\xi} \left|e^{-\pi(R^2+2iRz-z^2)}\right|\,\mathrm{d}z \le \xi e^{-\pi R^2} \xrightarrow{R\to\infty} 0$$

Similarly, the last integral on the right hand side of equation (5) tends to 0 as $R \to \infty$. Finally,

$$\int_{R}^{-R} e^{-\pi(z+\imath\xi)^2}\,\mathrm{d}z = -\int_{-R}^{R} e^{-\pi(z^2+2iz\xi-\xi^2)}\,\mathrm{d}z = -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi z^2}e^{-2\pi iz\xi}\,\mathrm{d}z$$

Ergo taking the limit as $R \to \infty$ in equation (5) yields

$$0 = 1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi z^2}e^{-2\pi iz\xi}\,\mathrm{d}z$$

The desired result follows.

Lecture 6

Homotopies and simply connected domains. We now generalize Cauchy's theorem from disks to simply connected domains.

Definition 2.5. *Given an open set Ω and continuous curves $\gamma_0, \gamma_1 : [a, b] \to \Omega$ with common end-points, we say that γ_0, γ_1 are* ***homotopic*** *if there exists a function $\gamma : [0, 1] \times [a, b] \to \Omega$ satisfying the following conditions:*

(1) $\gamma(0, t) = \gamma_0(t)$, $\gamma(1, t) = \gamma_1(t)$ for all $t \in [a, b]$

(2) $\gamma(x, a) = \gamma_0(a) = \gamma_1(a)$ and $\gamma(x, b) = \gamma_0(b) = \gamma_1(b)$ for all $x \in [0, 1]$

(3) γ is jointly-continuous in both variables.

Definition 2.6. *A region Ω is said to be simply connected if any two continuous curves in Ω with the same end-points are homotopic.*

Lemma 2.7. *Given homotopic curves $\gamma_0, \gamma_1 : [a, b] \to \Omega \subseteq \mathbb{C}$ (where Ω is open) and a holomorphic function if $f : \Omega \to \mathbb{C}$, it holds that*

$$\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

Proof. By definition, there exists a function $\gamma : [0, 1] \times [a, b] \to \Omega$ which satisfies the properties in definition 2.5. Note that the image of γ is compact and Ω is open. We may therefore find some $\varepsilon > 0$ such that for every point in the image of γ, the ε-neighbourhood of that point is entirely contained in Ω. Indeed, if no such ε exists then for all $n \in \mathbb{N}$ there exists $z_n \in K$, $w_n \notin \Omega$ such that $|z_n - w_n| < 1/n$. By compactness of K, z_n has a convergent sub-sequence, say $z_{n_k} \to z \in K$. By construction, it would then hold that $w_{n_k} \to z$. But this contradicts the assumption that Ω^c is closed.

Now, since γ is continuous and defined on a compact set, γ is uniformly continuous. Thus, for ε as above, there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|\gamma(x, t) - \gamma(y, t)| < \varepsilon$ for all $t \in [a, b]$.

Fix x, y such that $|x - y| < \delta$. We now construct 2 finite sequences as follows; $z_0 = w_0 = \gamma_0(a) = \gamma_1(a)$. For each following $n \in \mathbb{N}$, pick z_n on $\gamma(x, t)$ and $w_n \in \gamma(y, t)$ such that all the points $z_{n-1}, z_n, w_{n-1}, w_n$ are contained in some disk D_n of radius $\varepsilon/2$. Proceed as such up to some $z_N = w_N = \gamma_0(b) = \gamma_1(b)$. In each disk D_n, consider the primitive of f which we denote F_n. By the fundamental theorem of calculus, for any $0 \le n \le N - 1$

$$F_{n+1}(z_n) - F_{n+1}(w_n) = F_n(z_n) - F_n(w_n)$$

Ergo,

$$\begin{aligned}
\int_{\gamma(x,t)} f(z)\,\mathrm{d}z - \int_{\gamma(y,t)} f(z)\,\mathrm{d}z &= \sum_{n=1}^{N} \left([F_n(z_n) - F_n(z_{n-1})] - [F_n(w_n) - F_n(w_{n-1})]\right) \\
&= \sum_{n=1}^{N-1} \left([F_n(z_n) - F_n(w_n)] - [F_n(z_{n-1}) - F_n(w_{n-1})]\right) \\
&= [F_N(z_N) - F_N(w_N)] - [F_1(z_0) - F_1(w_0)] \\
&= 0
\end{aligned}$$

since $w_0 = z_0, w_N = z_N$.

We may therefore pick a finite sequence x_n such that $x_0 = 0, x_N = 1$ and $|x_{n+1} - x_n| < \delta$ for each $0 \le n \le N - 1$ to conclude that

$$\int_{\gamma_0} f = \int_{\gamma(x_0,t)} f = \int_{\gamma(x_1,t)} f = \cdots = \int_{\gamma(x_N,t)} f = \int_{\gamma_1} f$$

□

Theorem 2.8. *Every holomorphic function on a simply connected domain has a primitive.*

Proof. The proof is left as an exercise. □

Corollary 2.9 (General Cauchy theorem). *Given a simply connected domain Ω, a holomorphic function f defined in Ω and any closed curve $\gamma \subseteq \Omega$, it holds that*

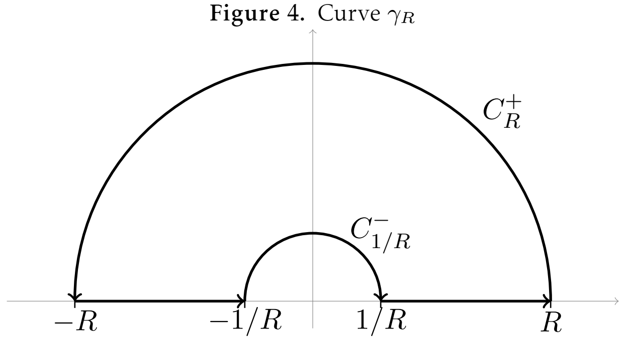
$$\int_\gamma f = 0$$

Example 2.2. We wish to evaluate the improper Riemann integral;

$$\int_0^\infty \frac{1-\cos(x)}{x^2} = \frac{\pi}{2}$$

Let $f(x) = \frac{1-e^{iz}}{z^2}$ which is holomorphic on $\mathbb{C} \setminus \{0\}$.

Figure 4. Curve γ_R



By Cauchy's theorem for a disk 2.9, for any $R > 1$, integrating f over γ_R we find

$$0 = \int_{1/R}^{R} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z + \int_{C_R^+} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z + \int_{-R}^{-1/R} \frac{1-e^{iz}}{z^2} + \int_{C_{1/R}^-} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z \tag{6}$$

We now evaluate each integral individually as $R \to \infty$.

$$\left| \int_{C_R^+} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z \right| \le \int_{C_R^+} \left| \frac{1-e^{iz}}{z^2} \right| \mathrm{d}z \le \int_{C_R^+} \frac{2}{|z|^2}\,\mathrm{d}z = \frac{\pi}{R} \xrightarrow{R\to\infty} 0$$

The fourth integral on the right hand side is given by

$$\int_\pi^0 \frac{1-e^{i\varepsilon e^{it}}}{\varepsilon e^{it}} i\,\mathrm{d}t = -i \int_0^\pi \frac{1-\sum_{n=0}^\infty (i\varepsilon e^{it})^n/n!}{\varepsilon e^{it}}\,\mathrm{d}t = -\int_0^\pi \sum_{n=1}^\infty \frac{(i\varepsilon e^{it})^{n-1}}{n!}\,\mathrm{d}t$$

where ε denotes $1/R$. Thus,

$$\lim_{R\to\infty} \int_{C_{1/R}^-} \frac{1-e^{iz}}{z^2}\,\mathrm{d}z = -\int_0^\pi \sum_{n=1}^\infty \lim_{\varepsilon\to 0} \frac{(i\varepsilon e^{it})^{n-1}}{n!}\,\mathrm{d}t = -\int_0^\pi 1\,\mathrm{d}t = -\pi$$

where we can justify passing the limit under the integral and the sum by the dominated convergence theorem (analysis). Finally, taking the limit as R tends to infinity in equation

(6) yields
$$\int_0^\infty \frac{1-e^{iz}}{z^2}\,\mathrm{d}z + \int_{-\infty}^0 \frac{1-e^{iz}}{z^2}\,\mathrm{d}z = \pi$$
Taking the real parts of either side of the above equation and dividing by 2 concludes the exercise.

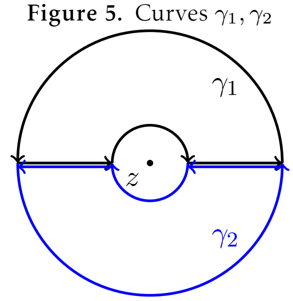
Lecture 7

Cauchy's Intergral Formula.

Theorem 2.10. *Let f be a holomorphic function in an open set $\Omega \supseteq \overline{D}$ where D is a disk and let C denote the boundary of D in the counter-clockwise direction (positive orientation), then*
$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi \quad \textit{for all } z \in D \tag{7}$$

Proof. For fixed $z \in D$, the function $f(\xi)/(\xi - z)$ is holomorphic on $D \setminus \{z\}$. For all ε sufficiently small, we find by Cauchy's theorem 2.9 that
$$\int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi\,\mathrm{d}\xi = \int_{C_\varepsilon} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi\,\mathrm{d}\xi = \int_{C_\varepsilon} \frac{f(z)}{\xi - z}\,\mathrm{d}\xi + \int_{C_\varepsilon} \frac{f(\xi)-f(z)}{\xi - z}\,\mathrm{d}z$$
where C_ε is the circle centered at z with radius ε. Indeed, we may see that this is true by integrating over the paths γ_1, γ_2 as depicted below:

Figure 5. Curves γ_1, γ_2



Since f is holomorphic, the term $\frac{f(\xi)-f(z)}{\xi - z}$ is bounded by some constant $M > 0$ for all ξ close to z. In particular, for all sufficiently small ε
$$\left|\int_{C_\varepsilon} \frac{f(\xi)-f(z)}{\xi - z}\,\mathrm{d}z\right| \leq M2\pi\varepsilon \xrightarrow{\varepsilon \to 0} 0$$
Finally, taking $\varepsilon \to 0$ in;
$$\int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi = \int_{C_\varepsilon} \frac{f(z)}{\xi - z}\,\mathrm{d}\xi + \int_{C_\varepsilon} \frac{f(\xi)-f(z)}{\xi - z}\,\mathrm{d}z = 2\pi i f(z) + \int_{C_\varepsilon} \frac{f(\xi)-f(z)}{\xi - z}\,\mathrm{d}z$$
yields the desired result. □

Generalizing Cauchy's integral formula. We now provide Cauchy's Integral formula in it's full generality which gives an explicit form to every derivative of a holomorphic function f. This theorem will have some useful consequences. For instance it will allow us to prove that all holomorphic functions are infinitely differentiable and analytic. We will also use the result to prove *Liouville's Theorem*, a famous theorem which is known, in part, for it's role in the proof of the *Fundamental theorem of Algerba*. Another application of this generalization will be the proof of the *identity theorem*, or *analytic continuation* which we have mentioned earlier in this text.

Theorem 2.11. *Let f be a holomorphic function, then f has infinitely many complex derivatives in it's domain Ω. Furthermore, if D is a disk such that $\overline{D} \subseteq \Omega$, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} \,\mathrm{d}\xi \qquad \forall z \in D \tag{8}$$

where C denotes the boundary of the disk (with positive orientation).

Proof. We prove the above by induction. The case $n = 0$ is exactly Cauchy's integral formula 7. We now assume the result holds for $n - 1$ where $n \in \mathbb{N}$. Fix $z \in D$ and let h be small

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\xi)}{h} \left[\frac{1}{(\xi - z - h)^n} - \frac{1}{(\xi - z)^n} \right]$$

By the identity $A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1})$ we obtain

$$\frac{(n-1)!}{2\pi i} \int_C \frac{f(\xi)}{h} \frac{h}{(\xi - z - h)(\xi - z)} \left[\frac{1}{(\xi - z - h)^{n-1}} + \cdots + \frac{1}{(\xi - z)^{n-1}} \right]$$

Note that for all sufficiently small h, the integrand is bounded. Thus, by the bounded convergence theorem we may take the limit as $h \to 0$ by passing it under the integral to find that

$$f^{(n)}(z) = \frac{(n-1)!}{2\pi i} \int_C f(\xi) \frac{n}{(\xi - z)^{n+1}} = \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{n+1}} \,\mathrm{d}\xi$$

□

Corollary 2.12 (Cauchy's inequality). *If f is a holomorphic function in an open set containing the closure of a disk $D_R(z_0)$, then*

$$\left| f^{(n)}(z_0) \right| \leq \frac{n!}{R^n} \|f\|_C \tag{9}$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$.

Proof. Indeed, we have

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}} \right| \leq \frac{n!}{2\pi} \int_C \frac{|f(\xi)|}{|\xi - z_0|^{n+1}} \leq \frac{n!}{2\pi R^{n+1}} \|f\|_C \, 2\pi R$$

□

Fundamental Theorem of Algebra.

Theorem 2.13 (Liouville). *If f is entire and bounded, then it is constant.*

Proof. Let $M > 0$ be an upper bound of f, then any $z \in \mathbb{C}, R > 0$, we find from Cauchy's inequality (9) that

$$\left|f'(z)\right| \leq \frac{1}{R}\|f\|_C \leq \frac{M}{R}$$

Letting R tend to infinity, we conclude that $f'(z) = 0$ for all $z \in \mathbb{C}$, i.e. f is constant. □

Corollary 2.14. *Every non-constant polynomial $P(z) = a_n z^n + \cdots + z_0$ has a root in $\mathbb{C}$.*

Proof. We show that if P has no roots in $\mathbb{C}$ then it must be constant. For any $z \neq 0$,

$$\frac{P(z)}{z^n} = a_n + \left(\frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)$$

Since the right hand side tends to $|a_n|$ as $|z|$ tends to infinity, there exists $R > 0$ such that for all $z \in \mathbb{C}, |z| \geq R$,

$$\left|\frac{P(z)}{z^n}\right| \geq \frac{|a_n|}{2} \iff P(z) \geq \frac{|a_n|}{2}|z^n|$$

In particular $1/P(z)$ is bounded outside of the disk $D_R(0)$. Furthermore, since P has no root, $1/P(z)$ is holomorphic on $\overline{D}_R(0)$ and thus bounded on that disk. We infer that $1/P(z)$ is a bounded entire function. By Liouville's theorem, $P(z)$ is constant. □

Theorem 2.15 (Fundamental Theorem of Algebra). *Every non-constant polynomial $P(z) = a_n z^n + \cdots + z_0$ of degree n has n roots in $\mathbb{C}$*

Proof. By the above corollary $P(z)$ must have k roots for some $1 \leq k \leq n$. Let $w_1, \ldots w_k$ denote these roots and write

$$P(z) = (z - w_1)(z - w_2)\cdots(z - w_k)Q(z)$$

where Q is a polynomial of degree $n - k$. If $k < n$ then by the above corollary Q must have at least 1 root, which is also a root of P. By induction, we conclude $k = n$. □

Analytic continuation.

Definition 2.16. *A function $f : \Omega \to \mathbb{C}$ where Ω is an open set is said to be analytic if for every $z_0 \in \Omega$, there exists a disk D centered about z_0 and coefficients $\{a_n\}_{n=1}^{\infty}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \forall z \in D$$

Theorem 2.17. *Let f be holomorphic in an open set Ω and let D be a disk centered at z_0 whose closure is contained in Ω. Then f is analytic and has the power series expansion*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \qquad \forall z \in D$$

Proof. Let $z \in D$ and denote by C the boundary of D, then there exists $0 < r < 1$ such that

$$\frac{z - z_0}{\xi - z_0} < r \qquad \forall \xi \in C$$

Ergo, for all $\xi \in C$, the below series converges absolutely;

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0}\left(\frac{1}{1 - \frac{z - z_0}{\xi - z_0}}\right) = \frac{1}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n$$

Finally, by Cauchy's integral formula 8,

$$\begin{aligned} f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi &= \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z_0}\sum_{n=0}^{\infty}\left(\frac{z - z_0}{\xi - z_0}\right)^n \mathrm{d}\xi \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z_0)^{n+1}}\right](z - z_0)^n\ \mathrm{d}\xi \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z - z_0)^n \end{aligned}$$

□

Theorem 2.18 (Identity Theorem). *Suppose that f is a holomorphic function in a connected region Ω. Suppose furthermore that there exists a sequence of distinct points $\{w_k\}_{k=1}^{\infty}, w_k \to w \in \Omega$ such that $f(w_k) = 0$ for all $k \in \mathbb{N}$, then $f \equiv 0$.*

Proof. We will first show that $f \equiv 0$ in any disk $D \subseteq \Omega$ centered about w. By the previous theorem, there exists a sequence of complex numbers $\{a_n\}_{n=1}^{\infty}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - w)^n \qquad \forall z \in D$$

If $f \not\equiv 0$, then define m to be the minimal positive integer such that $a_m \neq 0$. Write

$$f(z) = a_m (z - w)^m \left(1 + a_m^{-1}\sum_{n=m+1}^{\infty} a_n (z - w)^{n-m}\right) := a_m (z - w)^m (1 + g(z))$$

Note that $g(z) \to 0$ as $z \to w$. Thence, we may pick an element w_k in our sequence such that $|g(w_k)| \leq 1/2$. Finally,

$$0 = |f(w_k)| = |a_m (w_k - w)^m (1 + g(w_k))| \geq \frac{|a_m|}{2} > 0$$

which is absurd.

We now prove that $f \equiv 0$ on Ω. Define U to be the (non-empty) interior of $\{z \in \Omega \mid f(z) = 0\}$ and $V = \Omega \setminus U$. If both U and V are open (with respect to the subspace topology on Ω), then since U is non-empty, $U \cup V = \Omega$ and Ω is connected, $U = \Omega$. In other words, $f \equiv 0$

on Ω. The set U is open by definition and it remains to show that V is open, or equivalently that U is closed. Let $z_n \to z$ be a sequence in U converging to $z \in \Omega$. By the above argument, there exists a neighbourhood of z on which $f \equiv 0$. Whence, z is in the interior of $\{z \in \Omega \mid f(z) = 0\}$, i.e. $z \in U$. □

Corollary 2.19. *If f, g are holomorphic in Ω and $f(z) = g(z)$ for all z in a some non-empty subset of Ω containing a cluster point, then $f \equiv g$ in Ω.*

Definition 2.20. *If $f : \Omega \supseteq \Omega' \to \mathbb{C}$ and $F : \Omega' \to \mathbb{C}$ are holomorphic functions such that $f \equiv F$ in Ω, then F is called the analytic continuation of f.*

Lecture 8

Theorem 2.21 (Morera's theorem)**.** *If f is a continuous function in a disk D such that*

$$\int_T f = 0$$

for every triangle T contained in D, then f is holomorphic.

Proof. Note that this theorem is a converse of Goursat's theorem 2.1. Since it suffices to show that f has a primitive, the proof is identical to that of Cauchy's theorem. □

Theorem 2.22. *If there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ of holomorphic functions that converge uniformly to f in every compact subset of Ω, then f is holomorphic in Ω.*

Proof. Let D be a disk whose closure is contained in Ω. For any triangle T in D and any $n \in \mathbb{N}$, by Goursat's theorem 2.1

$$\int_T f_n = 0$$

Since $f_n \to f$ uniformily, we may take the limit as $n \to \infty$ by interchanging between the limit and the integral which yields

$$\int_T f = 0$$

Hence, by Morera's theorem, f is holomorphic. □

Theorem 2.23. *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of holomorphic functions that converge uniformly to f in every compact subset of Ω, then $\{f_n'\}$ tends to f' uniformly on every compact subset of Ω.*

Proof. Without loss og generality, suppose that $\{f_n\}_{n\in\mathbb{N}} \to f$ uniformly in all of Ω. Fix $\delta > 0$ and define

$$\Omega_\delta = \left\{z \in \Omega \mid \overline{D}_\delta(z) \subseteq \Omega\right\}$$

By Cauchy's inequality 9,

$$\sup_{z\in\Omega_\delta} \left|(f - f_n)'(z)\right| \leq \frac{1}{\delta} \sup_{z\in\Omega} |f(z) - f_n(z)| \xrightarrow{n\to\infty} 0$$

□

Schwartz reflection principle.

Theorem 2.24. *Let $\Omega = \Omega^+ \cup I \cup \Omega^-$ be such that for all $z \in \Omega^+$, $\Im(z) > 0$ (upper half-plane), for all $z \in I$, $\Im(z) = 0$ and for all $z \in \Omega^-$, $\Im(z) < 0$ (lower half-plane). Suppose that $z \in \Omega^+$ if and only if $\overline{z} \in \Omega^-$. If f^+ is holomorphic in Ω^+ and f^- is holomorphic in Ω^- and*

$$\lim_{z \in \Omega^+ \to x} f(z) = \lim_{z \in \Omega^- \to x} f(z)$$

for all $x \in I$, then the function

$$f(z) = \begin{cases} f^+(z) & z \in \Omega^+ \\ f^-(z) & z \in \Omega^- \\ \lim_{z \in \Omega^+ \to x} f(z) & \textit{otherwise} \end{cases}$$

is holomorphic.

Proof. Consider an arbitrary triangle T in Ω. By Morera's theorem, it suffices to show that $\int_T f = 0$. There are three interesting cases:

(1) If a vertex of T is on I and the interior of T is either in Ω^+ or Ω^-. In this case we construct triangles T_n tending to T entirely in Ω^+ or Ω^- and take the limit.

(2) An edge of T is on I and the interior of T is either in Ω^+ or Ω^-. We proceed as in the previous case.

(3) If the interior of T is split between Ω^+ and Ω^-, then we may separate T into finitely many triangles such that the interior of each triangle is either in Ω^+ or Ω^-.

□

Theorem 2.25 (Schwartz). *If f is holomorphic in Ω^+ and extends continuously to I and f is real-valued on I, then f has an analytic continuation F to Ω.*

Proof. We define

$$F(z) = \overline{f(\overline{z})} \quad \forall z \in \Omega$$

For $z \in \Omega^+$ and appropriate $z_0 \in \Omega^+$ we may write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Hence,

$$F(z) = \overline{\sum_{n=0}^{\infty} a_n (\overline{z} - z_0)^n} = \sum_{n=0}^{\infty} \overline{a}_n (z - \overline{z_0})^n$$

which is indeed holomorphic. Furthermore, F agrees with f on I where both are real-valued. It follows from the previous theorem that the function defined by f on Ω^+ and by F on Ω^- may be extended to a holomorphic function on Ω. □

Exercises and solutions

Problem 1. Show that

$$\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x = \frac{\pi}{2}$$

Solution.

$$\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x = \frac{1}{2i}\int_0^\infty \left[\frac{e^{ix}}{x} - \frac{e^{-ix}}{x}\right]\mathrm{d}x = \frac{1}{2i}\left[\int_0^\infty \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{-\infty}^0 \frac{e^{ix}}{x}\,\mathrm{d}x\right] = \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}}{x}\,\mathrm{d}x$$

By Cauchy's theorem that the integral of of $\frac{e^x}{x}$ over the indented semi-circle equals 0. We therefore obtain

$$\begin{aligned}
\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,\mathrm{d}x + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,\mathrm{d}x &= \int_{C_\varepsilon} \frac{e^{ix}}{x}\,\mathrm{d}x - \int_{C_R} \frac{e^{ix}}{x}\,\mathrm{d}x \\
&= \int_0^\pi \frac{\exp\left\{i\varepsilon e^{it}\right\}}{\varepsilon e^{it}} i\varepsilon e^{it}\,\mathrm{d}t - \int_0^\pi \frac{\exp\left\{iRe^{it}\right\}}{Re^{it}} iRe^{it}\,\mathrm{d}t \\
&= \underbrace{i\int_0^\pi \exp\left\{i\varepsilon e^{it}\right\}\,\mathrm{d}t}_{(1)} - \underbrace{i\int_0^\pi \exp\left\{iRe^{it}\right\}\,\mathrm{d}t}_{(2)}
\end{aligned}$$

where R denotes the radius of the large circle and ε the radius of the small circle. We first evaluate integral (2)

$$\begin{aligned}
\left|\int_0^\pi \exp\left\{iRe^{it}\right\}\,\mathrm{d}t\right| \leq \int_0^\pi \left|\exp\left\{iR\cos(t) - R\sin(t)\right\}\right|\,\mathrm{d}t &= \int_0^\pi \left|\exp\left\{-R\sin(t)\right\}\right|\,\mathrm{d}t \\
&= \int_0^{\pi/2} \left|\exp\left\{-R\sin(t)\right\}\right|\,\mathrm{d}t
\end{aligned}$$

Furthermore, since sin is concave on the interval $[0, \pi/2]$ and $\sin(0) = 0, \sin(\pi/2) = 1$, it holds that $\sin(t) \geq \frac{t}{\pi/2}$ for all $t \in [0, \pi/2]$. The above integral may therefore be bounded by

$$\int_0^{\pi/2} \left|\exp\left\{-\frac{2Rt}{\pi}\right\}\right|\,\mathrm{d}t = \frac{-\pi}{2R}\left[e^{-R} - 1\right] \xrightarrow{R\to\infty} 0$$

We now evaluate integral (1);

$$\begin{aligned}
\int_0^\pi \exp\left\{i\varepsilon e^{it}\right\}\,\mathrm{d}t = \int_0^\pi \exp\left\{i\varepsilon\cos(t) - \varepsilon\sin(t)\right\}\,\mathrm{d}t &= \int_0^\pi \exp\left\{i\varepsilon\cos(t)\right\}\exp\left\{-\varepsilon\sin(t)\right\}\,\mathrm{d}t \\
&\xrightarrow{\varepsilon\to 0} \int_0^\pi 1\,\mathrm{d}t = \pi
\end{aligned}$$

Ergo, taking the limits as $R \to \infty, \varepsilon \to 0$, we obtain

$$\int_{-\infty}^\infty \frac{e^{ix}}{x}\,\mathrm{d}x = i\pi$$

Finally, equating the imaginary parts of the left and right hand side we conclude that

$$\int_0^\infty \frac{\sin(x)}{x} = \frac{1}{2i}\int_{-\infty}^\infty \frac{e^{ix}}{x}\,\mathrm{d}x = \pi/2$$

□

Problem 2. Evaluate the following integrals

(1) $\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$,

(2) $\int_{|z|=1} \frac{e^z}{z^n}\,\mathrm{d}z$.

Solution.

(1) We have

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \int_0^{2\pi} \frac{\left|\rho i e^{it}\right|}{|\rho e^{it}-a|^2}\,\mathrm{d}t = \rho\int_0^{2\pi} \frac{1}{(\rho e^{it}-a)(\rho e^{-it}-\overline{a})}\,\mathrm{d}t$$

In the hopes of recovering the integral in terms of z, we write

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{\rho}{i}\int_0^{2\pi} \frac{i\rho e^{it}}{(\rho e^{it}-a)(\rho^2-\overline{a}\rho e^{it})}\,\mathrm{d}t = \frac{\rho}{i}\int_{|z|=\rho} \frac{1}{(z-a)(\rho^2-z\overline{a})}\,\mathrm{d}z$$

Since $|a| < \rho$, we know that the $\left|\rho^2 - z\overline{a}\right| > 0$ in the disk D_ρ of radius ρ about the origin. I follows that function

$$g(z) := \frac{1}{\rho^2 - z\overline{a}}$$

is holomorphic in D_ρ. Hence, by the Cauchy integral formula,

$$g(a) = \frac{1}{2\pi i}\int_{|z|=\rho} \frac{g(z)}{z-a}\,\mathrm{d}z = \frac{1}{2\pi i}\frac{i}{\rho}\frac{\rho}{i}\int_{|z|=\rho} \frac{1}{(z-a)(\rho^2-z\overline{a})}\,\mathrm{d}z = \frac{1}{2\pi\rho}\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

Finally,

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = 2\pi\rho g(a) = \frac{2\pi\rho}{\rho^2 - |a|^2}$$

(2) If $n \leq 0$, then f is holomorphic on the entire disk of radius 1 about the origin and from Cauchy's theorem we find that the integral evaluates to 0. Otherwise, i.e. $n \geq 1$, consider the function $g(z) := e^z$ which is holomorphic in the disk of radius 1 about the origin. By Cauchy integral formula,

$$1 = g^{(n-1)}(0) = \frac{(n-1)!}{2\pi i}\int_{|z|=1} \frac{g(z)}{z^n}\,\mathrm{d}z = \frac{(n-1)!}{2\pi i}\int_{|z|=1} \frac{e^z}{z^n}\,\mathrm{d}z$$

Multiplying both sides of the above equation by $\frac{2\pi i}{(n-1)!}$ concludes the exercise.

□

Probelm 3. Suppose that f is a holomorphic function on the strip $\{z = x + iy \mid -1 < y < 1, x \in \mathbb{R}\}$ such that

$$|f(z)| \leq A\,(1 + |z|)^{\eta}$$

for some constants $A > 0$ and $\eta \in \mathbb{R}$. Show that there exists a sequence $\{A_n\}_{n=1}^{\infty}$ such that for all $x \in \mathbb{R}$,

$$\left|f^{(n)}(x)\right| \leq A_n\,(1 + |x|)^{\eta}$$

Solution. Fix $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $r < 1$. By Cauchy's inequality

$$\left|f^{(n)}(x)\right| \leq \frac{n!}{r^n}\,\|f\|_D \leq \frac{n!}{r^n} A \sup_{|z-x|=r} (1 + |z|)^{\eta}$$

If η is non-negative then for all z such that $|z - x| = r$, $|z| \leq |x| + r \leq |x| + 1$. Thence,

$$\left|f^{(n)}(x)\right| \leq \frac{n!}{r^n} A\,(2 + |x|)^{\eta} \leq \frac{n!}{r^n} A\,(2 + 2\,|x|)^{\eta} \xrightarrow{r \to 1} n! A 2^{\eta}\,(1 + |x|)^{\eta}$$

In particular, $A_n = n!A2^{\eta}$ solves the problem. Otherwise, i.e. if η is negative then $|z| \geq |x| - r \geq |x| - 1$ and

$$(1 + |z|)^{\eta} \leq (1 + |z|/2)^{\eta} \leq \left(1 + \frac{|x| - 1}{2}\right)^{\eta} = \frac{1}{2^{\eta}}\,(1 + |x|)^{\eta}$$

Letting r tend to 1,

$$\left|f^{(n)}(x)\right| \leq n! A \frac{1}{2^{\eta}}\,(1 + |x|)^{\eta} = A_n\,(1 + |x|)^{\eta}$$

where $A_n = n!A/2^{\eta}$. □

Problem 4. Show that if f is holomorphic on an open set containing the closure of $D_R(0)$, then for all $z \in D_R(0)$,

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\varphi}) \Re\left(\frac{Re^{i\varphi} + z}{Re^{i\varphi} - z}\right) \mathrm{d}\varphi$$

Solution. By Cauchy's integral formula,

$$\begin{aligned} f(z) = \frac{1}{2\pi i} \int_{|\xi|=R} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(Re^{i\varphi})}{Re^{\varphi} - z} iRe^{i\varphi}\,\mathrm{d}\varphi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\varphi})}{Re^{i\varphi} - z} Re^{i\varphi}\,\mathrm{d}\varphi \qquad (\dagger) \end{aligned}$$

In order to compare the above to the desired result, we write

$$\begin{aligned}\Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) &= \frac{1}{2}\left[\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}+\frac{Re^{-i\varphi}+\overline{z}}{Re^{-i\varphi}-\overline{z}}\right] \\ &= \frac{1}{2}\left[\frac{\left(Re^{i\varphi}+z\right)\left(Re^{-i\varphi}-\overline{z}\right)+\left(Re^{i\varphi}-z\right)\left(Re^{-i\varphi}+\overline{z}\right)}{\left(Re^{i\varphi}-z\right)\left(Re^{-i\varphi}-\overline{z}\right)}\right] \\ &= \frac{R^2-|z|^2}{\left(Re^{i\varphi}-z\right)\left(Re^{-i\varphi}-\overline{z}\right)} \qquad (2)\end{aligned}$$

Furthermore, note that

$$\Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) = \frac{Re^{i\varphi}\left(|z|^2-R^2\right)/\overline{z}}{\left(Re^{i\varphi}-z\right)\left(Re^{i\varphi}-R^2/\overline{z}\right)} = \frac{Re^{i\varphi}}{\left(Re^{i\varphi}-z\right)}-\frac{Re^{i\varphi}}{\left(Re^{i\varphi}-R^2/\overline{z}\right)}$$

Since $R>|\overline{z}|$, the function

$$\frac{Re^{i\varphi}}{\left(Re^{i\varphi}-R^2/\overline{z}\right)}$$

is holomorphic in the disk of radius R about the origin. The product of holomorphic function being again holomorphic, we conclude by Cauchy's theorem that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)\,\mathrm{d}\varphi &= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\left[\frac{Re^{i\varphi}}{\left(Re^{i\varphi}-z\right)}-\frac{Re^{i\varphi}}{\left(Re^{i\varphi}-R^2/\overline{z}\right)}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\varphi})}{Re^{i\varphi}-z}Re^{i\varphi}\,\mathrm{d}\varphi\end{aligned}$$

Finally, by (†), we find that

$$f(z)=\frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\Re\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)\,\mathrm{d}\varphi$$

□

Problem 5. Suppose that f is an analytic function defined on $\mathbb{C}$ such that for every $z_0\in\mathbb{C}$, the expansion about z_0,

$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n \qquad \forall z \text{ near } z_0$$

has at least one zero coefficient (i.e. there exists $k\in\mathbb{N}$ such that $c_k=0$), then f is a polynomial.

Solution. Fix $z_0\in\mathbb{C}$. If $c_n=0$, then

$$f^{(n)}(z_0)=c_n n!=0$$

In order to prove that f is a polynomial, we equivalently need to show that $f^{(n)}\equiv 0$ for some $n\in\mathbb{N}$. By the identity theorem 2.18 in the second chapter of the book, it suffices to show that for some $z \in \mathbb{C}$ there exists a sequence $z_k \to z$ of distinct point such that $f^{(n)}(z_k) = 0$ for all $k \in \mathbb{N}$. By assumption, for each $z_0 \in \mathbb{C}$ there must exists an $n \in \mathbb{N}$ such that the n^{th} term of the power series expansion about z_0 is 0. For each natural number n, consider the set

$$S_n = \left\{ z \in \mathbb{C} \mid \text{the } n^{\text{th}} \text{ term of the power series expansion about } z \text{ is } 0 \right\}$$

It follows from our assumption that

$$\bigcup_{n \in \mathbb{N}} S_n = \mathbb{C}$$

Since the countable union of countable sets is again countable, there must exist a natural number $n_0 \in \mathbb{N}$ such that S_{n_0} is uncountable. Thence, S_{n_0} must have an accumulation point $z \in S_{n_0}$ and we may therefore pick a sequence $z_k \to z$ in S_{n_0}. By definition of S_{n_0} along wiht our initial remarks, it holds that

$$f^{(n_0)}(z_k) = 0 \quad \forall k \in \mathbb{N}$$

Finally, by the identity theorem, we indeed have $f^{n_0} \equiv 0$, i.e. f is a polynomial. $\square$

Chapter 3

Meromorphic functions

Lecture 9

The Residue theorem. We would like to generalize some of the results from the previous chapter to functions that are *almost holomorphic*.

Definition 3.1 (Isolated singularities). *A point singularity, or isolated singularity, of a function f is a point z_0 such that f is defined in some neighbourhood of z_0 but not at z_0.*

Given an open set $\Omega \subseteq \mathbb{C}$ and a function $f : \Omega \setminus z_0 \to \mathbb{C}$, we define three types of singularities below. A singularity at a point z_0 is said to be isolated if there exists a neighbourhood U of z_0 such that f is holomorphic on $U \setminus \{z_0\}$.

Removable: The point z_0 is called a removable singularity if there exists $a \in \mathbb{C}$ such that

$$f^*(z) = \begin{cases} f(z) & z \neq z_0 \\ a & z = z_0 \end{cases}$$

is holomorphic in a neighbourhood of z_0.

Poles: If there exists a neighbourhood V of z_0 such that

$$f^*(z) = \begin{cases} 1/f(z) & z \in V \setminus \{z_0\} \\ 0 & z = z_0 \end{cases}$$

is well-defined and holomorphic, then z_0 is called a **pole** of f.

Essential: If z_0 is not a removable singularity nor a poles, then it is called an essential singularity.

Lemma 3.2. *Suppose $f \not\equiv 0$ is a holomorphic function on an open connected set Ω, then for any $z_0 \in \Omega$, there exists a neighbourhood U of z_0 on which*

$$f(z) = (z - z_0)^n g(z)$$

for some unique positive integer n and non-vanishing function g.

Proof. First note that by the identity theorem 2.18, there must exist a neighbourhood U of z_0 such that f vanishes exclusively at z_0. Since f is analytic, we may write

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k$$

By assumption, there exists a minimal positive integer n such that $a_n \neq 0$. Then,

$$f(z) = (z - z_0)^n g(z) \quad \text{where} \quad g(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}$$

By results derived in chapter 1, g is holomorphic and it's power series representation has the same radius of convergence as that of f. Furthermore, $g(z_0) = a_n \neq 0$ and $g(z) \neq 0$ for all $z \in U$ since f is non-zero on U.

In order to prove uniqueness, it suffices to show that our choice of n was unique. Suppose m is a positive integer such that the decomposition $f(z) = (z - z_0)^m h(z)$ satisfies the conditions of the lemma. If $m > n$, then $a_n = g(z_0) = (z - z_0)^{m-n} h(z) = 0$ which contradicts our choice of n. Likewise, if $m < n$, $h(z_0) = (z - z_0)^{n-m} g(z) = 0$ which contradicts the assumption made on h. We infer that the choice of n was indeed unique. □

Theorem 3.3. *If f is a function with a pole at z_0 then there exists a neighbourhood U of z_0, a unique positive integer n and a non-vanishing holomorphic function $h : U \to \mathbb{C}$ such that*

$$f(z) = (z - z_0)^{-n} h(z) \qquad \forall z \in U \setminus \{z_0\}$$

*We call n the **degree** of the pole. If $n = 1$, then the pole is said to be simple.*

Proof. By definition of a pole, there exists a function g which is holomorphic in a neighbourhood U of z_0 and equal to $1/f$ away from z_0. By the previous lemma 3.2, there exists a unique positive integer n and a non-vanishing function holomorphic h such that

$$g(z) = (z - z_0)^n h(z) \qquad \forall z \in U \setminus \{z_0\}$$

Taking the reciprocal of either side of the above equation concludes the proof. □

Proposition 3.4. *If f has a pole of order n at z_0 then there exists a neighbourhood U of z_0 such that*

$$f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k = \underbrace{\sum_{k=-n}^{-1} a_k (z - z_0)^k}_{\text{principle part}} + \underbrace{\sum_{k=0}^{\infty} a_k (z - z_0)^k}_{\text{residue}} \qquad \forall z \in U \setminus \{z_0\}$$

We call the first n terms of the above series the ***principle part*** *of f and the rest of the series the* ***residue*** *of f. Furthermore, the term a_{-1} is called the residue of f at z_0, denoted* $\operatorname{res}_{z_0}(f)$.

Proof. By the previous theorem, there exists a neighbourhood U of z_0 and a holomorphic function h such that $f(z) = (z - z_0)^{-n}h(z)$ for all $z \in U \setminus \{z_0\}$. By theorem 2.17, we may expand h into it's power series about z_0;

$$f(z) = (z - z_0)^{-n} \sum_{k=0}^{\infty} b_k (z - z_0)^k = \sum_{k=-n}^{\infty} a_k (z - z_0)^k \qquad \forall z \in U \setminus \{z_0\}$$

where $a_k = b_{k-n}$. □

Theorem 3.5. *Suppose that f is holomorphic in an open set Ω except for at z_0 which is a pole. If C is a circle contained with its interior in Ω and such that z_0 is in the interior of C. Then*

$$\frac{1}{2\pi i} \int_C f(z)\,\mathrm{d}z = \operatorname{res}_{z_0}(f)$$

Proof. Let $P(z)$ and $R(z)$ denote the principle part and the residue of f about z_0. Then for some constant $\{a_k\}_{k=1}^{n}$,

$$\int_C f(z)\,\mathrm{d}z = \int_C P(z)\,\mathrm{d}z + \int_C R(z)\,\mathrm{d}z = \int_C \sum_{k=1}^{n} a_k \frac{1}{(z - z_0)^k}\,\mathrm{d}z = \sum_{k=1}^{n} a_k \int_C \frac{1}{(z - z_0)^k}\,\mathrm{d}z$$

where the integral of $R(z)$ vanishes since $R(z)$ is holomorphic by theorem 1.18. To complete the proof, it suffices to note that

$$\int_C \frac{1}{(z - z_0)^n}\,\mathrm{d}z = i \int_0^{2\pi} e^{it(1-n)}\,\mathrm{d}z = \begin{cases} 0 & n \geq 2 \\ 2\pi i & n = 1 \end{cases}$$

□

Theorem 3.6. *If f has a pole at z_0 of order n, then*

$$\operatorname{res}_{z_0}(f) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z - z_0)^n f(z)\right]$$

Corollary 3.7 (Residue formula). *If f is holomorphic in an open set containing a circle C and it's interior, except for a finite number of poles $z_1, \ldots, z_n$ then*

$$\frac{1}{2\pi i} \int_C f = \sum_{k=1}^{n} \operatorname{res}_{z_k} f$$

Corollary 3.8. *If f is holomorphic in an open set containing a toy contour C and it's interior, except for a finite number of poles $z_1, \ldots, z_n$ then*

$$\frac{1}{2\pi i} \int_C f = \sum_{k=1}^{n} \operatorname{res}_{z_k} f$$

Proof. By theorem 3.3, there exists a neighbourhood U of z_0 and a function h which is holomorphic on U such that $(z-z_0)^n f(z) = h(z)$ for all $z \in U \setminus \{z_0\}$. By theorem 2.17,

$$h(z) = \sum_{k=0}^{\infty} \frac{h^{(k)}(z_0)}{k!}(z-z_0)^k \qquad \forall z \in U$$

In particular, the term a_{-1} in the power series expansion of f corresponds to

$$\mathrm{res}_{z_0}(f) = \frac{h^{(n-1)}(z_0)}{(n-1)!} = \lim_{z\to z_0} \frac{1}{(n-1)!} h^{(n-1)}(z) = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[(z-z_0)^n f(z)\right]$$

□

Lecture 10

Example 3.1. We wish to use the Resiude theorem to show that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,\mathrm{d}x = \pi$$

We define

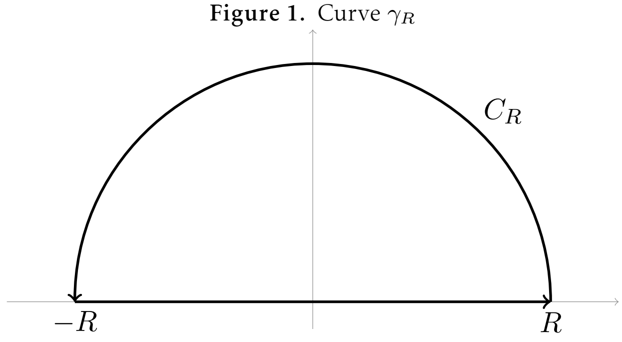
$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z+i)}$$

Then f holomorphic away from $i, -i$ which are poles and

$$\mathrm{res}_i(f) = \lim_{z\to i} \frac{z-i}{(z-i)(z+i)} = \pm\frac{1}{2i}$$

Let $R > 2$ and integrate over the curve γ_R of radius R;

Figure 1. Curve γ_R



By the residue theorem, the integral over γ_R evaluates to π. Furthermore,

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| \leq \int_{C_R} \frac{1}{|1+z^2|}\,\mathrm{d}z \leq \pi R \frac{1}{R^2-1} \xrightarrow{R\to\infty} 0$$

Combining our results and letting R tend to infinity, we conclude that

$$\pi = \int_{\gamma_R} f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} \frac{1}{1+z^2}\,\mathrm{d}z$$

Example 3.2. We wish to evaluate the integral

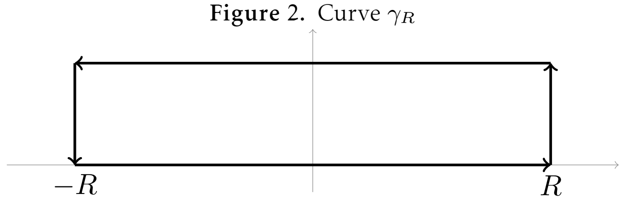
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} \qquad \text{for some } 0 < a < 1$$

Let

$$f(z) = \frac{e^{az}}{1+e^z}$$

which has a pole whenever $e^z = -1$, i.e. $z = i(2n\pi + \pi)$ for some integer n. For any $R > 0$, consider the rectangle contour of height 2π;

Figure 2. Curve γ_R



The residue at πi is given by

$$\lim_{z\to\pi i}(z-\pi i)\frac{e^{az}}{1+e^z} = \lim_{z\to\pi i} e^{az}\frac{1-\pi i}{-e^{\pi i}+e^z} = -e^{i\pi a}$$

where we computed the limit by using the classical l'Hospital's rule. By the residue theorem, the integral of f over the entire rectangle evaluates to $-2\pi i e^{\pi i a}$. We now compute the integral over the upper segment of the rectangle;

$$-\int_{-R}^{R} f(x+2\pi i)\,\mathrm{d}x = -\int_{-R}^{R}\frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,\mathrm{d}x = -e^{2\pi i a}\int_{-R}^{R}\frac{e^{ax}}{1+e^x}\,\mathrm{d}x = -e^{2\pi i a}\int_{-R}^{R} f(x)\,\mathrm{d}x$$

since $e^{2\pi i} = 1$. Furthermore, the integral of f over the left and right sides of the rectangle ia bounded by

$$\left|\int_0^{2\pi}\frac{e^{a(R\pm it)}}{1+e^{R\pm it}}\,\mathrm{d}t\right| \le \int_0^{2\pi}\frac{\left|e^{a(R\pm it)}\right|}{\left|1+e^{R\pm it}\right|}\,\mathrm{d}t \le \int_0^{2\pi}\frac{e^{aR}}{e^R-1}\,\mathrm{d}t \le 2\pi\frac{e^{(a-1)R}}{1-e^{-R}} \xrightarrow{R\to\infty} 0$$

since $a < 1$. Finally, taking the limit as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x - e^{2\pi i a}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = -2\pi i e^{\pi i a}$$

Thus, using the identity $\sin(z) = \frac{e^{iz}-e^{iz}}{2i}$, we conclude

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,\mathrm{d}x = \frac{-2\pi i e^{\pi i a}}{1-e^{2\pi i a}} = \frac{\pi}{\sin(\pi a)}$$

Remark 3.3 (L'Hospital's rule). The general form of l'Hospitals rule does not hold for complex functions. But, if we suppose that f, g are holomorphic in a neighbourhood of c, $f(c) = g(c) = 0$ and g, g' are non-zero in that neighbourhood then

$$\lim_{z\to c}\frac{f(z)}{g(z)} = \lim_{z\to c}\frac{(f(z)-f(c))/(z-c)}{(g(z)-g(c))/(z-c)} = \lim_{z\to c}\frac{f'(z)}{g'(z)}$$

Theorem 3.9 (Riemann's theorem on removable discontinuities). *If f has a singularity at z_0 and there exists a neighbourhood U of z_0 such that f is holomorphic and bounded on $U \setminus \{z_0\}$, then z_0 is a removable singularity.*

Proof. We mimic the argument of Cauchy's integral formula 7. Let C denote the boundary of U (a circle with positive orientation) and fix $z \in D$, $z \neq z_0$. By Cauchy's theorem, for all sufficiently small $\varepsilon > 0$

$$\int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi = \int_{C_\varepsilon(z)} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi + \int_{C_\varepsilon(z_0)} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi \tag{10}$$

where $C_\varepsilon(z)$ and $C_\varepsilon(z_0)$ are the circles with positive orientation of radius ε about z and z_0 respectively. By Cauchy's integral formula,

$$\int_{C_\varepsilon(z)} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi = 2\pi i f(z)$$

Let $M > 0$ be such that $|f(\xi)| < M$ for all $\xi \in U$ and note that for $\xi \in C_\varepsilon(z_0)$,

$$|\xi - z| \geq |z - z_0| - |\xi - z_0| = |z - z_0| - \varepsilon$$

Thus,

$$\left|\int_{C_\varepsilon(z_0)} \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi\right| \leq \int_{C_\varepsilon(z_0)} \frac{M}{|\xi - z|}\,\mathrm{d}\xi \leq \frac{M}{|z - z_0| - \varepsilon} 2\pi\varepsilon \xrightarrow{\varepsilon\to 0} 0$$

Combining our results into equation (10) and letting ε tend to 0, we infer that

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi \qquad \forall z \in U \setminus \{z_0\}$$

To complete the proof of the theorem, it suffices to show that the function

$$h(z) := \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi$$

is holomorphic on U. Let T be a triangle contained in U, then by Fubini-Tonelli,

$$\int_T h(z)\,\mathrm{d}z = \int_T \int_C \frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi\,\mathrm{d}z = \int_C \int_T \frac{f(\xi)}{\xi - z}\,\mathrm{d}z\,\mathrm{d}\xi = 0$$

Indeed, for fixed $\xi \in C$, the function $\frac{f(\xi)}{\xi - z}$ is holomorphic and by Goursat's theorem 2.1, the integral evaluates to 0. By Morera's theorem 2.21, h is indeed holomorphic. □

Corollary 3.10. *Suppose that f has an isolated singularity at z_0, then f has a pole at z_0 if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof. The forward direction is clear. We prove the converse by a contrapositive argument. If $|f|$ tends to infinity, then $1/f(z) \to 0$ as $z \to z_0$ and is bounded about z_0. Thus, $1/f$ has a removable singularity at z_0 and f has a pole at z_0. □

Lecture 11

Essential singularities.

Example 3.4. Consider $f(z) = e^{1/z}$ defined on $\mathbb{C} \setminus \{0\}$. If z is real, then

$$\lim_{z \to 0^+} f(z) = \infty \quad \text{and} \quad \lim_{z \to 0^-} f(z) = -\infty$$

On the other hand, if z is purely imaginary, i.e. $z = \varepsilon i$, then $f(z)$ is bounded (by 1) as $\varepsilon \to 0$. Therefore, the point $z_0 = 0$ is an essential singularity.

Theorem 3.11 (Casorati-Weierstrass). *Suppose that f is holomorphic in the punctured disk $D_r(z_0) \setminus z_0$ and has an essential singularity at z_0. Then the image of $D_r(z_0) \setminus \{z_0\}$ under f is dense in $\mathbb{C}$.*

Proof. By way of contradiction, suppose that $f(D_r(z_0) \setminus \{z_0\})$ is not dense in $\mathbb{C}$. Then there exists $w \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - w| > \delta$$

for all $z \in D_r(z_0) \setminus \{z_0\}$. Thus, the function

$$g(z) = \frac{1}{f(z) - w}$$

is holomorphic and bounded in $D_r(z_0)$. By Riemann's theorem on removable discontinuities 3.9, z_0 is a removable discontinuity of g. Ergo, we may assign to $g(z_0)$ a value such that g is holomorphic. If $g(z_0) = 0$, then $1/g(z) = f(z) + w$ and $f(z)$ have a pole at z_0, contradicting the assumption that z_0 is an essential singularity. On the other hand, if $g(z_0) \neq 0$, then $1/g(z) = f(z) = w$ is holomorphic at z_0 which is again a contradiction. □

Meromorphic functions.

Definition 3.12. *A function f on an open set Ω is meromorphic if there exists a sequence $\{z_k\}_{k=1}^{\infty}$ with no convergent subsequence in Ω such that f is holomorphic on $\Omega \setminus \{z_k\}$ and f has a pole at each z_k.*

It will be useful to define these types of functions on the extended complex plane. To do so, it suffices to define what it means for f to be holomorphic or have a pole at infinity. To this end, suppose that f is holomorphic for all large $z \in \mathbb{C}$ and define $F(z) = f(1/z)$ which is holomorphic near the origin. We say that f has a pole at infinity if F has a pole at the origin. Likewise, f is holomorphic at infinity if$F$ is holomorphic at the origin and f has a removable singularity at infinity if F has a removable singularity at the origin.

Thus, a function is **meromorphic** on the extended complex plane if it has finitely many poles. Indeed, if f has infinitely many poles $\{z_k\}_{k=1}^{\infty}$, then there must exist a subsequence $\{z_{k_n}\}$ which converges in $\overline{\mathbb{C}}$.

Theorem 3.13. *A function f is meromorphic on the extended complex plane if and only is it is rational (i.e. of the form $P(z)/Q(z)$ for some polynomials P, Q).*

Proof. Clearly, all rational functions are meromorphic on the extended complex plane. To prove the forward direction, suppose that f is meromorphic on the extended complex plane and let $\{z_k\}_{k=1}^{n}$ denote the finite poles of f.

About every pole z_k, the function f may be separated between it's principle part f_k and it's (holomorphic) residue g_k. If the point at infinity is a pole, then $F(z) = f(1/z)$ has a pole at the origin and may also be written in terms of it's principal part $F_\infty(z)$ and residue $G_\infty(z)$. Thus, there exists a "neighbourhood of the point at infinity" on which $f \equiv f_\infty + g_\infty$ where $f_\infty(z) = F_\infty(1/z)$ and $g_\infty(z) = G_\infty(1/z)$. By neighbourhood of the point at infinity, we mean a set of the form $\{z \in \mathbb{C} \mid |z| > R\}$ for some $R > 0$. If f is holomorphic at infinity, let $f_\infty = g_\infty = 0$.

Now, note that

$$f_\infty(z) + \sum_{k=1}^{\infty} f_k(z)$$

is a rational function. Hence, to complete the proof it suffices to show that

$$H(z) = f(z) - f_\infty(z) - \sum_{k=1}^{\infty} f_k(z)$$

is constant.

First note that H is entire. Indeed, H is clearly holomorphic away from any pole and since the principle part about each pole has been subtracted, H is also holomorphic at every pole.

We also claim that H is bounded. Recall that we have separated the function $F(z) = f(1/z)$ into it's principle part F_∞ and residue G_∞. Let $\overline{D}_r(0)$ be the closed disk on which $F \equiv F_\infty + G_\infty$. Since continuous function on compact sets are bounded, G_∞ is bounded on $\overline{D}_r(0)$. Thus, on the set $\{z \in \mathbb{C} \mid |z| > 1/r\}$ the function $g_\infty(z) = G_\infty(1/z)$ is bounded and $f \equiv f_\infty + g_\infty$. Moreover, we may pick r sufficiently small so that $|z_k| \leq 1/r$ for each $k = 1, \ldots, n$. Then f_k is bounded on $\{z \in \mathbb{C} \mid |z| > 1/r\}$ for each k and, in particular, H is bounded in that set. Finally, since H is entire, it is also bounded in $\{z \in \mathbb{C} \mid |z| \leq 1/r\}$.

We conclude that H is entire and bounded. By Liouville's theorem 2.13, H is indeed constant. ☐

Principle of argument and it's applications.

Theorem 3.14 (Principle of argument). *Suppose that f is a meromorphic function in an open set containing a toy contour γ and it's interior. If f has no zeros/poles on γ, then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \#\{\textit{zeros in } \gamma\} - \#\{\textit{poles in } \gamma\}$$

where the zeros and poles are counted with their multiplicities.

Proof. Consider the function f which is holomorphic with a zero of order n at z_0. Then we may write $f(z) = (z - z_0)^n g(z)$ for some non-vanishing holomorphic function g in which case

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)}$$

Similarly, if f has a pole at z_0, we may find a holomorphic function H such that

$$\frac{f'(z)}{f(z)} = \frac{-n}{z - z_0} + H(z)$$

Finally, the theorem follows from the residue theorem. □

Theorem 3.15 (Rouché's Theorem). *Consider two functions f and g which are holomorphic in an open set Ω containing a circle C and it's interior. If $|f(z)| > |g(z)|$ for all $z \in C$, then f and $f + g$ have the same number of zeros on the interior of C.*

Proof. Let $f_t(z) = f(z) + tg(z)$ for any $t \in [0, 1]$ and let n_t denote the number of zeros of f_t inside C (counting multiplicities). By the argument principle, we have

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}\,\mathrm{d}z$$

We observe that f_t'/f_t is continuous in $t \in [0, 1]$ and $z \in C$. It follows that n_t is continuous in t. Then since n_t is integer, it must be constant. □

Theorem 3.16 (Open mapping theorem). *If f is a non-constant holomorphic function in some region Ω then f is open. That is, f maps open sets to open sets.*

Proof. Given $w_0 = f(z_0)$, we need to show that there exists a neighbourhood of w_0 which is contained in $f(\Omega)$.

By the identity theorem 2.18, there exists $\delta > 0$ such that if $0 < |z - z_0| < \delta$ then $f(z) \neq w_0$. Indeed, if it were not the case then we could find a sequence $z_k \to z_0$ such that $f(z_k) = w_k$ for all $k \in \mathbb{N}$ and by the theorem, $f \equiv w_0$ on Ω. Pick $\varepsilon > 0$ such that $|f(z) - w_0| \geq \varepsilon$ whenever $|z - z_0| = \delta$. Consider now any w such that $|w_0 - w| < \varepsilon$ and define $F(z) := f(z) - w_0$ and $G(z) := w - w_0$. By Rouché's theorem, $F + G$ and F have the same number of zeroes on the interior of the circle $C = \{z \in \mathbb{C} \mid |z - z_0| = \delta\}$. Since $F(z_0) = 0$, there exists z in the interior of C such that

$$0 = F(z) + G(z) = f(z) - w$$

That is, $w \in f(\Omega)$. Since w was an arbitrary element in $D_\varepsilon(w_0)$, we have $D_\varepsilon(w_0) \subseteq f(\Omega)$ which conclude the proof. □

Lecture 12

Theorem 3.17 (Maximum modulus principle). *If f is a non-constant holomorphic on a open set Ω, then f does not achieve it's maximum on that region.*

Proof. This theorem follows almost immediately from the open-mapping theorem. Indeed, given $z \in \Omega$, there exists a neighbourhood of $f(z)$ which is also in the image of f. Since this neighbourhood must contain some point w such that $|w| > |f(z)|$, f cannot reach it's maximum at z. Since z was an arbitrary point in Ω, f cannot reach it's maximum on that set. □

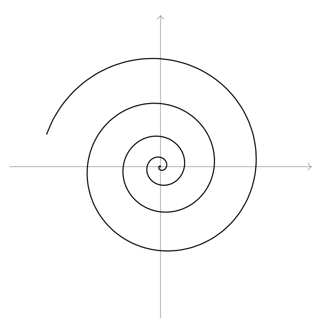
Corollary 3.18. *If f is a non-constant holomorphic on a bounded open set Ω, then*

$$\sup_{z\in\partial\Omega} |f(z)| \geq \sup_{z\in\Omega} |f(z)|$$

Complex Logarithm. So far, we have generalized many functions to the complex plane. What about the logarithm? Does there exist a *continuous* function $\log(z)$ such that $e^{\log(z)} = z$ for every $z \in \mathbb{C}$? Note first that at 0, the logarithm cannot be defined. Is it possible to define a continuous function on $\mathbb{C} \setminus \{0\}$ which maps any $z = re^{i\theta}$ to

$$\log(r) + i\theta?$$

Since θ may always be picked up to some multiple of 2π, the complex logarithm is certainly not unique. What if we restrict θ? Then can we find a continuous logarithm? Consider a spiral turning about the origin



Given any restriction of θ, the logarithm cannot be continuous since it will have to "jump" after every full loop. For instance, if we restrict $\theta \in [0, 2\pi)$, then after one full turn, we will see the logarithm take on values near $\log(r) + 2\pi i$ and then jump to values near $\log(r) + 0i$.

Theorem 3.19. *Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain 2.6 which contains 1 but not the origin. Then there exists a function F such that*

(1) *F agrees with the usual logarithm for all real numbers near 1,*
(2) $z = e^{F(z)}$ *for every* $z \in \Omega$,
(3) *F is holomorphic in* Ω.

Proof. Consider the function $f(z) = 1/z$ and we claim that

$$F(z) = \int_\gamma f(z)\,\mathrm{d}z$$

where γ is any path contained in Ω connecting 1 to z satisfies the 3 properties of the theorem. F is well-defined since Ω is simply connected and holomorphic since f is holomorphic. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}z} ze^{-F(z)} = e^{-F(z)} - ze^{-F(z)}\frac{1}{z} = 0$$

and $1e^{F(1)} = e^0 = 1$. Thus, $ze^{-F(z)}$ is a constant function equal to 1, rearranging the terms establishes property (2). Finally, since Ω is open, there exists a neighbourhood of 1 which is entirely contained in Ω. For any real number r in that neighbourhood, the segment from 1 to r is contained in Ω. Thence,

$$F(r) = \int_1^r \frac{1}{z}\,\mathrm{d}z = \log(r)$$

□

Remark 3.5. Note that in general, the identity $\log(xy) = \log(x) + \log(y)$ does not hold for the complex logarithm. Consider for instance $x = e^{3i\pi/2}$ and $y = e^{i\pi/4}$.

We now provide a construction of the logarithm which is more useful in practice.

Theorem 3.20. *If* f *is a non-vanishing holomorphic on and open, simply-connected set* Ω. *Then there exists a function* F *such that*

(1) $e^F(z) = f(z)$ *for all* $z \in \Omega$,
(2) *F is holomorphic in* Ω.

Proof. Fix $z_0 \in \Omega$, let c_0 be such that $e^{c_0} = f(z_0)$ and define

$$F(z) = \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z + c_0$$

Then F is holomorphic, $e^F(z_0) = e^c_0 = f(z_0)$ and since

$$\left(f(z)e^{-F(z)}\right)' = f'(z)e^{-F(z)} - f(z)\frac{f'(z)}{f(z)}e^{-F(z)} = 0$$

we conclude that $e^F(z) = f(z)$ for all $z \in \Omega$. □

Remark 3.6. Suppose we would like to find the "complex logarithm" explicitly on the disk of radius 1 about 1. The function

$$f(z) = f(re^{i\theta}) = \log(r) + i\theta$$

where θ is restricted to the interval $(-\pi, \pi)$ is a satisfying solution and we call this the **principle branch** of the logarithm. One can show that any branch of the logarithm is of the above form, with possibly different restrictions on θ. Since f is holomorphic in a disk, it has a power series representation. Specifically, for all $z \in \mathbb{D}$ (the unit disk), we find that

$$f(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n$$

If $z \notin \mathbb{D}$, the we cannot guarantee that the above sum will converge (see proposition 1.17).

Exercises and solutions

Problem 1. Evaluate

$$\int_{-\infty}^{\infty} \frac{1}{1+z^4}\,\mathrm{d}z$$

Solution. Define $f(z) := 1/(1+z^4)$ has poles whenever $1+z^4 = 0$ which has roots $\left\{e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}\right\}$ each of degree 1.

$$\mathrm{res}_{e^{i\pi/4}}(f) = \lim_{z \to e^{i\pi/4}} \frac{\left(z - e^{i\pi/4}\right)}{1+z^4} = \frac{1}{4e^{3i\pi/4}}$$

and

$$\mathrm{res}_{e^{3i\pi/4}}(f) = \lim_{z \to e^{3i\pi/4}} \frac{\left(z - e^{3i\pi/4}\right)}{1+z^4} = \frac{1}{4e^{9i\pi/4}} = \frac{1}{4e^{i\pi/4}}$$

Define C_R to be the semi-circle parameterized by Re^{it} for $t \in [0, \pi]$. By the residue theorem,

$$\int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi i \left(\frac{1}{4e^{3i\pi/4}} + \frac{1}{4e^{i\pi/4}}\right) = 2\pi i \left(\frac{e^{3i\pi/4} + e^{i\pi/4}}{4e^{3i\pi/4}e^{i\pi/4}}\right) = \frac{\pi}{\sqrt{2}}$$

Furthermore, for any $R > 1$,

$$\left|\int_{C_R} f(z)\,\mathrm{d}z\right| = \left|\int_0^{\pi} \frac{Rie^{it}}{1+(Re^{it})^4}\,\mathrm{d}t\right| \le \int_0^{\pi} \left|\frac{Rie^{it}}{1+R^4e^{4it}}\right| \mathrm{d}t \le \int_0^{\pi} \frac{R}{R^4-1}\,\mathrm{d}t = \frac{R\pi}{R^4-1}$$

Finally, taking that limit as $R \to \infty$, we infer that

$$\lim_{R\to\infty} \int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} \frac{1}{1+z^4}\,\mathrm{d}z = \frac{\pi}{\sqrt{2}}$$

□

Problem 2. Evaluate

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}$$

for any $a > 0$.

Solution. Define the meromorphic function

$$f(z) = \frac{ze^{iz}}{z^2 + a^2}$$

which has poles of degree 1 at $z = ia$ and $z = -ia$. By the residue theorem,

$$\int_{-R}^{R} f(z)\,\mathrm{d}z + \int_{C_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname{res}_{ia}(f) = 2\pi i \lim_{z \to ia} \frac{(z - ia)\, ze^{iz}}{z^2 + a^2} = i\pi e^{-a}$$

for any $R > 0$, where C_R to be the semi-circle parameterized by Re^{it} for $t \in [0, \pi]$. Now, note that for sufficiently large R we have

$$\begin{aligned} \left| \int_{C_R} f(z)\,\mathrm{d}z \right| = \left| \int_0^{\pi} \frac{iRe^{it} Re^{it} e^{iRe^{it}}}{(Re^{it})^2 + a^2}\,\mathrm{d}t \right| &\leq \int_0^{\pi} \left| \frac{iR^2 e^{2it} e^{iR(cos(t) + i\sin(t))}}{R^2 e^{2it} + a^2} \right| \mathrm{d}t \\ &\leq \int_0^{\pi} \frac{R^2 e^{-R\sin(t)}}{R^2 - a^2}\,\mathrm{d}t \\ &= \frac{R^2}{R^2 - a^2} \int_0^{\pi} e^{-R\sin(t)} \end{aligned}$$

Noting that the function $\sin(x)$ is symmetric about π and concave from 0 to $\pi/2$, we write

$$\begin{aligned} \left| \int_{C_R} f(z)\,\mathrm{d}z \right| \leq \frac{2R^2}{R^2 - a^2} \int_0^{\pi/2} e^{-R\sin(t)}\,\mathrm{d}t &\leq \frac{2R^2}{R^2 - a^2} \int_0^{\pi/2} e^{-R2t/\pi}\,\mathrm{d}t \\ &= \frac{2R^2}{R^2 - a^2} \frac{-\pi}{2R} \left(e^{-R} - 1\right) \xrightarrow{R \to \infty} 0 \end{aligned}$$

Ergo, taking the limit as $R \to \infty$ yields

$$\int_{-\infty}^{\infty} f(z)\,\mathrm{d}z = \int_{-\infty}^{\infty} \frac{ze^{iz}}{z^2 + a^2}\,\mathrm{d}z = i\pi e^{-a}$$

Finally, equating the imaginary parts on either sides of the above equation, we conclude that

$$\int_{-\infty}^{\infty} \frac{z \sin(z)}{z^2 + a^2}\,\mathrm{d}z = \pi e^{-a}$$

□

Problem 3. Evaluate

$$\int_0^{2\pi} \frac{1}{(a+\cos(\theta))^2}\, \mathrm{d}\theta$$

for any real number $a > 1$.

Solution. By Euler's formula, the above integral is given by

$$\int_0^{2\pi} \frac{4}{\left(2a+e^{i\theta}+e^{-i\theta}\right)^2}\, \mathrm{d}\theta = \int_{|z|=1} \frac{4}{iz\left(2a+z+z^{-1}\right)^2}\, \mathrm{d}z = \int_{|z|=1} \frac{-4zi}{\left(2az+z^2+1\right)^2}\, \mathrm{d}z$$

Define

$$f(z) := \frac{-4zi}{\left(2az+z^2+1\right)^2}$$

A series of algebraic manipulations show that f has a single pole inside the unit circle at $-a+\sqrt{a^2-1}$. By the residue theorem,

$$\int_0^{2\pi} \frac{1}{(a+\cos(\theta))^2}\, \mathrm{d}\theta = \int_{|z|=1} \frac{-4i}{\left(2az^2+z^3+z\right)^2}\, \mathrm{d}z = 2\pi i\left(\mathrm{res}_{a_1}(f)\right) = \frac{2\pi a}{\left(a^2-1\right)^{3/2}}$$

□

Problem 4. How many zeroes does

$$z^7 - 2z^5 + 6z^3 - z + 1$$

have inside the unit circle?

Solution. Define

$$f(z) := 6z^3 - z, \quad g(z) := z^7 - 2z^5 + 1$$

On the circle $|z| = 1$,

$$|f(z)| = \left|6z^3 - z\right| \geq 5 > 4 = \left|z^7\right| + \left|2z^5\right| + 1 \geq \left|z^7 - 2z^5 + 1\right| = |g(z)|$$

Therefore, by Rouché's theorem, f and $f+g$ have the same number of zeros in the unit circle. Now,

$$f(z) = z\left(6z^2-1\right) = z\left(\sqrt{6}z-1\right)\left(\sqrt{6}z+1\right)$$

has three zeroes inside the unit circle. We conclude that $f+g$ has exactly three roots inside the unit circle. □

Problem 5. Prove that any entire injective function is linear.

Solution. Let f be entire and injective and define $F(z) = f(1/z)$ which is also injective and holomorphic away from the origin. Suppose first that F has an essential singularity at 0. By the open-mapping theorem, the set $U := F(\{z \mid 1 < |z| < 2\})$ is open and by the Casorati-Weierstrass theorem, $V := F(\{z \mid 0 < |z| < 1\})$ is dense in the complex plane. Thus, $U \cap V$ is non-empty. Let $y \in U \cap V$, then $F(x_1) = y = F(x_2)$ for some $0 < |x_1| < 1$ and $1 < |x_2| < 2$, which contradicts the injectivity of F.

Since $F(z)$ does not have an essential singularity at the origin, f is meromorphic in the extended complex plane. By theorem 3.13, there exists polynomials P, Q such that

$$f(z) = \frac{P(z)}{Q(z)}$$

Without loss of generality, we suppose that P, Q do not share any roots.

Since f is entire, Q must be constant. Otherwise, Q has at least one root at which f would have a pole. Thus f is a polynomial. By injectivity, f cannot be constant. By the fundamental theorem of algebra, f has at least one root. Since f is injective, the roots of f cannot be distinct. We deduce that $f(z) = (z - c)^n$ for some $c \in \mathbb{C}$ and positive integer $n \geq 1$. For $x_1 = e^{2i\pi/n} + c$ and $x_2 = 1 + c$, we have

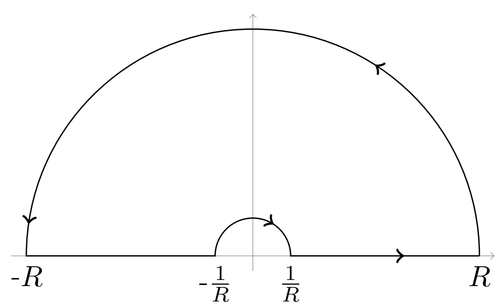
$$f(x_1) = (x_1 - c)^n = \left(e^{2i\pi/n}\right)^n = e^{2i\pi} = 1 = (1 + c - c)^n = (x_2 - c)^n = f(x_2)$$

By injectivity of f, we must have $x_1 = x_2$, i.e. $n = 1$. □

Problem 6. Evaluate

$$\int_0^\infty \frac{\log(x)}{x^2 + a^2}\,\mathrm{d}x$$

Solution. Define $f(x) = \log(x)/(x^2 + a^2)$ and consider the contour γ_R



By the residue theorem, the integral over this entire path is

$$2\pi i \operatorname{res}_{ia} f = 2\pi i \lim_{z \to ia} (z - ia) \frac{\log(z)}{z^2 + a^2} = 2\pi i \frac{\log(ia)}{2ia} = \pi \frac{\log(a)}{a} + \pi^2 \frac{i}{2a}$$

Now, the integral over the large semi-circle is bounded by

$$\left|\int_0^\pi \frac{iRe^{it}\log\left(Re^{it}\right)}{\left(Re^{it}\right)^2+a^2}\,\mathrm{d}t\right| \le \int_0^\pi \left|\frac{iRe^{it}\log(R)-tRe^{it}}{\left(Re^{it}\right)^2+a^2}\right|\,\mathrm{d}t \xrightarrow{R\to\infty} 0$$

Likewise, if $\varepsilon = 1/R$, the integral over the small semi-circle is bounded by

$$\left|\int_0^\pi \frac{i\varepsilon e^{it}\log\left(\varepsilon e^{it}\right)}{\left(\varepsilon e^{it}\right)^2+a^2}\,\mathrm{d}t\right| \le \int_0^\pi \left|\frac{i\varepsilon e^{it}\log(\varepsilon)-t\varepsilon e^{it}}{\left(\varepsilon e^{it}\right)^2+a^2}\right|\,\mathrm{d}t \xrightarrow{R\to\infty} 0$$

Integrating over the first line segment yields

$$\int_{-R}^{-1/R} \frac{\log(x)}{x^2+a^2}\,\mathrm{d}x = \int_{-R}^{-1/R} \frac{\log(|x|)+i\pi}{x^2+a^2}\,\mathrm{d}x \xmapsto{-x\mapsto x} \int_{1/R}^{R} \frac{\log(x)}{x^2+a^2}\,\mathrm{d}x + i\pi\int_{1/R}^{R}\frac{1}{x^2+a^2}\,\mathrm{d}x$$

Finally, combining what we have so far, we find that as $R\to\infty$ the integral over the entire path is given by

$$2\int_0^\infty \frac{\log(x)}{x^2+a^2}\,\mathrm{d}x + i\pi\int_0^\infty \frac{1}{x^2+a^2}\,\mathrm{d}x = \pi\frac{\log(a)}{a} + \pi^2\frac{i}{2a}$$

Equating the real parts of the above equation yields the desired result. $\square$

Chapter 4

Entire functions

In this section, we study entire functions, that is functions that are holomorphic on the whole complex plane. We adopt the following notation; given $R > 0$, D_R denotes the open disk of radius R about the origin and $C_R = \partial D_R$ the circle of radius R about the origin.

Lecture 13

Jensen's Formula.

Theorem 4.1. *Let Ω be an open set containing D_R and C_R for some $R > 0$. Suppose that f is a holomorphic function on Ω which does not vanish at the origin nor on C_R and let $z_1, \dots z_N$ denote the zeroes of f inside D_R (counted with multiplicities). Then*

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right| \, \mathrm{d}\theta \tag{11}$$

Proof. We let $J_{\Omega,R}$ denote the set of functions which satisfy the conditions of our theorem and for which Jensen's formula (11) holds true. The proof will be separated into four steps.

(1) We first show that $J_{\Omega,R}$ is closed under multiplication. That is, if $f_1, f_2 \in J_{\Omega,R}$, then $f_1 f_2 \in J_{\Omega,R}$.

(2) If g is holomorphic on Ω and does not vanish on $\overline{D_R}$ (the closure of D_R), then $g \in J_{\Omega,R}$.

(3) Every function of the form $z - w$ where $w \in D_R$, $w \neq 0$ is in $J_{\Omega,R}$

(4) Given a function f satisfying the condition to our theorem, there exists a non-vanishing holomorphic function g such that

$$f(z) = (z - z_1)\cdots(z - z_N)g(z)$$

on $\overline{D_R}$. Thus, by (1), (2) and (3), we conclude that $f \in J_{\Omega,R}$.

Step 1. Given $f_1, f_2 \in J_{\Omega,R}$, let $z_1, \ldots, z_N$ denote the zeroes of f_1 and $z_{N+1}, \ldots, z_M$ denote the zeroes of f_2. Then $f_1 f_2$ has zeroes $z_1, \ldots, z_N, z_{N+1}, \ldots, z_M$ and

$$\begin{aligned}\log|f_1 f_2(0)| &= \log|f_1(0)| + \log|f_2(0)| \\ &= \sum_{k=1}^{M} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f_1\left(Re^{i\theta}\right)\right| + \log\left|f_2\left(Re^{i\theta}\right)\right| \,\mathrm{d}\theta \\ &= \sum_{k=1}^{M} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|[f_1 f_2]\left(Re^{i\theta}\right)\right| \,\mathrm{d}\theta\end{aligned}$$

That is, $f_1 f_2 \in J_{\Omega,R}$.

Step 2. Suppose that g is holomorphic and does not vanish on the closure of D_R, then by theorem 3.20, there exists a holomorphic function h such that $g(z) = e^{h(z)}$ for all $z \in D_R$. By Cauchy's integral formula 7,

$$h(0) = \frac{1}{2\pi i}\int_{|\xi|=R} \frac{h(\xi)}{\xi}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_0^{2\pi} h(Re^{it})\,\mathrm{d}t$$

Ergo,

$$\log|g(0)| = \Re(h(0)) = \frac{1}{2\pi}\int_0^{2\pi} \Re\left(h(Re^{it})\right)\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} \log\left|g\left(Re^{it}\right)\right|\,\mathrm{d}t$$

Step 3. We first leave it is an exercise to show that whenever $|a| < 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} - a\right|\,\mathrm{d}\theta = 0$$

Then given $w \in D_R \setminus \{0\}$, $f(z) = z - w$ we have

$$\log\left(\frac{|w|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|Re^{i\theta} - w\right|\,\mathrm{d}\theta = \log|w| + \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} - \frac{w}{R}\right|\,\mathrm{d}\theta = \log|w|$$

which is precisely Jensen's formula for f. □

Lemma 4.2. *Let Ω be an open set containing D_R and C_R for some $R > 0$. Suppose that f is a holomorphic function on Ω which does not vanish at the origin nor on C_R and let $z_1, \ldots z_N$ denote the zeroes of f inside D_R (counted with multiplicities). Then*

$$\int_0^R \frac{n(r)}{r}\,\mathrm{d}r = \sum_{k=1}^{N} \log\left(\frac{R}{|z_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(Re^{i\theta}\right)\right|\,\mathrm{d}\theta - \log|f(0)|$$

Proof. The second equality is an immediate consequence of Jensen's formula. For the first equality, define

$$\eta_k(r) = \begin{cases} 0 & |z_k| \le r \\ 1 & |z_k| > r \end{cases}$$

and note that

$$n(r) = \sum_{k=1}^{N} \eta_k(r)$$

Thus,

$$\sum_{k=1}^{N} \log\left(\frac{R}{|z_k|}\right) = \sum_{k=1}^{N} \int_{|z_k|}^{R} \frac{1}{r}\,\mathrm{d}r = \sum_{k=1}^{N} \int_0^R \eta_k(r)\frac{1}{r}\,\mathrm{d}r = \int_0^R \sum_{k=1}^{N} \eta_k(r)\frac{1}{r}\,\mathrm{d}r = \int_0^R \frac{n(r)}{r}\,\mathrm{d}r$$

□

Functions of finite order.

Definition 4.3 (Growth order). *An entire function f is said to be of finite order if there exists $\rho > 0$ and constants $A, B > 0$ such that*

$$|f(z)| \le Ae^{B|z|^\rho} \qquad \forall z \in \mathbb{C} \tag{12}$$

The growth order of f is $\rho_f = \inf \rho$ where the infimum is taken over all $\rho > 0$ for which there exists constants A, B such that equation (12) *holds.*

Theorem 4.4. *If f is an entire function with growth order ρ, then*

(1) *There exists a constants $C, R > 0$ such that $n(r) \le Cr^\rho$ for all $r > R$ ($n(r)$ denotes the number of zeroes in D_r).*

(2) *If $z_1, z_2, \ldots$ denote the zeroes of f and $z_k \neq 0$ for all $k \in \mathbb{N}$, then for all $s > \rho$,*

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty$$

Proof. Suppose first that $f(0) \neq 0$ and note that since $n(r)$ is an increasing function,

$$n(r) = \frac{n(r)}{\log(2)} \int_r^{2r} \frac{1}{x}\,\mathrm{d}x \le \frac{1}{\log(2)} \int_r^{2r} \frac{n(x)}{x}\,\mathrm{d}x$$

By the previous lemma,

$$\int_r^{2r} \frac{n(x)}{x}\,\mathrm{d}x \le \frac{1}{2\pi} \int_0^{2\pi} \log\left|f\left(2re^{it}\right)\right|\,\mathrm{d}t - |f(0)|$$

Then there exists $R > 0$ such that for all $r > R$,

$$\int_0^{2\pi} \log|f(2re^{it})|dt \le \int_0^{2\pi} \log|A\exp\{B(2r)^\rho\}|\,\mathrm{d}t \le Cr^\rho$$

for some constant C. We conclude that $n(r) \leq Cr^\rho$ for all $r > R$. Suppose now that f has a pole of degree m at the origin and define

$$g(z) = \frac{f(z)}{z^m}$$

which has the same zeroes as f and a removable discontinuity at the origin. Thus, $n(r) \leq m + Cr^\rho$ for all $r > R$. An adjustment in C and/or R yields $n(r) \leq Cr^\rho$ for all $r > R$.

For the second part, let $n = \lceil \log(R) \rceil$ and since $s > \rho$ we have

$$\sum_{|z_k| \geq R} \frac{1}{|z_k|^s} = \sum_{j=n}^{\infty} \sum_{2^j \leq |z_k| < 2^{j+1}} \frac{1}{|z_k|^s} \leq \sum_{j=n}^{\infty} 2^{-js} n(2^{j+1}) \leq C \sum_{j=n}^{\infty} 2^{-js} 2^{\rho(j+1)} < \infty$$

□

Lecture 14

Infinite products.

Proposition 4.5. *Consider a sequence of complex numbers $\{a_n\}_{n=1}^\infty$. If $\sum_{n=1}^\infty |a_n|$ converges then the product*

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges. Furthermore, the above product tends to zero if and only if one of the factors is 0.

Proposition 4.6. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of holomorphic functions on an open set Ω. If there exists constants $c_n > 0$ such that $\sum_{n=1}^\infty c_n < \infty$ and $|F_n(z) - 1| \leq c_n$ for all $z \in \Omega$, then*

(1) $\prod_{n=1}^\infty F_n(z)$ converges uniformly in Ω to some holomorphic function F,

(2) If $F_n(z)$ does not vanish for any n, then

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$$

Proof. Let $\{a_n\}_{n=1}^\infty$ be the sequence define by $|F_n(z)| = 1 + a_n(z)$ and note that $|a_n(z)| \leq c_n$. By the previous proposition, the product $\prod_{n=1}^\infty F_n(z)$ converges. Since the constants c_n are independent of z, the product converges uniformly. In particular, by theorem 2.23, F is holomorphic and for every compact set $K \subseteq \Omega$, $G_N \to F$ uniformly and $G_N' \to F'$ uniformly where

$$G_N(z) = \prod_{n=1}^{N} F_n(z)$$

Since F_n does not vanish for any n, G_N is bounded below on K. Thus,

$$\sum_{n=1}^{N} \frac{F_n'(z)}{F_n(z)} = \frac{G_N'(z)}{G_N(z)} \to \frac{F'(z)}{F(z)} \qquad \forall z \in K$$

Recalling that K was an arbitrary compact set in Ω, the above must hold for all points in Ω. □

Example 4.1. An application of the above theorem is the product formula for sin. We aim to establish the identity

$$\frac{\sin(\pi z)}{\pi} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

First, we shall prove that

$$\pi\cot(\pi z) = \sum_{n=-\infty}^{\infty}\frac{1}{z+n} = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z+n} = \frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

We will show that the left and right hand sides of the above equation satisfy

(1) $F(z+1) = F(z)$ whenever $z \notin \mathbb{Z}$,
(2) $F(z) = \frac{1}{z} + F_0(z)$ where F_0 is holomorphic near 0,
(3) $F(z)$ has simple poles at each $z \in \mathbb{Z}$ and no other singularities.

For the left hand side, one can see that the above properties holds by writing

$$\pi\cot(\pi z) = \pi\frac{\cos(\pi z)}{\sin(\pi z)}$$

As for the right hand side, it is clear that

$$\sum_{n=-\infty}^{\infty}\frac{1}{z+n}$$

satisfies (1). Furthermore, by considering the representation

$$\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

one find that the right hand side clearly satisfies (2) and (3). Now, consider

$$\Delta(z) = F(z) - \sum_{n=1}^{\infty}\frac{1}{z+n}$$

We claim that Δ is constant. By (2), the singularity of Δ at 0 is removable. By periodicity (property (1) above), Δ has a removable singularity at every integer and we may conclude that Δ is entire. Furthermore, using again the periodicity of Δ, it suffices to show that Δ is bounded when $|\Re(z)| \leq 1/2$ in order to show that Δ is bounded in $\mathbb{C}$. Since Δ is holomorphic, it is bounded when $|\Im(z)| \leq 1$. Consider the case $\Im(z) > 1$;

$$|\cot(\pi z)| = \left|\frac{e^{-2\pi y}+e^{-2\pi ix}}{e^{-2\pi y}-e^{-2\pi ix}}\right| \leq \frac{e^{-2\pi y}+1}{e^{-2\pi y}-1}$$

where $z = x + iy$. The above is clearly bounded for $y > 1$. Similarly, we bound the right hand side;

$$\left|\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2+n^2}\right| = \left|\frac{1}{x+iy} + \sum_{n=1}^{\infty} \frac{2x+2iy}{x^2+2ixy-y^2+n^2}\right| \leq C + C\sum_{n=1}^{\infty}\left|\frac{y}{y^2+n^2}\right|$$
$$\leq C + C\int_1^{\infty} \frac{y}{y^2+n^2}\,\mathrm{d}n$$

for some constant C. We conclude that Δ is entire and bounded. By Liouville's theorem 2.13, Δ must be constant. Finally, since Δ is constant, in order to conclude the proof for the cotangent identity, it suffices to show that it holds for a single point. Indeed, at $z = 1/2$ we have $\pi\cot(z\pi) = 0$ and

$$\sum_{n=-\infty}^{\infty} \frac{1}{1/2+n} = \lim_{N\to\infty} \sum_{n=0}^{N}\left[\frac{1}{1/2+n} - \frac{1}{1/2-(n+1)}\right] = 0$$

Finally, let $G(z) = \sin(\pi z)/\pi$ and

$$P(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

By proposition 4.6 and the product rule,

$$\frac{P'(z)}{P(z)} = \frac{P(z)}{zP(z)} + \sum_{n=1}^{\infty} \frac{\left(1-\frac{z^2}{n^2}\right)'}{\left(1-\frac{z^2}{n^2}\right)} = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2} \qquad \forall z \notin \mathbb{Z}$$

Ergo, by the cotangent identity,

$$\left(\frac{P(z)}{G(z)}\right) = \frac{P(z)}{G(z)}\left[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right] = 0$$

That is, for all $z \notin \mathbb{Z}$, $G(z) = cP(z)$ for some constant c ($z \in \mathbb{Z}$, then it is clear that $P(z) = G(z) = 0$). Finally,

$$c = \lim_{z\to 0}\frac{G(z)}{P(z)} = \lim_{z\to 0}\frac{\sin(\pi z)/\pi z}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)} = 1$$

Weierstrass infinite product.

Lemma 4.7. *Define the* canonical factors E_k *of degree* $k \geq 0$ *by*

$$E_k(z) = \begin{cases} (1-z) & k = 0 \\ (1-z)\exp\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^k}{k}\right) & k \geq 1 \end{cases}$$

If $|z| \leq 1/2$*, then* $|1 - E_k(z)| \leq c\,|z|^{k+1}$ *for constant some* $c > 0$ *independent of* k*.*

Proof. Using the principle branch of the logarithm, we may define (for fixed $k \geq 0$)

$$w = \log(1-z) + z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^k}{k}$$

Then $E_k = e^w$. By remark 3.6, it holds for all $|z| \leq 1/2$ that

$$|w| = \left| \sum_{n=k+1}^{\infty} \frac{z^n}{n} \right| \leq |z|^{k+1} \sum_{j=0}^{\infty} \frac{1}{2^j} \leq 2\,|z|^{k+1} \leq 1$$

Thence, there exists a constant $c > 0$ independent of k for which

$$|1 - E_k(z)| = |1 - e^w| \leq \frac{c}{2}\,|w| \leq c\,|z|^{k+1}$$

□

Theorem 4.8 (Weierstrass)**.** *Given a sequence $\{a_n\}$ of complex numbers with $|a_n| \to \infty$, there exists and entire function f which vanishes at a_n for each $n \in \mathbb{N}$ and nowhere else. Furthermore, any other such function is of the form $f(z)e^{g(z)}$ where g is entire.*

Proof. Suppose first that both f_1 and f_2 have zeroes at a_n for each $n \in \mathbb{N}$ and nowhere else. Then f_1/f_2 has removable singularities at each a_n. It is therefore nowhere vanishing entire function and by theorem 3.20 there exists an entire function g such that

$$\frac{f_1}{f_2} = e^g$$

We now prove existence. Since $|a_n| \to \infty$, we may suppose that we are given the order of the zero at the origin (m) and $a_n \neq 0$ for all $n \in \mathbb{N}$. Then define the Weierstrass product

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(z/a_n\right)$$

where the E_n's denote the canonical factors as in the above lemma. Clearly, $f(a_n) = 0$ Fix $R > 0$ and let z be a complex number such that $|z| < R$. We may decompose,

$$\prod_{n=1}^{\infty} E_n\left(z/a_n\right) = \prod_{|a_n|<2R}^{\infty} E_n\left(z/a_n\right) \prod_{|a_n|\geq 2R}^{\infty} E_n\left(z/a_n\right)$$

The first term on the right hand side is finite. Moreover, by the above lemma

$$|1 - E_n\left(z/a_n\right)| \leq c\,|z/a_n|^{n+1} \leq \frac{c}{2^{n+1}}$$

Ergo, by proposition 4.5, the product for $f(z)$ converges whenever $|z| < R$. Letting R tend to infinity, we conclude that the Weierstrass product converges for every $z \in \mathbb{C}$. Furthermore, since $E_n(1) = 0$, it is clear by the proposition that f has a zero at each a_n. Furthermore, the degree of each zero z_0 is precisely the number of times z_0 shows up in the sequence $\{a_n\}_{n=1}^{\infty}$. Finally, by proposition 4.6, f is entire. □

Lecture 15

Hadamard Factorization Theorem.

Theorem 4.9 (Hadamard's factorization theorem). *Let f be an entire function with order of growth p_0 and let $k \in \mathbb{Z}$ be such that $k \leq p_0 < k+1$. If $a_1, a_2, \ldots$ denote the (non-zero) zeroes of f then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(z/a_n\right)$$

where P is a polynomial of degree at most k and m is the order of the zero at the origin.

In order to prove this result, we will first need a few preliminary results.

Lemma 4.10. *The canonical factors E_k (see lemma 4.7) satisfy*

(1) $|E_k(z)| \geq e^{-c|z|^{k+1}}$ *for all $|z| < \frac{1}{2}$ and some constant c,*

(2) $|E_k(z)| \geq |1-z|\, e^{-c'|z|^{k+1}}$ *for all $|z| \geq \frac{1}{2}$ and some constant c'.*

Proof. Suppose first that $|z| < \frac{1}{2}$, then

$$E_k(z) = \exp\left(\log\left(1-z\right) + \sum_{n=1}^{k} \frac{z^n}{n}\right) = \exp\left(-\sum_{n=k+1}^{\infty} \frac{z^n}{n}\right) := e^w$$

Hence, noting that $|e^w| \geq e^{-|w|}$ and $|w| \leq c\,|z|^{k+1}$ we find that (1) holds true. Suppose now that $|z| \geq \frac{1}{2}$, then

$$|E_k| = |(1-z)| \left| e^{z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^k}{k}} \right| \geq |1-z|\, e^{-c'|z|^k}$$

for some constant c'. □

Lemma 4.11. *For any s with $p_0 < s < k+1$,*

$$\left|\prod_{n=1}^{\infty} E_k\left(z/a_n\right)\right| \geq e^{-c|z|^s}$$

except, possibly, when z is on the ***forbidden disks****;*

$$\left\{x \in \mathbb{C} \mid |x-a_n| < |a_n|^{-(k+1)}\right\}_{n=1}^{\infty}$$

Proof. We write

$$\prod_{n=1}^{\infty} E_k\left(z/a_n\right) = \prod_{|a_n| \leq 2|z|} E_k\left(z/a_n\right) \prod_{|a_n| > 2|z|} E_k\left(z/a_n\right) \tag{13}$$

It follows from the previous lemma that

$$\left|\prod_{|a_n|>2|z|} E_k\left(z/a_n\right)\right| = \prod_{|a_n|>2|z|} \left|E_k\left(z/a_n\right)\right| \geq \prod_{|a_n|>2|z|} e^{-c|z/a_n|^{k+1}}$$
$$= \exp\left(-c\left|z\right|^{k+1} \sum_{|a_n|>2z} \left|a_n\right|^{-k-1}\right)$$

By theorem 4.4, $\sum_{n=1}^{\infty} |a_n|^{-s} < \infty$ Since $s < k+1$ and $|a_n| > 2|z|$, there exists a constant $C > 0$ such that $|a_n|^{k+1} \geq \frac{1}{C} |a_n|^s |z|^{k+1-s}$ or equivalently $|a_n|^{-k-1} \leq C |a_n|^{-s} |z|^{s-k-1}$. Ergo,

$$\exp\left(-c\left|z\right|^{k+1} \sum_{|a_n|>2z} \left|a_n\right|^{-k-1}\right) \geq \exp\left(-c\left|z\right|^{s} \sum_{n=1}^{\infty} \left|a_n\right|^{-s}\right) = e^{-c|z|^s}$$

where in the last step we have adjusted the constant c. In order to complete the proof, we need to establish a bound on the first product of equation (13). A similar procedure yields

$$\left|\prod_{|a_n|\leq 2|z|} E_k\left(z/a_n\right)\right| = \prod_{|a_n|\leq 2|z|} \left|E_k\left(z/a_n\right)\right| \geq \prod_{|a_n|\leq 2|z|} \left|1 - \frac{z}{a_n}\right| \prod_{|a_n|\leq 2|z|} e^{-c'|z/a_n|^{k}}$$
$$\geq e^{-c|z|^s} \prod_{|a_n|\leq 2|z|} \left|1 - \frac{z}{a_n}\right|$$

Whenever z is <u>not</u> in the forbidden disks, $|a_n - z| \geq |a_n|^{-k-1}$. Therefore,

$$\prod_{|a_n|\leq 2|z|} \left|1 - \frac{z}{a_n}\right| = \prod_{|a_n|\leq 2|z|} \left|\frac{a_n - z}{a_n}\right| \geq \prod_{|a_n|\leq 2|z|} |a_n|^{-k-2}$$

The negative of the logarithm of the above is (by theorem 4.4);

$$(k+2) \sum_{|a_n|\leq 2|z|} \log|a_n| \leq (k+2) n(2|z|) \log(2|z|) \leq c|z|^s \log(2|z|) \leq c' |z|^{s'}$$

for any $s' > s > \rho_0$ and some constant c'. Since the above holds for any $s > \rho_0$, the bound also holds with s instead of s'. Ergo,

$$\prod_{|a_n|\leq 2|z|} \left|1 - \frac{z}{a_n}\right| \geq \prod_{|a_n|\leq 2|z|} |a_n|^{-k-2} \geq e^{-c'|z|^s}$$

Combining our bounds concludes the proof. □

Corollary 4.12. *There exists a sequence of radii $r_1, r_2, \ldots$ with $r_n \to \infty$ such that*

$$\prod_{n=1}^{\infty} E_k\left(z/a_n\right) \geq e^{c|z^2|} \qquad \textit{for each } |z| = r_m$$

Proof. Since the sum $\sum |a_n|^{-(k+1)}$ converges, we may find $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |a_n|^{-(k+1)} < \frac{1}{10}$$

Consider two consecutive large integers $\ell, \ell+1$ so that the *forbidden disks* about a_n do not intersect $[\ell, \ell+1]$ for all $n < N$. Then we may find $\ell \leq r < \ell+1$ such that the circle of radius r does not intersect any of the forbidden disks. Indeed, if no such r exists, then the union of intervals

$$I_n = \left(|a_n| - \frac{1}{|a_n|^{k+1}}, |a_n| + \frac{1}{|a_n|^{k+1}}\right)$$

covers $[\ell, \ell+1]$. But then, it holds that

$$1 \leq \sum_{n=N}^{\infty} m\left(I_n\right) = 2\sum_{n=N}^{\infty} \frac{1}{|a_n|^{-(k+1)}} \leq \frac{2}{10}$$

which is absurd. □

Lemma 4.13. *Suppose that g is an entire function and $u = \Re(g)$ satisfies*

$$u(z) \leq Cr_k^s \qquad |z| = r_n$$

for some constant C and a sequence $r_k \to \infty$. Then g is a polynomial of degree at most s.

Proof. Since g is entire, it has a power series centered at the origin;

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$a_n = \frac{g^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_{C_r} \frac{g(\xi)}{\xi^{n+1}}\,\mathrm{d}\xi = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(re^{it})}{(re^{it})^{n+1}} ire^{it}\,\mathrm{d}t = \frac{1}{2\pi r^n}\int_0^{2\pi} g(re^{it})e^{-int}\,\mathrm{d}t$$

To obtain the second equality we have used Cauchy's integral formula 8. The above also holds for $n < 0$, that is for all $n < 0$

$$0 = \frac{1}{2\pi r^n}\int_0^{2\pi} g(re^{it})e^{-int}\,\mathrm{d}t = \frac{1}{2\pi r^n}\int_0^{2\pi} \overline{g(re^{it})e^{-int}}\,\mathrm{d}t = \frac{1}{2\pi r^n}\int_0^{2\pi} \overline{g(re^{it})}e^{int}\,\mathrm{d}t$$

Thus, for $n > 0$,

$$\frac{1}{\pi r^n}\int_0^{2\pi} u(re^{it})e^{-int}\,\mathrm{d}t = \frac{1}{2\pi r^n}\int_0^{2\pi} \left(g(re^{it}) + \overline{g(re^{it})}\right) e^{-int}\,\mathrm{d}t = a_n$$

If $n = 0$, then the above evaluates to $2\Re(a_0)$. Finally, since the integral of e^{-int} over any circle about the origin vanishes, if $r = r_k$ for some $k \in \mathbb{N}$ then

$$|a_n| = \left| \frac{1}{\pi r^n} \int_0^{2\pi} \left[u(re^{it}) - Cr^s \right] e^{-int} \, \mathrm{d}t \right| = \frac{1}{\pi r^n} \int_0^{2\pi} Cr^s - u(re^{it}) \, \mathrm{d}t = 2Cr^{s-n} - 2\Re(a_0)$$

For all $n > s$, the above tends to 0 as $r = r_k \to \infty$. Thus, g is indeed a polynomial of degree $\leq s$. □

Proof of Hadamard's factorization theorem. Define

$$E(z) = z^m \prod_{n=1}^{\infty} E_k \left(z/a_n \right)$$

Repeating the argument of Weierstrass's theorem, one can show that E is entire. Furthermore, since E and f have the same zeroes, $f/E = e^g$ for some entire function g. By corollary 4.12, there exists a sequence $r_k \to \infty$ such that

$$e^{\Re(g(z))} = \left| \frac{f(z)}{E(z)} \right| \leq c' e^{c|z|^s} \qquad \forall \, |z| = r_k$$

Thus, $\Re(g(z)) \leq C \, |z|^s$ whenever $|z| = r_k$ and by the above lemma, g is indeed a polynomial of degree $\leq s$. □

Exercises and Solutions

Problem 1. In this section, we have often disguised the use of the **mean value property** for holomorphic function. Prove this property, i.e. if f is holomorphic in a disk $D_R(z)$ then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + reit) \, \mathrm{d}t \qquad \forall 0 < r < R$$

Hint: Use Cauchy's integral formula. Note also how we use this identity in lemma 4.13.

Problem 2. Let $t > 0$ and define

$$F(z) = \prod_{n=1}^{N} \left(1 - e^{-2\pi n t} e^{2\pi i z} \right)$$

then show that f is a holomorphic function of growth rate 2 and vanishes exactly when $z = m - int$ for any $n, m \in \mathbb{Z}$. If z_n is an enumeration of the zeroes of F, show that

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^2} = \infty$$

Note that by theorem 4.4,

$$\sum_{n=1}^{\infty} \frac{1}{|z_n|^{2+\varepsilon}} < \infty$$

for any $\varepsilon > 0$.

Solution. It follows from theorem 4.6 that F is holomorphic. Fix $z \in \mathbb{C}$ and define $N = \left\lfloor \frac{|z|}{t} \right\rfloor = c\,|z|$ for some $c > 0$. We have $F(z) = F_1(z)F_2(z)$ where

$$F_1(z) = \prod_{n=1}^{N} \left(1 - e^{-2\pi nt} e^{2\pi iz}\right) \qquad F_2(z) = \prod_{n=N+1}^{\infty} \left(1 - e^{-2\pi nt} e^{2\pi iz}\right)$$

then we find that

$$|F_2(z)| \leq \prod_{n=N+1}^{\infty} \left(1 + e^{-2\pi nt} e^{2\pi|z|}\right) \leq \prod_{n=1}^{\infty} \left(1 + e^{-2\pi nt}\right) := A$$

where A is a constant independent of z. On the other hand,

$$|F_1(z)| \leq \prod_{n=1}^{N} \left(1 + e^{-2\pi nt} e^{2\pi|z|}\right) \leq e^{2N\pi|z|} \prod_{n=1}^{N} \left(e^{-2\pi|z|} + e^{-2\pi nt}\right) \leq e^{2N\pi|z|} \prod_{n=1}^{N} 2$$

Recalling $N = c\,|z|$,

$$|F_1(z)| \leq e^{2N\pi|z|} e^{N} = e^{2c\pi|z|^2 + c|z|} \leq e^{a|z|^2}$$

for some constant a. Finally, we conclude that

$$|F(z)| = |F_1(z)|\,|F_2(z)| \leq A e^{a|z|^2}$$

Thus the growth order of F is at most 2. Since F is of the form $\prod (1 + a_n)$ where $\sum a_n < \infty$, we know from class results that $F(z) = 0$ if and only if one of the terms is 0. That is, for some $n \geq 1$,

$$1 - e^{-2\pi nt} e^{2\pi iz} = 0 \iff e^{-2\pi i(z + int)} = 1 \iff z + int = m \quad \text{for some } m \in \mathbb{Z}$$

We have showed that the zeros of F are exactly the point $z = m - int$ where $n \geq 1, m$ are integers. Now, let z_k be an enumeration of these zeroes, then

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^2} \geq \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} \frac{1}{m^2 + n^2 t^2} \sim \int_1^{\infty} \int_{-\infty}^{\infty} \frac{1}{(m^2 + n^2 t^2)} \, \mathrm{d}m \, \mathrm{d}n = \frac{\pi}{t} \int_1^{\infty} \frac{1}{n} \, \mathrm{d}n = \infty$$

By theorem 4.4 we conclude that F must have growth order 2. □

Problem 3. Prove Wallis' product formula;

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n \cdot 2n}{(2n-1)(2n+1)}$$

Solution. The product formula for sin at 1/2 yields

$$\frac{1}{\pi} = \frac{\sin\left(\pi \cdot \frac{1}{2}\right)}{\pi} = \frac{1}{2} \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^2 n^2}\right) = \frac{1}{2} \prod_{n=1}^{\infty} \frac{(2n+1)(2n-1)}{2n \cdot 2n}$$

Thus,

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n \cdot 2n}{(2n-1)(2n+1)}$$

as desired. □

Problem 4. Find the Hadamard's products of $e^z - 1$ and $\cos(\pi z)$.

Solution. The function $f(z) := e^z - 1$ has a zeroes of order 1 at $2\pi i n$ for all integers n. Furthermore, it is clear that the growth order of f is 1. By Hadamard's factorization theorem,

$$f(z) = e^{az+b} z \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} E_1\left(\frac{z}{2\pi i n}\right) = e^{az+b} z \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{2\pi i n}\right) e^{\frac{z}{2\pi i n}}$$

We may divide both sides of the above equation by z and as $z \to 0$, the left hand side, $f(z)/z = \frac{e^z - 1}{z}$, tends to 1 while the right hand tends to e^b. Ergo, $b = 0$. Moreover, we may combine the positive and negative terms of the product to find

$$g(z) := \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{2\pi i n}\right) e^{\frac{z}{2\pi i n}} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

Now, remark that $g(z)$ is even, i.e. $g(z) = g(-z)$ for all $z \in \mathbb{C}$. Thus,

$$\frac{f(z)}{f(-z)} = \frac{e^{az} z g(z)}{e^{-az}(-z) g(-z)} = -e^{2az}$$

Substituting z for $\log(2)$ in the above we deduce that $a = 1/2$. We conclude that

$$f(z) = e^{z/2} z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{4\pi^2 n^2}\right)$$

For the second part of the problem, define $f(z) := \cos(\pi z)$ which has growth order 1 and roots of order 1 at $\frac{2n+1}{2}$ for every integer n. By Hadamard's factorization theorem,

$$f(z) = e^{az+b} \prod_{n \in \mathbb{Z}} E_1\left(\frac{2z}{2n+1}\right) = e^{az+b} \prod_{n \in \mathbb{Z}} \left(1 - \frac{2z}{2n+1}\right) e^{\frac{2z}{2n+1}}$$

At $z = 0$, the left hand side is $f(0) = 1$ while the right hand side is given by e^b, thence $b = 0$. Furthermore, multiplying together each term in the multiplication for which n is non-negative with the term corresponding to $-(n+1)$, we find that

$$f(z) = e^{az} \prod_{n=0}^{\infty} \left(1 - \frac{4z^2}{(2n+1)^2}\right)$$

Finally, remark that for any z, we have $\frac{f(z)}{f(-z)} = e^{2az}$ and letting $z = 1$, yields $1 = e^{2a}$, thus $a = 0$. □

Problem 5. Prove that there exists infinitely many complex number z for which $e^z = z$.

Solution. By way of contradiction, suppose that $f(z) = e^z - z$ has finitely many zeroes $\{a_1, \ldots, a_N\}$. The function f is holomorphic of growth order 1 and is non-zero at $z = 0$, therefore by Hadamard's theorem,

$$f(z) = e^{az+b} \prod_{n=1}^{N} E_1\left(\frac{z}{a_n}\right) = e^{az+b} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right) e^{z/a_n} := e^{Az+b} g(z)$$

where g is a polynomial. Then,

$$g(z) = \frac{e^z - z}{e^{Az+b}} \in \Theta\left(e^{1-A}\right)$$

Now, if polynomial and an exponential have the same growth rate, then both are constant. Thence, $g(z)$ is constant and $A = 1$. Returning to our formula for f, we find that $f(z) = Ce^z$ for some constant C, which is absurd. $\square$

Chapter 5

The Gamma and Zeta functions

Lecture 16

The Gamma function. The Gamma function is denoted $\Gamma : \mathbb{C} \to \mathbb{C}$ and extends the *factorial* operation. We will define the Γ and show that is entire with zeros at the non-positive integers and growth rate at most 1.

Definition 5.1. *For $s > 0$,*

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, \mathrm{d}t$$

Remark 5.1. To see that the above integral converges, note that for all $s > 0$,

$$\int_0^\infty e^{-t} t^{s-1} \, \mathrm{d}t = \int_0^1 e^{-t} t^{s-1} \, \mathrm{d}t + \int_1^\infty e^{-t} t^{s-1} \, \mathrm{d}t \leq \underbrace{\int_0^1 t^{s-1} \, \mathrm{d}t}_{<\infty} + \int_1^\infty e^{-t} t^{s-1} \, \mathrm{d}t$$

Furthermore, the second integral also converges since e^{-t} tends to 0 *very fast*. That is, there exists $C > 0$ such that

$$\int_1^\infty e^{-t} t^{s-1} \, \mathrm{d}t < C \int_1^\infty e^{-t} e^{t/2} \, \mathrm{d}t < \infty$$

Proposition 5.2. *The Γ function can be extended to a holomorphic function on the uppoer half-plane ($s \in \mathbb{C}$, $\Re(s) > 0$) which is still given by the integral as in the definition of Γ.*

Proof. We will show that the integral is a holomorphic function in each strip

$$S_{\delta,M} = \{s \in \mathbb{C} \mid \delta < \Re(s) < M\} \qquad 0 < \delta < M$$

Fix s in some strip $\in S_{\delta,M}$ and define $\sigma = \Re(s)$. Note that for any $t \geq 0$, we have

$$\left|e^{-t}t^{s-1}\right| = e^{-t}t^{\sigma-1}$$

Define also

$$\Gamma_\varepsilon(s) := \int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^{s-1}\,\mathrm{d}t$$

which is a holomorphic function. Thus, to see that Γ is holomorphic it suffices to prove that Γ_ε converges uniformly to Γ for each $s \in S_{\delta,M}$. We have

$$|\Gamma(s) - \Gamma_\varepsilon(s)| \leq \int_0^\varepsilon e^{-t}t^{\sigma-1}\,\mathrm{d}t + \int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{\sigma-1}\,\mathrm{d}t$$

The first integral on the right hand side tends to 0 uniformly as $\varepsilon \to 0$. Moreover

$$\left|\int_{\frac{1}{\varepsilon}}^\infty e^{-t}t^{\sigma-1}\,\mathrm{d}t\right| \leq C' \int_{\frac{1}{\varepsilon}}^\infty e^{-t/2}\,\mathrm{d}t = Ce^{\frac{-1}{2\varepsilon}}$$

Ergo, Γ_ε converges to Γ uniformly and by theorem 2.23 we conclude that Γ is holomorphic on $S_{\delta,M} > 0$ for every $0 < \delta < M$ and therefore on the upper half plane. □

Lemma 5.3. *If* $\Re(s) > 0$, *then* $\Gamma(s+1) = s\Gamma(s)$. *In particular,* $\Gamma(n+1) = n!$ *for all* $n \in \mathbb{N}$.

Proof. We have

$$\int_\varepsilon^{\frac{1}{\varepsilon}} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t}t^s\right)\,\mathrm{d}t = -\int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^s\,\mathrm{d}t + s\int_\varepsilon^{\frac{1}{\varepsilon}} e^{-t}t^{s-1}\,\mathrm{d}t$$

As $\varepsilon \to 0$, we find that $0 = -\Gamma(s+1) + s\Gamma(s)$. For the second part of the lemma, it suffices to note that

$$\Gamma(1) = \int_0^\infty e^{-t}t^{1-1}\,\mathrm{d}t = 1$$

□

Theorem 5.4. *The* Γ *function has an analytic continuation to a meromorphic function on* $\mathbb{C}$ *whose singularities are simple poles at the non-positive integers. Furthermore, the residue of* Γ *at each point* $-n$ *for* $n \in \mathbb{N}$ *is given by* $(-1)^n/n!$

Remark 5.2. It follows from the principle of analytic continuation (identity theorem 2.18) that the extension of Γ is unique.

First proof. It suffices to extend Γ to $\Re(s) > -m$ for each integer $m \in \mathbb{N}$. Consider first $m = 1$ and define

$$F_1(s) = \frac{\Gamma(s+1)}{s}$$

Since Γ is holomorphic on $\Re(s) > 0$, we find that F_1 is meromorphic on $\Re(s) > -1$ with a simple pole at the origin. Furthermore, the residue of F_1 at 0 is

$$\lim_{z\to 0} z\frac{\Gamma(z+1)}{z} = \Gamma(1) = 1$$

Finally, by the lemma we find that for $\Re(s) > 0$, $F_1(s)$ agrees with Γ.

We proceed as above iteratively. For $m \geq 2$, we write

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)\,(s+m-2)\cdots s}$$

Each F_m is meromorphic on the half plane $\Re(s) > -m$ and agrees with Γ on the right half-plane. Furthermore, the residue at each integer $-n$ for $n \leq m$ is given by

$$\frac{\Gamma(-n+m)}{(m-n-1)\,(m-n-2)\cdots(1)(-1)(-2)\cdots(-n)} = \frac{(m-n-1)!}{(m-n-1)!(-1)^n n!} = \frac{(-1)^n}{n!}$$

□

Remark 5.3. In the continues version of Γ,

(1) The identity $\Gamma(s+1) = s\Gamma(s)$ holds for all $s \in \mathbb{C}$, $s \neq -n$ for $n \in \mathbb{N}$.

(2) For $n \in \mathbb{N}$, we have

$$\operatorname{res}_{-n+1}\Gamma(s) = \operatorname{res}_{-n}\Gamma(s+1) = -n\operatorname{res}_{-n}\Gamma(s)$$

Furthermore, $\Gamma(1) = \lim_{s\to 0} s\Gamma(s) = \operatorname{res}_0 \Gamma(s)$.

Second proof. Recall that for all $s > 0$,

$$\Gamma(s) = \int_0^1 e^{-t}t^{s-1}\,\mathrm{d}t + \int_1^\infty e^{-t}t^{s-1}\,\mathrm{d}t$$

Note first that the second integral is an entire function. Furthermore, for s such that $\Re(s) > 0$,

$$\int_0^1 e^{-t}t^{s-1}\,\mathrm{d}t = \int_0^1 \sum_{n=0}^\infty \frac{(-1)^n t^n}{n!} t^{s-1}\,\mathrm{d}t = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\int_0^1 t^{s+n-1}\,\mathrm{d}t = \sum_{n=0}^\infty \frac{(-1)^n}{(n+s)n!}$$

The above sum is a meromorphic function on $\mathbb{C}$ with poles at the non-positive integers. To see this, fix $R > 0$ and let $N \geq 2R$ be an integer. Then

$$\sum_{n=0}^\infty \frac{(-1)^n}{(n+s)n!} = \sum_{n=0}^N \frac{(-1)^n}{(n+s)n!} + \sum_{n=N+1}^\infty \frac{(-1)^n}{(n+s)n!}$$

The sum from 0 to N is a rational function and thus meromorphic. The second sum converges uniformly to 0 in $|s| < R$ since

$$\left|\sum_{n=N+1}^\infty \frac{(-1)^n}{(n+s)n!}\right| \leq \sum_{n=N+1}^\infty \frac{1}{Rn!}$$

Ergo, taking $R \to \infty$, we find that $\sum_{n=0}^\infty \frac{(-1)^n}{(n+s)n!}$ is indeed meromorphic on $\mathbb{C}$. Thence,

$$\Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{(n+s)n!} + \int_1^\infty e^{-t}t^{s-1}\,\mathrm{d}t \qquad \forall s \in \mathbb{C} \tag{14}$$

□

Lecture 17

Properties of Gamma.

Theorem 5.5. *For any $s \in \mathbb{C}$, we have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

Lemma 5.6. *For $0 < a < 1$, it holds that*

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,\mathrm{d}v = \int_{-\infty}^\infty \frac{e^{ax}}{1+e^x} = \frac{\pi}{\sin \pi a}$$

(This is example 3.2)

Proof of Theorem 5.5. Note that both side are meromorphic with simple poles for all $s \in \mathbb{Z}$. By the principle of analytic continuation it suffices to prove the theorem for $0 < s < 1$. For $0 < s < 1$, we may fix $t > 0$ and find that

$$\Gamma(1-s) = \int_0^\infty e^{-u}u^{-s}\,\mathrm{d}u \xmapsto{v:=u/t} t\int_0^\infty e^{-vt}(vt)^{-s}\,\mathrm{d}v$$

Thus,

$$\begin{aligned}\Gamma(1-s)\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\Gamma(1-s)\,\mathrm{d}t &= \int_0^\infty e^{-t}t^{s-1}t\int_0^\infty e^{-vt}(vt)^{-s}\,\mathrm{d}v\,\mathrm{d}t \\ &= \int_0^\infty\int_0^\infty e^{-t(v+1)}v^{-s}\,\mathrm{d}v\,\mathrm{d}t \\ &= \int_0^\infty \frac{v^{-s}}{1+v}\,\mathrm{d}v\end{aligned}$$

By the lemma, the above is equal to

$$\frac{\pi}{\sin\pi(1-s)} = \frac{\pi}{\sin \pi s}$$

□

Corollary 5.7.

$$\Gamma(1/2) = \sqrt{\pi}$$

Theorem 5.8. *The following holds true;*

(1) The function $1/\Gamma$ is entire with zeros exactly at the non-positive integers,
(2) $1/\Gamma$ has growth $|1/\Gamma(s)| \leq c_1 e^{c_2|s|\log|s|}$. In particular, Γ has growth order 1 since for every $\varepsilon > 0$, there exists a constant c such that $|1/\Gamma(s)| \leq ce^{c_2|s|^{1+\varepsilon}}$

Proof. By the previous theorem,

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin(\pi s)}{\pi}$$

which is entire since for all $s \in \mathbb{N}$ the function $\Gamma(1-s)$ has a simple pole while $\sin(\pi s)/\pi$ has a zero of degree 1. Furthermore, it is evident that $1/\Gamma$ has zeroes at $s = 0, -1, -2, \ldots$.

To prove the estimate we have on the growth rate of Γ, consider first $s \in \mathbb{C}$ with $\sigma = \Re(s) > 0$. Then we may find $n \in \mathbb{N}$ such that $\sigma \leq n \leq \sigma + 1$ in which case

$$\int_1^\infty e^{-t}t^{-\sigma} \leq \int_0^\infty e^{-t}t^{-n} = \Gamma(n+1) = n! \leq n^n = e^{n\log(n)} \leq e^{(\sigma+1)\log(\sigma+1)}$$

By equation (14) and theorem 5.5,

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin(\pi s)}{\pi} = \frac{\sin(\pi s)}{\pi}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1-s)n!} + \frac{\sin(\pi s)}{\pi}\int_1^\infty e^{-t}t^{-s}\,\mathrm{d}t$$

For any $s \in \mathbb{C}$, we know by our previous comments that the above integral is bounded by $e^{(|\sigma|+1)\log(|\sigma|+1)}$. Furthermore, it follows from Euler's formula for sin that $|\sin(\pi s)| \leq e^{\pi|s|}$. In order to bound the infinite sum, we consider two distinct cases; $\Im(s) > 1$ and $\Im(s) \leq 1$. In the first case, the sum is bounded by $ce^{\pi|s|}$ for some constant c. In the second case, pick k such that $k - 1/2 \leq \Re(s) \leq k + 1/2$. If $k \geq 1$, then

$$\frac{\sin(\pi s)}{\pi}\sum_{n=0}^\infty \frac{(-1)^n}{(n+1-s)n!} = \frac{(-1)^{k-1}\sin(\pi s)}{(k-1)!(k-s)\pi} + \sum_{n\neq k-1}^\infty (-1)^n\frac{\sin(\pi s)}{\pi(n+1-s)n!}$$

Since $\sin(\pi s) = 0$ when $s = k$, the first term on the right is bounded. Furthermore, the second term is bounded by $c'\sum_{n=0}^\infty \frac{1}{n!} = c$ for some constant c. If $k \leq 0$, then $\Re(s) \leq 1/2$ and we find once more that the sum is bounded by c. □

Theorem 5.9. *For all $s \in \mathbb{C}$,*

$$\frac{1}{\Gamma(s)} = e^{\gamma s}\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)e^{-s/n}$$

*where γ is the **Euler-Mascheroni** constant and is define as*

$$\gamma = \lim_{N\to\infty}\sum_{n=1}^N \frac{1}{n} - \log N$$

Proof. By Hadamard's factorization theorem, there exist constants A, B such that

$$\frac{1}{\Gamma(s)} = e^{As+B}s\prod_{n=1}^\infty E_1\left(\frac{s}{n}\right) = e^{As+B}s\prod_{n=1}^\infty \left(1 + \frac{s}{n}\right)e^{-s/n}$$

Since $s\Gamma(s) \to 1$ as $s \to 0$, we must have $B = 0$. Furthermore, letting $s = 1$ yields

$$1 = e^A \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n} = e^A \lim_{N\to\infty} \left[\prod_{n=1}^{N} \left(\frac{n+1}{n}\right) \prod_{n=1}^{N} e^{-1/n}\right]$$
$$= e^A \lim_{N\to\infty} \left[(N+1) \prod_{n=1}^{N} e^{-1/n}\right]$$

Therefore, for some $k \in \mathbb{Z}$,

$$A + 2\pi ik = -\lim_{N\to\infty} \log \left[(N+1) \prod_{n=1}^{N} e^{-1/n}\right] = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log(N+1)$$
$$= \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{n} - \log(N) - \lim_{N\to\infty} \log\left(1 + \frac{1}{N}\right)$$

That is, $A + 2\pi ik = \gamma$. Furthermore, since Γ is real valued whenever $s \in \mathbb{R}$, k must be 0 which concludes the proof. □

Lecture 18

The zeta function.

Definition 5.10. *For all $s > 1$, we define the ζ function by*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Proposition 5.11. *The series defining ζ converges for all $s \in \mathbb{C}$ with $\Re(s) > 1$ and ζ is holomorphic in that half-plane.*

Proof. Indeed, if $s = \sigma + it$ then

$$\left|n^{-s}\right| = \left|e^{-s\log n}\right| = e^{-\sigma \log n} = n^{-\sigma}$$

□

Definition 5.12. *The theta function is defined as*

$$\vartheta(t) = \sum_{n=-\infty}^{n=\infty} e^{-\pi n^2 t}$$

By example 2.1, the fourrier inverse of $f(x) = e^{-\pi x^2 t}$ $(t > 0)$ is

$$\int_{-\infty}^{\infty} e^{-\pi x^2 t} e^{-2\pi i x\xi}\, \mathrm{d}x \xmapsto{y:=xt^{1/2}} \int_{-\infty}^{\infty} e^{-\pi y^2} e^{-2\pi i x\left(t^{-1/2}\xi\right)} t^{-1/2}\, \mathrm{d}y = t^{-1/2} e^{-\pi\xi^2/t}$$

By Plancherel's theorem,

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) = t^{-1/2} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} = t^{-1/2}\vartheta(1/t)$$

Thus, there exists a constant $C > 0$ such that

$$\vartheta(t) \leq Ct^{-1/2} \qquad \text{as } t \to 0$$

Furthermore, for all $t \geq 1$,

$$|\vartheta(t) - 1| = 2\sum_{n=1}^{\infty} e^{-\pi n^2 t} \leq 2\sum_{n=1}^{\infty} e^{-\pi n t} \leq Ce^{-\pi t}$$

Theorem 5.13. *If $\Re(s) > 1$, then*

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\,\zeta(s) = \frac{1}{2}\int_0^{\infty} u^{(s/2)-1}\left(\vartheta(u) - 1\right)\,\mathrm{d}u$$

Proof. Observe first that for all $n \geq 1$,

$$\int_0^{\infty} e^{-\pi n^2 u} u^{(s/2)-1}\,\mathrm{d}u \xmapsto{t:=\pi n^2 u} \left(\frac{1}{\pi n^2}\right)^{(s/2)-1} \int_0^{\infty} e^{-t} t^{(s/2)-1} \frac{1}{\pi n^2}\,\mathrm{d}t = \pi^{-s/2} n^{-s}\Gamma(s/2)$$

Therefore, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\infty} u^{(s/2)-1}\left(\vartheta(u) - 1\right)\,\mathrm{d}u = \int_0^{\infty} u^{(s/2)-1}\sum_{n=1}^{\infty} e^{-\pi n^2 t}\,\mathrm{d}u &= \sum_{n=1}^{\infty}\int_0^{\infty} u^{(s/2)-1} e^{-\pi n^2 t}\,\mathrm{d}u \\ &= \pi^{-1/2}\Gamma(s/2)\sum_{n=1}^{\infty} n^{-s}\end{aligned}$$

□

Theorem 5.14. *The function ξ is holomorphic in the half plane given by $\Re(s) > 1$ and has an analytic continuation to a meromorphic function in $\mathbb{C}$. The analytic continuation has simple poles at $s = 0, 1$ and*

$$\xi(s) = \xi(1 - s) \qquad \forall s \in \mathbb{C}$$

Proof. Define $\psi(u) := \left(\vartheta(u) - 1\right)/2$ and note that

$$\psi(u) = \frac{u^{-1/2}\vartheta\left(1/u\right) - 1}{2} = u^{-1/2}\psi(1/u) + \frac{u^{-1/2}}{2} - \frac{1}{2}$$

We compute

$$\begin{aligned}\int_0^1 u^{(s/2)-1}\psi(u)\,\mathrm{d}u &= \int_0^1 u^{(s/2)-1}\left(u^{-1/2}\psi(1/u) + \frac{u^{-1/2}}{2} - \frac{1}{2}\right)\,\mathrm{d}u \\ &= \frac{1}{s-1} + \frac{1}{s} + \int_0^1 u^{(s/2)-3/2}\psi(1/u)\,\mathrm{d}u\end{aligned}$$

A change of variables yields

$$\int_0^1 u^{(s/2)-1}\psi(u)\,\mathrm{d}u = \frac{1}{s-1} + \frac{1}{s} + \int_1^\infty u^{(-s/2)-1/2}\psi(u)\,\mathrm{d}u$$

Ergo, whenever $\Re(s) > 1$,

$$\begin{aligned}\xi(s) = \int_0^\infty u^{(s/2)-1}\psi(u)\,\mathrm{d}u &= \int_0^1 u^{(s/2)-1}\psi(u)\,\mathrm{d}u + \int_1^\infty u^{(s/2)-1}\psi(u)\,\mathrm{d}u \\ &= \frac{1}{s-1} + \frac{1}{s} + \int_1^\infty \left(u^{(-s/2)-1/2} + u^{(s/2)-1}\right)\psi(u)\,\mathrm{d}u\end{aligned}$$

Since ψ has exponential decay, the above integral defines an entire function. Thus, ξ is a meromorphic function with simple poles at $s = 0, 1$. Moreover, it is immediate from the above representation that $\xi(s) = \xi(1-s)$. □

Theorem 5.15. *The zeta function has an analytic continuation to the entire complex plane whose only pole is a simple pole at $s = 1$.*

Proof. By the above theorems, the analytic continuation of ζ is given by

$$\zeta(s) = \frac{\xi(s)}{\pi^{-s/2}\Gamma(s/2)}$$

Since $1/\Gamma$ is entire with simple zeroes at $0, -1, -2, \ldots$ and ξ has simple poles at 0 and 1, the zeta function must be meromorphic with a simple pole at $s = 1$. □

Theorem 5.16. *There exists a sequence of entire function $\{\delta_n\}\, n = 1^\infty$ which satisfy $|\delta_n(s)| \le |s|/n^{\sigma+1}$ and $|\delta_n(s)| \le 2/n^\sigma$, where $s = \sigma + it$, such that*

$$\sum_{n=1}^N \delta_n(s) = \sum_{n=1}^N \frac{1}{n^s} - \int_1^N \frac{1}{x^s}\,\mathrm{d}x$$

Proof. Clearly, this sequence of functions is given by

$$\delta_n(s) = \int_n^{n+1}\left(\frac{1}{n^s} - \frac{1}{x^s}\right)\mathrm{d}x$$

By the mean value theorem applies to $f(x) = x^{-s}$, whenever $n \le x \le n+1$, there exists $n \le y \le x$ such that

$$\left|\frac{1}{n^s} - \frac{1}{x^s}\right| = \frac{1}{|n^s x^s|}\,|x^s - n^s| = \frac{|x-n|}{n^\sigma x^\sigma}\,|f'(y)| = \frac{|x-n|}{n^\sigma x^\sigma}\,|s|\,y^{\sigma-1}$$

Then

$$|\delta_n(s)| = \frac{|x-n|}{n^\sigma x^\sigma}\,|s|\,y^{\sigma-1} \le \frac{|s|}{n^\sigma x} \le \frac{|s|}{n^{\sigma+1}}$$

The second bound is trivially true since

$$|\delta_n(s)| = \left|\frac{1}{n^s} - \frac{1}{x^s}\right| \le \frac{1}{|n^s|} + \frac{1}{|x^s|} = \frac{1}{n^\sigma} + \frac{1}{|x|^\sigma} \le \frac{2}{n^\sigma}$$

□

Corollary 5.17. *For* $\Re(s) > 0$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \delta_n(s)$$

Proposition 5.18. *If* $s = \sigma + it$ *then for every* $0 \leq \sigma_0 \leq 1, \varepsilon > 0$ *there exists a constant* c_ε *such that*

$$|\zeta(s)| \leq c_\varepsilon |t|^{1-\sigma_0+\varepsilon} \qquad \text{if } \sigma_0 \leq \sigma, t \geq 1$$

and

$$\left|\zeta'(s)\right| \leq c_\varepsilon |t|^{\varepsilon} \qquad \text{if } 1 \leq \sigma, t \geq 1$$

Proof. Combining the two estimates in the above corollary,

$$|\delta_n(s)| \leq \left(\frac{|s|}{n^{\sigma+1}}\right)^{\alpha} \left(\frac{2}{n^{\sigma}}\right)^{1-\alpha} \leq \frac{2|s|^{\alpha}}{n^{\sigma_0+\alpha}} \qquad \forall \alpha \geq 0, \sigma_0 \leq \sigma$$

Thus, letting $\alpha = 1 - \sigma_0 + \varepsilon$ and using the second part of the corollary we find that

$$|\zeta(s)| \leq \frac{1}{|s-1|} + 2|s|^{1-\sigma_0+\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$$

This completes the proof of the first bound. For the second bound, recall that by Cauchy's integral formula 8

$$\zeta'(s) = \frac{1}{2\pi r} \int_0^{2\pi} \zeta\left(s + re^{i\theta}\right) e^{i\theta} \, \mathrm{d}\theta$$

For $r = \varepsilon$, if $\sigma \geq 1$ then$\Re(s + re^{i\theta}) \geq 1 - \varepsilon$ and using the first bound we find that

$$\left|\zeta'(s)\right| \leq \frac{1}{2\pi\varepsilon} \int_0^{2\pi} c_\varepsilon |t + \varepsilon \sin(\theta)|^{1-(1-\varepsilon)+\varepsilon} \, \mathrm{d}\theta \leq c_\varepsilon |t|^{\varepsilon}$$

□

Exercises and Solutions

Problem 1. Prove that

$$\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)} = \prod_{n=1}^{\infty} \frac{n(a+b+n)}{(a+n)(b+n)}$$

Solution. Using the product for $1/\Gamma$, whenever $a, b > -1$ and $a+b \neq -1$,

$$\begin{aligned}
\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)} &= \frac{(a+b+1)}{e^\gamma(a+1)(b+1)} \prod_{n=1}^{\infty} \frac{\left(1+\frac{a+b+1}{n}\right)}{\left(1+\frac{a+1}{n}\right)\left(1+\frac{b+1}{n}\right)} e^{1/n} \\
&= \frac{(a+b+1)}{e^\gamma(a+1)(b+1)} \prod_{n=1}^{\infty} \frac{n\,(n+a+b+1)}{(n+a+1)\,(n+b+1)} e^{1/n} \\
&= \frac{(a+b+1)}{e^\gamma(a+1)(b+1)} \prod_{n=1}^{\infty} \frac{(n+1)\,(a+b+[n+1])}{(a+[n+1])\,(b+[n+1])} \frac{n}{n+1} e^{1/n} \\
&= \left[\prod_{n=1}^{\infty} \frac{n\,(a+b+n)}{(a+n)\,(b+n)}\right] \left[\frac{1}{e^\gamma} \prod_{n=1}^{\infty} \frac{n}{n+1} e^{1/n}\right]
\end{aligned}$$

Where γ is the Euler-Mascheroni constant. Noting that

$$\frac{1}{e^\gamma} \prod_{n=1}^{\infty} \frac{n}{n+1} e^{1/n} = \left[\lim_{N\to\infty} N \prod_{n=1}^{N} \frac{1}{e^{1/n}}\right] \prod_{n=1}^{\infty} \frac{n}{n+1} e^{1/n} = \lim_{N\to\infty} N \prod_{n=1}^{N} \frac{n}{n+1} = \lim_{N\to\infty} \frac{N}{N} = 1$$

concludes the problem. □

Problem 2. Given a function f, it's Mellin transform is

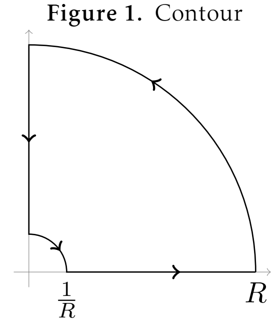
$$\mathcal{M}(f)(z) = \int_0^\infty f(t) t^{z-1}\,\mathrm{d}t$$

Find the Mellin transform of sin and cos on the strip $0 < \Re(z) < 1$. Use the Mellin transform of sin to show that

$$\int_0^\infty \frac{\sin(x)}{x^{1/2}}\,\mathrm{d}x = \frac{\pi}{2} \qquad \text{and} \qquad \int_0^\infty \frac{\sin(x)}{x^{3/2}}\,\mathrm{d}x = \sqrt{2\pi}$$

Solution. Let z be such that $0 < \Re(z) < 1$ and integrate $f(w) = e^{-w} w^{z-1}$ over the contour

Figure 1. Contour



By Cauchy's theorem,

$$\int_{1/R}^{R} e^{-w} w^{z-1}\,\mathrm{d}w$$
$$= \int_0^{\pi/2} e^{-\frac{e^{it}}{R}} \left(\frac{e^{it}}{R}\right)^{z-1} \frac{ie^{it}}{R}\,\mathrm{d}t - \int_{1/R}^{R} ie^{iw}\left(-iw\right)^{z-1}\,\mathrm{d}w - \int_0^{\pi/2} e^{-Re^{it}}\left(Re^{it}\right)^{z-1} Rie^{it}\,\mathrm{d}t$$

The left hand side tends to $\Gamma(z)$ as $R \to \infty$. The first integral and the third integral on the right hand side tends to 0 as $R \to \infty$. Thus

$$\Gamma(z) = \int_0^\infty e^{-w} w^{z-1}\,\mathrm{d}w = \frac{1}{i^z}\int_0^\infty e^{-iw} w^{z-1}\,\mathrm{d}w = \frac{1}{i^z}\mathcal{M}(e^{iz})(z)$$

That is, $\mathcal{M}(e^{iz})(z) = i^z\Gamma(z)$. Since $i^z = \cos(\pi z/2) + i\sin(\pi z/2)$, we have

$$\mathcal{M}(e^{iz})(z) = \mathcal{M}(\cos)(z) + i\mathcal{M}(\sin)(z) = \cos(\pi z/2)\,\Gamma(z) + i\sin(\pi z/2)\,\Gamma(z)$$

Ergo, after considering the cases where z is real, we deduce that

$$\mathcal{M}(\cos)(z) = \cos(\pi z/2)\,\Gamma(z) \quad \text{and} \quad \mathcal{M}(\sin)(z) = \sin(\pi z/2)\,\Gamma(z) \qquad \forall 0 < \Re(z) < 1$$

as desired.

For the second part of the problem, we first extend the Mellin transform of sin to the strip $-1 < \Re(z) < 1$. Since at $z = 0$ the Gamma function has a pole of degree 1 while sin has a zero of degree 1, $\sin(\pi z/2)\,\Gamma(z)$ in entire in the strip $-1 < \Re(z) < 1$. By the principle of analytic continuation, $\mathcal{M}(\sin)(z) = \sin(\pi z/2)\,\Gamma(z)$ for all $-1 < \Re(z) < 1$. Finally,

$$\mathcal{M}(\sin)(0) = \int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x = \lim_{z\to 0}\Gamma(z)\sin(\pi z/2) = \frac{\pi}{2}$$

and

$$\int_0^\infty \frac{\sin(x)}{x^{3/2}}\,\mathrm{d}x = \Gamma(-1/2)\sin(-\pi/4) = -2\Gamma(1/2)\left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2\pi}$$

□

Problem 3. Show that on the half-plane $\Re(z) > 1$,

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x$$

Solution. Since $e^{-x} < 1$ for all $x > 0$, we may use a geometric series to solve the problem;

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1}\,\mathrm{d}x = \int_0^\infty \sum_{n=1}^\infty e^{-nx}x^{s-1}\,\mathrm{d}x = \sum_{n=1}^\infty \int_0^\infty e^{-nx}x^{s-1}\,\mathrm{d}x \xmapsto{t:=nx} \sum_{n=1}^\infty \frac{1}{n^s}\int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t$$
$$= \sum_{n=1}^\infty \frac{1}{n^s}\Gamma(s) = \zeta(s)\Gamma(s)$$

□

Conformal Maps

Lecture 19

Definition 6.1. *Given open sets $U, V \in \mathbb{C}$, a conformal map (or bihomomorphism) $f : U \to V$ is a bijective holomorphic function. If such a function between U and V exists, we say that U and V are conformally equivalent or biholomorphic.*

In this chapter, the notation U, V will denote open subsets of $\mathbb{C}$.

Lemma 6.2. *If $f : U \to V$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in U$.*

Proof. By way of contradiction, let $z_0 \in U$ be such that $f'(z_0) = 0$. Without loss of generality, we may assume that $0 = z_0 = f(z_0) = f'(z_0)$ (otherwise consider $f(z + z_0) - f(z_0)$). Since f is holomorphic in some neighbourhood of z_0, we may find a power series expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots \quad \forall z \text{ near } 0$$

By the assumption $f(0) = f'(0) = 0$, the coefficient a_0, a_1 are 0. Let $N \geq 2$ be the least natural number such that $a_N \neq 0$ and define g to be a function such that $f(z) = z^N g(z)$. Since $g(0) \neq 0$, there exists some neighbourhood D_ε of 0 on which g does not vanish. Thus, there exists a holomorphic function G such that $e^G = g$ on D_ε. Define $h(z) = ze^{G(z)/N}$ and note that $f(z) = h(z)^N$. By the open mapping theorem, $h(D_\varepsilon)$ contains some 2δ-neighbourhood of 0. Therefore there exists z_1, z_2 such that $h(z_1) = \delta$, $h(z_2) = \delta e^{\frac{2\pi i}{N}}$ in which case

$$f(z_1) = h^N(z_1) = \delta^N = \left(\delta e^{\frac{2\pi i}{N}}\right)^N = h^N(z_2) = f(z_2)$$

But this contradicts the injectivity of f. □

Proposition 6.3. *The inverse of a conformal map is also a conformal map.*

Proof. Given a conformal map $f : U \to V$, it suffices to show that f^{-1} is holomorphic. For any $z, w \in V$ and $z_0, w_0 \in U$ such that $f(z_0) = z$, $f(w_0) = w$,

$$\frac{f^{-1}(z) - f^{-1}(w)}{z - w} = \frac{1}{(f(z_0) - f(w_0))/(z_0 - w_0)}$$

Since $f'(z_0) \neq 0$, we may take the limit as $w \to z$ which yields $[f^{-1}]'(z) = 1/f'(z_0)$. Since z was arbitrary, we conclude that f^{-1} is holomorphic. □

Theorem 6.4. *The map* $F : \mathbb{H} \to \mathbb{D}$, $z \mapsto \frac{i-z}{i+z}$ *from the upper complex plane* $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ *to the unit disk* $\mathbb{D}$ *is a conformal map with inverse* $G(w) = i\frac{1-w}{1+w}$.

Proof. It is clear that F is holomorphic in $\mathbb{H}$, it therefore remains to show that F indeed maps to $\mathbb{D}$, G maps to $\mathbb{H}$ and F is bijective. To see that F maps to $\mathbb{D}$, it suffices to note that any point $z \in \mathbb{H}$ is closer to i than to $-i$. Furthermore, to see that G maps to $\mathbb{H}$, pick $w = u + iv \in \mathbb{D}$ and write

$$\begin{aligned}
\Im(G(w)) = \Im\left(i\frac{1-u-iv}{1+u+iv}\right) = \frac{1-u-iv}{1+u+iv} &= \Re\left(\frac{1-u-iv}{1+u+iv}\right) \\
&= \Re\left(\frac{(1-u-iv)(1+u-iv)}{(1+u+iv)(1+u-iv)}\right) \\
&= \frac{1-u^2-v^2}{(1+u)^2+v^2} > 0
\end{aligned}$$

Finally, a few algebraic manipulations yield $F(G(w)) = w$ and $G(F(z)) = z$, hence F is bijective. □

Remark 6.1. This type of transformation is called a **Möbius transformation** (or fractional linear transformation). Specifically, it is any transformation of the form

$$z \mapsto \frac{az+b}{cz+d}$$

Some useful conformal maps.

(1) For any $0 < \alpha < 2$, then map $z \mapsto z^\alpha$ is a conformal map between the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ to the sector $S = \{z \in \mathbb{C} \mid 0 < \arg(z) < \alpha\pi\}$.

(2) The map $z \mapsto \frac{1+z}{1-z}$ is a conformal map between the upper half disk, i.e. the set $\{z \in \mathbb{C} \mid |z| < 1, \Im(z) > 0\}$, and the first quadrant $\{z \in \mathbb{C} \mid \Re(z) > 0, \Im(z) > 0\}$. The inverse map is given by $w \mapsto \frac{w-1}{w+1}$.

(3) If log is the principle branch of the logarithm, then the map $z \mapsto \log(z)$ is conformal from the upper half plane to the strip $\{z \in \mathbb{C} \mid 0 < \Im(z) < \pi\}$. Furthermore, the this map is also conformal from the upper half disk $\{z \in \mathbb{C} \mid \Im(z) > 0, |z| < 1\}$ to the strip $\{z \in \mathbb{C} \mid \Re(z) < 0, 0 < \Im(z) < \pi\}$. The inverse of this map is given by $w \mapsto e^w$.

(4) In a similar vein, the map $z \mapsto e^{iz}$ takes $\{z \in \mathbb{C} \mid -\pi/2 < \Re(z) < \pi/2, \Im(z) > 0\}$ to the upper half disk.

Lecture 20

The Dirichlet Problem. We wish to solve the *Dirichlet boundary problem*

$$\begin{cases} \triangle u = 0 & \text{in } \Omega \\ u = f & \text{on } \Omega \end{cases}$$

where $\triangle$ denotes the Laplacian ($\triangle u(\mathbf{x}) = \sum_{i=1}^{n} \partial_{x_i}^2 u$, where in our case we will usually have $\triangle = \partial_x^2 + \partial_y^2$). A function u for which $\triangle u = 0$ is called **harmonic**. If $\Omega = \mathbb{D}$, then the solution is known to be

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \, \mathrm{d}\theta = \frac{-i}{2\pi} \int_{|w|=1} f(w) \frac{1 - |z|^2}{z\,|z - w|^2} \, \mathrm{d}w \qquad \text{for } |z| < 1$$

Theorem 6.5. *If $F : V \to U$ is holomorphic and $u : U \to \mathbb{C}$ is harmonic for some open sets U, V, then $u \circ F$ is harmonic.*

Proof. This theorem can be proven by simply computing the Laplacian of $u \circ F$. □

As an application of the above theorem, we solve the Dirichlet problem in the case $\Omega = \{x + iy \mid x \in \mathbb{R}, 0 < x < 1\}$ with boundary condition $f : \partial\Omega \to \mathbb{C}$. Here, f is a continuous function such that $\lim_{|x|\to\infty} f(x + iy) = 0$ for $y = 0, 1$. We first find a conformal map from $\mathbb{D} \to \Omega$ and it's inverse. From results in the previous lecture, we know that there exists a conformal map from $\mathbb{D}$ to $\mathbb{H}$ given by $w \mapsto i\frac{1-w}{1+w}$ with inverse $z \mapsto \frac{i-z}{i+z}$. Furthermore, $z \mapsto \frac{1}{\pi}\log(z)$ where log is the principal branch of the logarithm is a conformal map from $\mathbb{H}$ to Ω. Finally,

$$F(z) = \frac{1}{\pi} \log\left(i\frac{1-z}{1+z}\right), \quad G(z) = \frac{i - e^{\pi z}}{i + e^{\pi z}}$$

are, respectively, the conformal $\mathbb{D} \to \Omega$ and it's inverse.

Now, we may define a function $\tilde{f}(z) = f(F(z))$ which is defined for all z on the boundary of $\mathbb{D}$. Note that at $1, -1$ the function $\tilde{f}$ is evaluated as a limit to be 0. Then the solution to the Dirichlet problem on the unit disk with boundary condition $\tilde{f}$ is

$$\tilde{u}(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \varphi) \tilde{f}(e^{i\varphi}) \, \mathrm{d}\varphi$$

where

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}$$

Ergo, the solution in our original space is $u(z) = G(\tilde{u}(z))$.

Schwartz Lemma.

Lemma 6.6 (Schwartz's lemma). *If $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic function with $f(0) = 0$, then*

(1) $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$*,*

(2) If for some $z_0 \in \mathbb{D}$, $|f(z_0)| = |z_0|$*, then* f *is a rotation, i.e.* $f(z) = cz$ *for some* c *with* $|c| = 1$*,*

(3) $|f'(0)| \leq 1$*. If equality holds then* f *is a rotation.*

Proof. Consider the power series expansion of f about 0;

$$a_0 + a_1 z + a_2 z^2 + \cdots$$

Since $f(0) = 0$, we must have $a_0 = 0$. Thus, $f(z)/z$ is holomorphic on the unit disk and since f maps to $\mathbb{D}$, we may pick $z \in \mathbb{D}$, $|z| = r$ and note that

$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{|z|} = \frac{1}{r}$$

By the maximum principle 3.17, $|f/z| \leq 1/r$ for all z with $|z| < r$. Letting r tend to 1, we conclude that $|f(z)/z| \leq 1$ for all $z \in \mathbb{D}$. If equality holds at some point z_0 inside the disk, then $f(z)/z$ attains a maximum inside the disk and by the maximum modulus principle must therefore be constant. Finally, define $g(z) = f(z)/z$ on $\mathbb{D}$ and note that

$$f'(0) = \lim_{z \to 0} \frac{f(z)}{z} = g(0)$$

If $|f'(z_0)| = 1$, the g attains it's maximum at the origin (recall that by (1), $g : \mathbb{D} \to \mathbb{D}$, thus $absg(z) \leq 1$ for all $z \in \mathbb{D}$). We conclude that $g \equiv c$ for some $|c| = 1$ and therefore $f(z) = cz$ is a rotation. □

Automorphisms of the unit disk. An automorphism is a conformal map from a set to itself. We denote the set of all automorphism on $\mathbb{D}$ by $\mathrm{Aut}(\mathbb{D})$. This is a group with composition being the group operation and the identity element is the map $z \mapsto z$. Obviously, any rotation map is an automorphism of the disk. Furthermore, for any $\alpha \in \mathbb{C}$ with $|\alpha| < 1$,

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

is an automorphism of $\mathbb{D}$. We have shown in problem 3 of chapter 1 that the above is indeed an automorphism of $\mathbb{D}$ and $\psi_\alpha^{-1} = \psi_\alpha$.

Theorem 6.7. *If f is an automorphism of the unit disk, then there exists $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta}\psi_\alpha(z)$.*

Proof. Since f is an automorphism of the unit disk, there exists $\alpha \in \mathbb{D}$ such that $f(\alpha) = 0$. Define $g = f \circ \psi_\alpha$ and note that g is also an automorphism of the disk and $g(0) = f(\psi_\alpha(0)) = f(\alpha) = 0$. Ergo, by Schwartz's lemma, $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$. Moreover, since $g \in \text{Aut}(\mathbb{D})$, we also have $g^{-1} \in \text{Aut}(\mathbb{D})$, $g^{-1}(0) = 0$ and again by Schwartz's lemma, $\left|g^{-1}(w)\right| \leq |w|$ for all $w \in \mathbb{D}$. Now, fix $z \in \mathbb{D}$, $w = g(z)$ and note that

$$|z| = \left|g^{-1} \circ g(z)\right| = \left|g^{-1}(w)\right| \leq |w| = |g(z)| \leq |z|$$

Ergo, by part (2) of Schwartz's lemma, there exists $\theta \in \mathbb{R}$ such that $g(z) = e^{i\theta}z$. Finally, we conclude that since $g = f \circ \psi_\alpha$, $f = g \circ \psi_\alpha^{-1} = g \circ \psi_\alpha$. That is, $f(z) = e^{i\theta}\psi_\alpha(z)$. □

Corollary 6.8. *The only automorphisms of the unit disk that fix the origin are rotations.*

Automorphisms of the upper half-plane. Fix the conformal map $F : \mathbb{H} \to \mathbb{D}$, $z \mapsto \frac{i-z}{i+z}$ and consider the homomorphism of groups

$$\Gamma : \text{Aut}(\mathbb{D}) \to \text{Aut}(\mathbb{H}), \quad \varphi \mapsto F^{-1} \circ \varphi \circ F$$

Furthermore, Γ has an inverse which is given by the map $\psi \mapsto F \circ \psi \circ F^{-1}$. Thus, Γ is bijective, i.e. Γ is an isomorphism and $\text{Aut}(\mathbb{D}) \cong \text{Aut}(\mathbb{H})$.

Define now

$$SL_2 = \left\{ M = \begin{pmatrix} `a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{R} \text{ and } \det(M) = ad - bc = 1 \right\}$$

For each $M \in SL_2$, it's associated map is

$$f_M(z) = \frac{az + b}{cz + d}$$

Theorem 6.9. *Every automorphism of $\mathbb{H}$ takes the form f_M for some $M \in SL_2$ and, conversely, every map associated to a matrix $M \in SL_2$ is an automorphism of the upper half plane.*

Proof. The proof is separated into multiple steps.

Step 1. If $M \in SL_2$, then f_M maps from $\mathbb{H}$ to itself. Furthermore, if $M, M' \in SL_2$, then $f_M \circ f_{M'} = f_{MM'}$.

Indeed, if $\Im(z) > 0$ then

$$\Im(f(z)) = \frac{(ad - bc)\Im(z)}{|cz + d|^2} = \frac{\Im(z)}{|cz + d|^2} > 0$$

We omit the proof of the second part which is also a straightforward calculation. Since every matrix in SL_2 is invertible, we conclude that $f_M \in \text{Aut}(\mathbb{H})$. Indeed, $f_M : \mathbb{H} \to \mathbb{H}$ is holomorphic and it's inverse is $f_M^{-1} = f_{M^{-1}}$.

Step 2. Given $z \in \mathbb{H}$, there exists $M \in SL_2$ such that $f_M(z) = i$.

Let $d = 0$ and note that (as in step 1)

$$\Im(f(z)) = \frac{\Im(z)}{|cz|^2}$$

Thus, let $c \in \mathbb{R}$ be such that the above evaluates to 1, then the matrix

$$M_1 = \begin{pmatrix} 0 & c^{-1} \\ c & 0 \end{pmatrix}$$

is in SL_2 and $\Im(f_{M_1}(z)) = 1$. Finally, if $b = -\Re(f_{M_1}(z))$, then $f_{M_2 M_1}(z) = 1$ where

$$M_2 = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

Step 3. If θ is real, then $\Gamma(f_{M_\theta}) = F \circ f_{M_\theta} \circ F^{-1}$ corresponds to the rotation of angle -2θ in the unit disk. Here,

$$M_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Indeed, it may be verified that $F \circ f_{M_\theta} = e^{-2i\theta} F$.

Step 4. Suppose $f \in \operatorname{Aut}(\mathbb{H})$ with $f(z_0) = i$ and consider the matrix $M \in SL_2$ such that $f_M(i) = z_0$. Then $g := f \circ f_M$ is an automorphism of the upper half-plane and $g(i) = i$. In particular, $F \circ g \circ F^{-1}$ is an automorphism of the unit disk which fixes the origin and by corollary 6.8 $F \circ g \circ F^{-1}$ is a rotation. By the previous step, there exists $\theta \in \mathbb{R}$ such that $F \circ g \circ F^{-1} = F \circ f_{M_\theta} \circ F^{-1}$, i.e. $g = f_{M_\theta}$. Recalling that $g = f \circ f_M$, we find that $f = f_{M^{-1} M_\theta}$. □

Remark 6.2. Note that $f_M = f_{-M}$, thus $\operatorname{Aut}(\mathbb{H}) \not\cong SL_2$. Therefore, we define PSL_2, the **Projective special linear group**, the group which identities the matrices $M \in SL_2$ with $-M$. Then $\operatorname{Aut}(\mathbb{H}) \cong PSL_2$.

Lecture 21

Riemann mapping theorem.

Theorem 6.10. *Suppose that Ω is a proper ($\Omega \neq \emptyset, \mathbb{C}$) simply connected open subset of $\mathbb{C}$, then for any $z_0 \in \Omega$ there exists a unique conformal map $F : \Omega \to \mathbb{D}$ such that*

$$F(z_0) = 0, \quad F'(z_0) > 0$$

Corollary 6.11. *Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.*

Some initial remarks. It is curious that the above theorem holds for sets such as the upper half-plane but cannot be extended to the entire complex plane. Indeed, if $f : \mathbb{C} \to \mathbb{D}$ is a conformal map, then since f bounded and entire, by Louisville's theorem 2.13 f must be constant which is absurd. Furthermore, by *invariance of domain*, if $f : \Omega \to \mathbb{D}$ is a conformal map, then since $\mathbb{D}$ is simply connected Ω must also be simply connected. This can easily be proven by associating any curve $\gamma \subseteq \Omega$ to a curve $\gamma' = f(\gamma) \subseteq \mathbb{D}$. Thus, the conditions of the theorem are not just sufficient, but also necessary for the existence of a conformal map. The theorem also states that if F, G are two conformal maps from

$\Omega \to \mathbb{D}$ which map some $z_0 \in \Omega$ to the origin, then $F = G$. Indeed, since $H = F \circ G^{-1}$ is an automorphism of the unit disk which fixes the origin, $H(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R}$. Moreover,

$$e^{i\theta} = H'(0) = F'(G^{-1}(0)) \left[G^{-1}\right]'(0) = \frac{F'(z_0)}{G'(z_0)} > 0$$

Ergo, $e^{i\theta}$. Then $F \circ G^{-1}(z) = H(z) = z$ and we conclude that $F = G$.

Over the remainder of the lecture, we will prove the existence part of the Riemann mapping theorem.

Montel's theorem.

Definition 6.12 (Normal family). *A family $\mathcal{F}$ of holomorphic functions is said to be normal if every sequence in $\mathcal{F}$ has a subsequence which converges uniformly on every compact subset of Ω. We do not require for the subsequence to converge to a function in $\mathcal{F}$, i.e. $\mathcal{F}$ is not necessarily closed.*

Definition 6.13. *A family $\mathcal{F}$ of holomorphic functions is said to be uniformly bounded on compact subsets of Ω if for each compact set $K \subseteq \Omega$, there exists a constant $B > 0$ such that $|f(z)| \leq B$ for all $z \in K, f \in \mathcal{F}$.*

Definition 6.14. *A family $\mathcal{F}$ of holomorphic functions is said to be equicontinuous on a compact set K if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $z, w \in \Omega$ with $|z - w| < \delta$, $|f(z) - f(w)| < \varepsilon$ for all $f \in \mathcal{F}$.*

Definition 6.15. *A sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of Ω is called an exhaustion if K_ℓ is contained in the interior of $K_{\ell+1}$ for all $\ell \in \mathbb{N}$ and any compact subset of Ω is contained in K_ℓ for some $\ell \in \mathbb{N}$.*

Theorem 6.16. *Suppose $\mathcal{F}$ is family of holomorphic functions on Ω which is uniformly bounded on every compact subset of Ω, then*

(1) *$\mathcal{F}$ is equicontinuous on every compact subset of Ω,*
(2) *$\mathcal{F}$ is a normal family.*

Proof. Let K be a compact subset of Ω and $B > 0$ such that $|f(z)| \leq B$ for all $z \in K, f \in \mathcal{F}$. Choose $r > 0$ such that $D_{3r}(z) \subseteq \Omega$ for all $z \in K$ and $z, w \in K$ be such that $|z - w| < r$. By Cauchy's integral formula 7, if γ is the circle of radius $2r$ about w then

$$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma f(\xi) \left(\frac{1}{\xi - z} - \frac{1}{\xi - w} \right) \mathrm{d}\xi$$

Thus,

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \int_\gamma |f(\xi)| \left| \frac{1}{\xi - z} - \frac{1}{\xi - w} \right| \mathrm{d}\xi \leq \frac{B}{2\pi} \int_\gamma \left| \frac{z - w}{(\xi - z)(\xi - w)} \right| \mathrm{d}\xi \leq \frac{B}{2\pi} 2\pi \frac{|z - w|}{r^2}$$

In particular, $|f(z) - f(w)| \le C|z - w|$ for some constant C which is independent of f, z, w. We conclude that $\mathcal{F}$ is indeed an equicontinuous family.

We prove the second part of the theorem with a *diagonalization* argument. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathcal{F}$ and $\{w_j\}_{j=1}^{\infty}$ a sequence of points which is dense in Ω. Since $\mathcal{F}$ is uniformly bounded, the sequence $f_n(w_j)$ is bounded for every $j \in \mathbb{N}$ ($\{w_j\}$ is itself a compact set). By the Bolzano-Weierstrass theorem, there exists a subsequence $\{f_{n,1}\}$ of $\{f_n\}$ such that $\{f_{n,1}(w_1)\}$ converges. Likewise, there exists a subsequence $\{f_{n,2}\}$ of $\{f_{n,1}\}$ such that $\{f_{n,2}(w_2)\}$ converges. Then note that $\{f_{n,2}(w_1)\}$ also converges. We continue as such and function a sequence $\{f_{n,j}\}_{j,n=1}^{\infty}$ such that $\{f_{n,j}(w_k)\}_{n=1}^{\infty}$ converges for every $k \le j$. Finally, let $g_n = f_{n,n}$ and note that $\{g_n(w_j)\}_{n=1}^{\infty}$ converges for every $j \in \mathbb{N}$.

We show that g converges uniformly on every compact subset K of Ω. For any $\varepsilon > 0$, we may pick $\delta > 0$ as in the definition of equicontinuity and let M be large so that the disks $D_\delta(w_1), \dots D_\delta(w_M)$ cover K. Fix N such that $|g_m(w_j) - g_n(w_j)| < \varepsilon$ for all $n, m \ge N$ and $j = 1, 2, \dots, M$. Therefore, for all $z \in K$, $z \in D_\delta(w_j)$ for some $1 \le j \le M$. Thence,

$$|g_n(z) - g_m(z)| \le |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| < 3\varepsilon$$

□

Proposition 6.17. *If $\Omega \subseteq \mathbb{C}$ is an open connected set and $\{f_n\}$ a sequence of holomorphic injective functions converging to f uniformly on every compact subset of Ω, then f is either injective or constant.*

Proof. By way of contradiction, suppose that f is neither injective nor constant and let $z_1, z_2 \in \mathbb{C}$ be distinct numbers such that $w := f(z_1) = f(z_2)$. Define the sequence $g(z) = f_n(z) - f_n(z_1)$ which,by injectivity, has no zero other than z_1. Furthermore, g_n tends to $f - w$ on every compact subset of Ω and g is not identically zero. Whence, z_2 must be an isolated zero of g and by the argument of principle,

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{g_n'(z)}{g_n(z)}\,\mathrm{d}z \xrightarrow{n\to\infty} \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,\mathrm{d}z = 1$$

for some sufficiently small circle γ. □

Lecture 22

Proof of the Riemann mapping theorem. Let Ω be a simply connected proper open subset of the complex plane. We prove the theorem in three steps;

(1) Ω is conformally equivalent to a simply connected open subset of $\mathbb{D}$ containing the origin.

(2) By step (1), we may assume without loss of generality that Ω is a simply connected open subset of $\mathbb{D}$ containing the origin. There exists a holomorphic injective function $f : \Omega \to \mathbb{D}$ which fixes the origin (i.e. $f(0) = 0$) such that if g is another holomorphic injective function which fixes the origin, then $|f'(0)| \ge |g'(0)|$.

(3) The function f as in step (2) is a conformal map from Ω to the unit disk.

To prove (1), since Ω is a proper subset, we may first pick $\alpha \in \mathbb{C} \setminus \Omega$. Then the function from Ω to $\mathbb{C}$ given by $z \mapsto z - \alpha$ has no roots and thus there exists a holomorphic function g on Ω such that $e^{g(z)} = z - \alpha$. Note also that g is injective since if $g(z_1) = g(z_2)$ then $z_1 - \alpha = z_2 - \alpha$. For fixed $w \in \Omega$, we claim that the map

$$G(z) = \frac{1}{g(z) - (g(w) + 2\pi i)}$$

is injective and holomorphic. Since g is injective, so is G. Furthermore, to see that G is holomorphic, it suffices to show that $g(z) \neq g(w) + 2\pi i$. Indeed, if equality holds then $z \neq w$ but $e^{g(z)} = z - \alpha = w - \alpha = e^{g(w)}$. Finally, we need to show that G maps to a bounded domain, so that a simple translation and re-scaling yields a conformal map from $\Omega \to \mathbb{D}$. By way of contradiction, suppose that G is not bounded, then there exists a sequence z_n such that $G(z_n) \to \infty$, i.e. $g(z_n) \to g(w) + 2\pi i$. But then z_n tends to w since $z_n - \alpha = e^{g(z_n)} \to e^{g(w)} = w - \alpha$ and by continuity of g, $g(z_n) \to g(w) \neq g(w) + 2\pi i$.

We now suppose that Ω is a simply connected open subset of the unit disk containing the origin and proceed to prove (2). Consider the family of functions

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ is injective and holomorphic and } f(0) = 0\}$$

Note first that $\mathcal{F}$ is not empty since it contains the map $z \mapsto z$ and $s = \sup_{f \in \mathcal{F}} |f'(0)| \geq 1$. By Cauchy's inequality, $f'(0)$ is uniformly bounded in $\mathcal{F}$. Hence, we may pick a sequence $\{f_n\}$ such that $|f_n'(0)| \to s$. Since every function in $\mathcal{F}$ maps to the unit disk, $\mathcal{F}$ is uniformly bounded and by Montel's theorem there exists a subsequence $\{f_{n_k}\}$ which converges to some function f uniformly on every compact set. By proposition 6.17, f is either injective or constant. Since $f'(0) = s \geq 1$, f must be injective and noting f is holomorphic and $f(0) = 0$ concludes the proof of (2).

For the last part, let f be as found in (2) and it suffices to prove that f is also surjective. We prove this by contradiction. Suppose there exists $\alpha \in \mathbb{D}$ such that $f(z) \neq \alpha$ for all $z \in \Omega$ and consider the automorphism of the unit disk

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

The set $U = \psi(f(\Omega))$ is a simply connected open subset of $\mathbb{D}$ which does not contain the origin. We may therefore define a square root function g on U (take e exponent half of the logarithm function defined on U). Define

$$h = \psi_{g(\alpha)} \circ g \circ \psi_\alpha \circ f$$

which we claim to be in the family

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f \text{ is injective and holomorphic and } f(0) = 0\}$$

The function h is clearly holomorphic, maps to the unit disk and $h(0) = 0$. To see that h is injective, note that $f, \psi_{g(\alpha)}$ and ψ_α are known to be injective and g is injective since if

$g(z) = g(w)$ then $z = g(z)^2 = g(w)^2 = w$. Now,

$$f = \psi_{\alpha}^{-1} \circ g^{-1} \circ \psi_{g(\alpha)}^{-1} \circ h = \psi_{\alpha} \circ g^{-1} \circ \psi_{g(\alpha)} \circ h := \Psi \circ h$$

and Ψ is a holomorphic map from $\mathbb{D} \to \mathbb{D}$ such that $\Psi(0) = 0$. Finally, by Schwartz' lemma $|\Psi'(0)| < 1$ and we conclude that

$$\left|f'(0)\right| = \left|\Psi'(h(0))h'(0)\right| = \left|\Psi'(0)\right| \left|h'(0)\right| < \left|h'(0)\right|$$

which contradicts our assumptions of f. □

Exercises and Solutions

Problem 1. Prove that $f(z) := -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from upper half disk to the upper half-plane

Solution. Define Ω to be the upper half disk and note that f is clearly holomorphic on Ω. Moreover, if $z \in \Omega$ then $z = re^{i\theta}$ for some $0 < r < 1$, $0 < \theta < \pi$ and

$$\begin{aligned} \Im(f(z)) = -\frac{1}{2}\left(\Im(re^{i\theta}) + \Im\left(r^{-1}e^{-i\theta}\right)\right) &= -\frac{1}{2}\left(r\sin(\theta) + r^{-1}\sin(-\theta)\right) \\ &= \frac{\sin(\theta)}{2}\left(\frac{1}{r} - r\right) > 0 \end{aligned} \tag{1}$$

The above it greater than 0 since $\sin(\theta) > 0$ whenever $0 < \theta < \pi$ and $r^{-1} > r$ for all $0 < r < 1$, thus f indeed maps to $\mathbb{H}$.

To see that f is injective from Ω to $\mathbb{H}$, suppose that $z \in \Omega$, $z \neq w$, then

$$f(z) = f(w) \iff z + \frac{1}{z} = w + \frac{1}{w} \iff z - w = zw(z - w) \iff zw = 1 \tag{2}$$

Ergo, $|w| = 1/|z| > 1$, i.e. $w \notin \Omega$.

Finally, we will prove that f is surjective. Let $w \in \mathbb{H}$, then

$$f(z) = w \iff z + \frac{1}{z} = -2w \iff z^2 + 2wz + 1 = 0$$

The above polynomial has two distinct roots, say z_1, z_2. Then $f(z_1) = f(z_2) = w$ and by equation (2) we know that $z_1 z_2 = 1$. Furthermore, it is clear from equation (1) that we cannot have $|z_1| = |z_2| = 1$. Ergo, one of z_1, z_2 is in $\mathbb{D}$. Without loss of generality, suppose $z_1 = re^{i\theta} \in \mathbb{D}$ the it remains to show that z_1 is in the upper half disk, i.e. $\Im(z_1) > 0$. If this were not the case, then equation (1) would yield a contradiction to the assumption $\Im(w) > 0$. □

Problem 2. Prove that

$$u(x,y) = \Re\left(\frac{i+z}{i-z}\right) \qquad z = x + iy$$

where $u(0,1) = 0$ is harmonic, unbounded and vanishes on the boundary.

Solution. The function u is harmonic since it is the real part of a holomorphic function. Furthermore, u vanishes on the boundary of $\mathbb{D}$ since

$$u(x,y) = \frac{1 - |z|^2}{1 - 2\Im(z) + |z|^2}$$

Finally, u is unbounded since $u(0,y)$ diverges as $y \to 1^+$. □

Problem 3. Let $F : \mathbb{H} \to \mathbb{D}$ be a holomorphic function from such that $|F(z)| \leq 1$ for all $z \in \mathbb{H}$ and $F(i) = 0$, then prove that

$$|F(z)| \leq \left|\frac{i-z}{i+z}\right| \quad \forall z \in \mathbb{H}$$

Solution. Recall that $G(w) = i\frac{1-w}{1+w}$ is a conformal map from $\mathbb{D}$ to $\mathbb{H}$ with inverse $G^{-1}(z) = \frac{i-z}{i+z}$. Thus, The function $F \circ G$ is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$ and

$$F \circ G(0) = F(G(0)) = F(i) = 0$$

By Schwartz's lemma, $|F \circ G(w)| \leq |w|$ for all $w \in \mathbb{D}$. Ergo, for any $z \in \mathbb{H}$

$$|F(z)| = \left|F\left(G\left(\frac{i-z}{i+z}\right)\right)\right| \leq \left|\frac{i-z}{i+z}\right|$$

□