Cantor's Theorem

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Theorem 1. *Let A be a set and $\mathcal{P}(A)$ the power set of A. The there does not exist a surjection $A \to \mathcal{P}(A)$.*

Before proving the theorem, I want to give an example to clarify what the power set is. Suppose for instance $A = \{1, 2\}$. Then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

The power set is a set of sets. The elements of the power set are themselves sets.

We include two more examples. If $A = \emptyset$ then $\mathcal{P}(A) = \{\emptyset\}$. Finally, if $A = \{0, 1, 2\}$ then

$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$$

You can make the observation that if A is a set with 2 elements then $\mathcal{P}(A)$ has 2^n elements.

Proof. Suppose for a contradiction there exists a set $f : A \to \mathcal{P}(A)$. Then for each $a \in A$, $f(a)$ is a subset of A. Now, we may consider the "anti-diagonal" set

$$D = \{a \in A : a \notin f(a)\}$$

That is, D is the subset of A containing all $a \in A$ such that a is not in the set $f(a)$.

Since D is a subset of A, we have $D \in \mathcal{P}(A)$. Since f is bijective (in particular surjective), there exists $x \in A$ such that $f(x) = D$. Now, there are exactly two possibilities: $x \in D$ or $x \notin D$. We consider what happens in both cases.

1. If $x \in D$, then by definition of the set D it must be the case that $x \notin f(x)$. But since $f(x) = D$, we then have $x \notin D$. This is absurd since we cannot have $x \in D$ and $x \notin D$.
2. If $x \notin D$. By definition of the set D the statement $x \notin D$ implies that we must have $x \in f(x)$. But $f(x) = D$. So we have $x \in D$ and $x \notin D$ which is absurd.

Either way we obtain a contradiction. This shows that there cannot exist a surjective function $A \to \mathcal{P}(A)$.[1] □

[1] Similarly, there does not exist an injective function from $\mathcal{P}(A)$ to A.