Real Floquet Factors of Linear Time-Periodic Systems ⋆

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Abstract

Floquet theory plays a ubiquitous role in the analysis and control of time-periodic systems. Its main result is that any fundamental matrix $\mathbf{X}(t,0)$ of a linear system with T-periodic coefficients will have a (generally complex) Floquet factorization with one of the two factors being T-periodic. It is also well known that it is always possible to obtain a real Floquet factorization for the fundamental matrix of a real T-periodic system by treating the system as having $2T$-periodic coefficients. The important work of Yakubovich in 1970 and Yakubovich and Starzhinskii in 1975 exhibited a class of real Floquet factorizations that could be found from computations on $[0,T]$ alone. Here we generalize these results to obtain other such factorizations. We delineate all factorizations of this form and show how they are related. We give a simple extension of the Lyapunov part of the Floquet-Lyapunov theorem in order to provide one way that the full range of real factorizations may be used based on computations on $[0,T]$ only. This new information can be useful in the analysis and control of linear time-periodic systems.

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1 Introduction: Complex Floquet Factors

The Floquet-Lyapunov theorem is a well-known and celebrated result in the field of linear time-periodic (LTP) systems (see e.g., [1–5]). The theorem consists of two main parts: the *Floquet representation theorem* and the *Lyapunov reducibility theorem*. Although the theorem applies to any fundamental matrix $\mathbf{X}(t,0)$ of solutions of an LTP system, in what follows we specialize our discussion in terms of the *state transition matrix*, namely the fundamental matrix of solutions $\mathbf{\Phi}(t,0)$ satisfying the initial condition $\mathbf{\Phi}(0,0) = \mathbf{I}$. This $\mathbf{\Phi}(t,0)$ is sometimes called the *principal matrix solution*.

In this section we briefly summarize the background theory that we require. We consider the homogeneous linear differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) \text{ given,} \tag{1}$$

where $\mathbf{A}(t) \in \mathbb{R}^{n\times n}$ is a continuous[4] matrix, $t \in \mathbb{R}$, and $\mathbf{x}(t) \in \mathbb{R}^n$. The state transition matrix of (1) is the solution of

$$\dot{\mathbf{\Phi}}(t,t_0) = \mathbf{A}(t)\cdot\mathbf{\Phi}(t,t_0), \quad \mathbf{\Phi}(t_0,t_0) = \mathbf{I}. \tag{2}$$

The standard theory shows that $\mathbf{\Phi}(t,t_0)$ exists, is unique, has a positive determinant, is continuous with a continuous derivative, and satisfies

$$\mathbf{\Phi}(t,t_0) = \mathbf{\Phi}(t,t_1)\cdot\mathbf{\Phi}(t_1,t_0). \tag{3}$$

Next (see e.g., [1–3]) the LTP system of the form (1) with

$$\mathbf{A}(t+T) = \mathbf{A}(t) \in \mathbb{R}^{n\times n}, \text{ for all } t, \text{ and some fixed period } T > 0, \tag{4}$$

has the following form of periodicity in its transition matrix:

$$\mathbf{\Phi}(t+T,t_0+T) = \mathbf{\Phi}(t,t_0) \quad \text{for all } t,\ t_0. \tag{5}$$

Without loss of generality, we take $t_0 = 0$ in the rest of the paper. Then (3) and (5) combine to show

$$\mathbf{\Phi}(t+T,0) = \mathbf{\Phi}(t,0)\cdot\mathbf{\Phi}(T,0) \quad \text{for all } t. \tag{6}$$

It is known (see e.g., [7, Chap.II, §2.1], or use (2) and the nonsingularity of $\mathbf{\Phi}(t,t_0)$) that $\mathbf{A}(t)$ is T-periodic if and only if (6) holds.

[4] This assumption is for simplicity only, see for example Hale [6, p.118], who also points out that the theory is valid for $\mathbf{A}(t)$ which is periodic and Lebesgue integrable if the differential equation holds almost everywhere. No changes in proofs are required. For a more formal and general presentation see [7].

The Floquet representation theorem provides an elegant representation of the state-transition matrix of a LTP system in terms of continuous and smooth factors. This requires matrix logarithms, and we refer to the theory as required.

The matrix equation $\mathrm{e}^{\mathbf{F}} = \mathbf{M} \in \mathbb{C}^{n\times n}$ has infinitely many solutions $\mathbf{F} \in \mathbb{C}^{n\times n}$ if and only if $\mathbf{M}$ is nonsingular, see e.g., [8, Thm.2.6h], [9, §6.4.15]. We call any such solution $\mathbf{F}$ a *logarithm* of $\mathbf{M}$, and write $\mathbf{F} = \log \mathbf{M}$. We will denote the set of all such solutions by $\mathcal{L}og\mathbf{M} \triangleq \{\mathbf{F} : \mathrm{e}^{\mathbf{F}} = \mathbf{M}\}$, and the subset of all real solutions by $\mathcal{R}Log\mathbf{M}$, which can be nonempty for some $\mathbf{M} \in \mathbb{R}^{n\times n}$. Since $\mathbf{\Phi}(T,0)$ is nonsingular, for any $\log \mathbf{\Phi}(T,0)$ we can take $\mathbf{F} \in \mathbb{C}^{n\times n}$ to be

$$\mathbf{F} = \frac{1}{T} \log \mathbf{\Phi}(T,0),$$

so that

$$\mathrm{e}^{T\mathbf{F}} = \mathbf{\Phi}(T,0). \tag{7}$$

We can use this $\mathbf{F}$ to define the nonsingular matrix function

$$\mathbf{L}_F(t,0) \triangleq \mathbf{\Phi}(t,0) \cdot \mathrm{e}^{-t\mathbf{F}} \in \mathbb{C}^{n\times n}. \tag{8}$$

Here the subscript $_F$ denotes the particular solution $T\mathbf{F}$ of (7) that we have chosen. Recalling that $\mathbf{L}_F(t,0)$ may be complex, we see

$$\mathbf{\Phi}(t,0) = \mathbf{L}_F(t,0) \cdot \mathrm{e}^{t\mathbf{F}}, \tag{9}$$

where $\mathbf{L}_F(T,0) = \mathbf{I} = \mathbf{L}_F(0,0)$ and $\mathbf{L}_F(t+T,0) = \mathbf{L}_F(t,0)$. Thus (9) is a factorization of $\mathbf{\Phi}(t,0)$ into a (possibly complex) T-periodic matrix $\mathbf{L}_F(t,0)$ that is continuous with a continuous derivative, and a matrix exponential $\mathrm{e}^{t\mathbf{F}}$. This is a Floquet factorization, and the Floquet representation theorem states the existence of these factors. For that reason (9) is also called a Floquet representation. Although the actual factors of $\mathbf{\Phi}(t,0)$ are $\mathbf{L}_F(t,0)$ and $\mathrm{e}^{t\mathbf{F}}$, it is common to refer to $\mathbf{L}_F(t,0)$ and $\mathbf{F}$ as the factors. We will follow this usage.

For practical applications we want to know what *real* factorizations exist. Previous results in this area have been mainly constructive, and have neither shown exactly what real factorizations exist, nor delineated the relationships between the possible real factorizations. In Section 3 we will fill in this gap by giving general results for *real* Floquet factorizations of the form (9).

Finally, the Lyapunov reducibility theorem states that the time-dependent change of variables

$$\mathbf{x}(t) = \mathbf{L}_F(t,0)\mathbf{z}(t) \tag{10}$$

transforms (1), with $t_0 = 0$ and (4), into the linear *time-invariant* system,

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t), \quad \mathbf{z}(0) = \mathbf{x}(0), \qquad \text{and so} \quad \mathbf{z}(t) = \mathrm{e}^{t\mathbf{F}}\mathbf{x}(0). \tag{11}$$

This shows the original system may be solved by finding an $\mathbf{F}$ in (7), and the corresponding $\mathbf{L}_F(t,0)$, and solving (11).

We will use the following concepts of periodicity.

Definition 1.1 *A function $f(t)$ is periodic with period T if there exists $T > 0$ such that $f(t+T) = f(t)$ for all t, and we will say $f(t)$ is T-periodic. In this case it has* primary *period T if T is the smallest such value, and then we will say it is primarily T-periodic.*

*A function $f(t)$ is T-*antiperiodic *if there exists $T > 0$ such that $f(t+T) = -f(t)$ for all t. In this case we will say it is primarily T-antiperiodic if T is the smallest such value.*

In the rest of the paper it should be kept in mind that if $\mathbf{A}(t)$ is primarily T-periodic then we would like to base all our computations on the interval $[0,T]$, rather than on a larger time interval.

2 Basic Real Floquet Factorizations

In general the state transition matrix of a real T-periodic matrix $\mathbf{A}(t)$ may have unavoidably complex Floquet factors in (9); see for example Section 5. We see from (8) that if $\mathbf{F}$ is real, the factors in (9) are real, so we would like to know when there are real solutions to (7). Culver [10] proved the following result (see also [9, Thm.6.4.15.c, p.475]).

Theorem 2.1 [10, Thm.1]. *Let $\mathbf{M}$ be a real square matrix. Then there exists a real solution $\mathbf{F}$ to the equation $\mathrm{e}^{\mathbf{F}} = \mathbf{M}$ if and only if $\mathbf{M}$ is nonsingular and each Jordan block of $\mathbf{M}$ belonging to a negative eigenvalue occurs an even number of times.*

It follows from (6) that $\mathbf{\Phi}(2T,0) = \mathbf{\Phi}(T,0)^2$ *always* has a real logarithm, since its only negative eigenvalues (if any) must come from purely imaginary eigenvalues of $\mathbf{\Phi}(T,0)$, and these must come in complex conjugate pairs of Jordan blocks because $\mathbf{\Phi}(T,0)$ is real. This leads to the most basic method of avoiding complex quantities using Floquet factorizations

Corollary 2.2 *It is always possible to obtain a real Floquet factorization of the state transition matrix of (1) with (4) by taking a $2T$-periodic factor via a* real *logarithm. Take any $\mathbf{F}_{2T}$ satisfying*

$$2T\mathbf{F}_{2T} \in \mathcal{R}Log\mathbf{\Phi}(2T,0), \quad \textit{so that} \quad \mathbf{\Phi}(T,0)^2 = \mathrm{e}^{2T\mathbf{F}_{2T}}. \tag{12}$$

Then $\mathbf{L}_{F_{2T}}(t,0) \triangleq \mathbf{\Phi}(t,0) \cdot \mathrm{e}^{-t\mathbf{F}_{2T}}$ is real and $2T$-periodic (but not necessarily T-periodic) with $\mathbf{L}_{F_{2T}}(0,0) = \mathbf{I}$. The disadvantage of this approach is that, at least with the analysis so far, two periods must always be used: for example $\mathbf{L}_{F_{2T}}(t,0)$ must be obtained for $0 \le t \le 2T$ in order to be used in (10)–(11). We will show how to avoid this disadvantage in Section 3.

In practice it is important to obtain real factorizations from computations on a single period. Yakubovich [11] and Yakubovich and Starzhinskii [7] address this problem, and in [7, Ch.2 §2.3] prove the following result (stated almost word for word here, but in the notation of the present paper). Notice that they use the more general assumptions of integrable and piecewise continuous $\mathbf{A}(t)$ *etc.*, and that our theory extends to such cases too; see Hale [6, p.118].

Theorem 2.3 *In the equation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t), \tag{13}$$

let $\mathbf{A}(t)$ *be a real matrix function, where* $\mathbf{A}(t)$ *is integrable and piecewise continuous on* $(0,T)$*, and* $\mathbf{A}(t+T) = \mathbf{A}(t)$ *almost everywhere. An arbitrary real matrix* $\mathbf{X}(t,0)$ *that is a fundamental solution of (13) may be expressed as*

$$\mathbf{X}(t,0) = \mathbf{L}(t,0) \cdot \mathrm{e}^{t\mathbf{F}}, \tag{14}$$

where $\mathbf{F}$ *is a real constant matrix,* $\mathbf{L}(t,0)$ *is a real matrix function such that*

$$\mathbf{L}(t+T,0) = \mathbf{L}(t,0) \cdot \mathbf{Y}, \tag{15}$$

and $\mathbf{Y}$ *some real matrix such that*

$$\mathbf{Y}^2 = \mathbf{I}, \quad \mathbf{FY} = \mathbf{YF}. \tag{16}$$

In particular,

$$\mathbf{L}(t+2T,0) = \mathbf{L}(t,0) \quad \textit{for all } t.$$

The function $\mathbf{L}(t,0)$ *is continuous with an integrable piecewise-continuous derivative.*

Conversely, let $\mathbf{L}(t,0)$*,* $\mathbf{F}$*, and* $\mathbf{Y}$ *be arbitrary real matrices satisfying conditions (15) and (16),* $\det \mathbf{L}(t,0) \neq 0$*, and let* $\mathbf{L}(t,0)$ *have an integrable piecewise-continuous derivative. Then (14) is a fundamental matrix for some equation of the form (13) with a real* T*-periodic matrix* $\mathbf{A}(t)$.

We will prove a more general result later, but both proofs use an instructive lemma [7, Ch.I, §2.7, Lemma II], for which we give a constructive proof.

Lemma 2.4 *For any real nonsingular matrix* $\mathbf{X}$ *there exist real matrices* $\mathbf{F}$ *and* $\mathbf{Y}$ *such that*

$$\mathrm{e}^{\mathbf{F}} = \mathbf{XY} = \mathbf{YX}, \quad \mathbf{FY} = \mathbf{YF}, \quad \mathbf{Y}^2 = \mathbf{I}.$$

Proof: Consider a real similarity transformation

$$\mathbf{S}^{-1}\mathbf{X}\mathbf{S} = \begin{bmatrix} \mathbf{J}_1 & 0 \\ 0 & \mathbf{J}_2 \end{bmatrix} = \mathbf{J}$$

where $\mathbf{J}_2$ contains all the negative real eigenvalues of $\mathbf{X}$ and no others ($\mathbf{J}$ could be the *real* Jordan canonical form of $\mathbf{X}$). With this partitioning define

$$\mathbf{K} \triangleq \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}, \quad \mathbf{Y} \triangleq \mathbf{S}\mathbf{K}\mathbf{S}^{-1},$$

so that $\mathbf{Y}^2 = \mathbf{I}$. We see $\mathbf{J}\mathbf{K} = \mathbf{K}\mathbf{J}$ has no negative real eigenvalues, so by Theorem 2.1 there exists real $\mathbf{F}$ such that

$$\mathbf{X}\mathbf{Y} = \mathbf{X}\mathbf{S}\mathbf{K}\mathbf{S}^{-1} = \mathbf{S}\mathbf{J}\mathbf{K}\mathbf{S}^{-1} = \mathbf{S}\mathbf{K}\mathbf{J}\mathbf{S}^{-1} = \mathbf{Y}\mathbf{X} = \mathrm{e}^{\mathbf{F}}.$$

Finally $\mathrm{e}^{\mathbf{S}^{-1}\mathbf{F}\mathbf{S}} = \mathbf{S}^{-1}\mathrm{e}^{\mathbf{F}}\mathbf{S} = \mathbf{J}\mathbf{K}$, so $\mathbf{S}^{-1}\mathbf{F}\mathbf{S}$ must have the same block structure, showing $\mathbf{K}\mathbf{S}^{-1}\mathbf{F}\mathbf{S} = \mathbf{S}^{-1}\mathbf{F}\mathbf{S}\mathbf{K}$, and so $\mathbf{F}\mathbf{Y} = \mathbf{Y}\mathbf{F}$. ■

The point of the approach of Yakubovich and Starzhinskii in [7] is that if $\mathbf{X} = \mathbf{\Phi}(T, 0)$ does *not* have a real logarithm, it is straightforward to find $\mathbf{Y}$ (as shown for example above) so $\mathbf{Y}\mathbf{\Phi}(T, 0)$ *does*; $\mathbf{\Phi}(2T, 0)$ is not required. Theorem 2.3 shows their factor $\mathbf{L}(t, 0)$ is a $2T$-periodic Floquet factor just as in Corollary 2.2. But their contribution is that $\mathbf{L}(t, 0)$ obeys (15) — a variant of T-periodicity that we call *near T-periodicity* — and the factors, and so any solutions, may thus be found from computations on a *single* period. This theorem marks a significant step in the characterization of real Floquet factorizations. It allows for a more concise representation of the real factors and efficiency gains in their computation.

In the context of (1), it is straightforward to show that $\mathbf{L}_F(t, 0)$ (in (8) with (7)) is T-periodic if and only if (6) holds, which we saw is true if and only if (4) holds. Any of these relations can be used to prove Corollary 2.2, but using just this could lead to significant drawbacks for control applications because it either forces the control engineer to use complex quantities or to work on a period that is potentially longer than necessary or otherwise undesirable. We also know [12] that a periodic feedback gain matrix $\mathbf{K}(t)$ can assign the whole matrix $\mathbf{F}$ in (9). Assuming a feedback of the form $\mathbf{u}(t) = \mathbf{K}(t)\mathbf{x}(t)$ in the LTP control system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{A}(t+T) = \mathbf{A}(t), \ \mathbf{B}(t+T) = \mathbf{B}(t)$$

for all t and some fixed $T > 0$, several proposed control schemes [13–16] attempt to first assign the factors $\mathbf{L}(t, 0)$, $\mathbf{F}$ of the LTP closed-loop system. It is an advantage to know and be able to assign the periodicity of $\mathbf{L}(t, 0)$; for example, it may be desirable to maintain the T-periodicity of the closed-loop system. By means of Theorem 3.1, we will be able to avoid the abovementioned drawback, and the control engineer will be able to work with real factors on one period and with a known periodicity of the factors.

Moreover, the development so far here, and apparently in the literature in general, has been essentially constructive, and has said nothing about what other real factorizations of the form (9) exist, nor about the relationships between them. In the next section we will complete this part of the theory by giving necessary and sufficient conditions for such factorizations. This work will allow us to answer the following questions (among others):

(1) Under exactly what circumstances will Corollary 2.2 or Theorem 2.3 produce T-periodic $\mathbf{L}_{F_{2T}}(t,0)$ or $\mathbf{L}(t,0)$? (This happens if and only if $\mathbf{\Phi}(T,0)$ has a real logarithm; see Theorem 3.1.)
(2) What is the relationship between the factorizations in Corollary 2.2 and those in Theorem 2.3? (The factorizations in Theorem 2.3 are a subset of those in Corollary 2.2.)
(3) Are there other real $2T$-periodic Floquet factorizations besides those in Corollary 2.2? (No, and one contribution of this work is to show *there are no others*. Other contributions are to show how *all* of these factorizations may be obtained from computations on just $[0,T]$, and to provide knowledge that Corollary 2.2 does not give.)
(4) Are there other useful real $2T$-periodic Floquet factorizations that can be obtained from calculations on just $[0,T]$ besides those in Theorem 2.3? (There are, making this paper useful in a practical sense, and not just of academic interest; see also [15,16].)

As part of this exercise we will characterize all real $2T$-periodic Floquet factorizations, show that Corollary 2.2 gives these, and show how those from Theorem 2.3 fit into this set.

3 General Real Floquet Factorizations

We wish to characterize *all* real Floquet factorizations $\mathbf{\Phi}(t,0) = \mathbf{L}(t,0)\mathrm{e}^{t\mathbf{F}}$ with T-periodic or $2T$-periodic $\mathbf{L}(t,0)$. To do this we will ignore the constraint (15). Then we will show that a constraint of this form leads to the subset of real $2T$-periodic factorizations given by Theorem 2.3.

Theorem 3.1 *In the equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ with $\mathbf{x}(0)$ given, let $\mathbf{A}(t)$ be a real matrix function, where $\mathbf{A}(t)$ is continuous on $(0,T)$, $T > 0$, and $\mathbf{A}(t+T) = \mathbf{A}(t)$ for all t. Let $\mathbf{\Phi}(t,0)$ be the corresponding (real, nonsingular) state transition matrix, and write $\mathbf{\Phi} \triangleq \mathbf{\Phi}(T,0)$. Let real $\mathbf{Y}$ be such that $\mathbf{Y\Phi}$ has a* *real logarithm (such a* $\mathbf{Y}$ *always exists; see for example Lemma 2.4), and take any* $\mathbf{F}_Y$ *satisfying*

$$T\mathbf{F}_Y \in \mathcal{R}Log(\mathbf{Y}\mathbf{\Phi}), \quad \textit{so } \mathbf{Y}\mathbf{\Phi} = \mathrm{e}^{T\mathbf{F}_Y}; \tag{17}$$

$$\mathbf{L}_{F_Y}(t,0) \triangleq \mathbf{\Phi}(t,0)\cdot \mathrm{e}^{-t\mathbf{F}_Y}, \ \textit{so } \mathbf{L}_{F_Y}(0,0) = \mathbf{I}. \tag{18}$$

Then in the real factorization $\mathbf{\Phi}(t,0) = \mathbf{L}_{F_Y}(t,0)\cdot \mathrm{e}^{t\mathbf{F}_Y}$, $\mathbf{L}_{F_Y}(t,0)$ *has a continuous derivative and*

$$\mathbf{L}_{F_Y}(t+T,0) = \mathbf{L}_{F_Y}(t,0)\cdot \mathrm{e}^{t\mathbf{F}_Y}\cdot \mathbf{L}_{F_Y}(T,0)\cdot \mathrm{e}^{-t\mathbf{F}_Y}, \tag{19}$$

the equivalent of (6) for $\mathbf{L}_{F_Y}(t,0)$*. The choice of* $\mathbf{Y}$ *affects* $\mathbf{L}_{F_Y}(t,0)$ *as follows:*

$$\mathbf{L}_{F_Y}(T,0) = \mathbf{Y}^{-1}; \tag{20}$$

$$\mathbf{L}_{F_Y}(t,0) \quad \textit{is } T\textit{-periodic if and only if } \mathbf{Y} = \mathbf{I}; \tag{21}$$

$$\mathbf{L}_{F_Y}(t,0) \quad \textit{is } T\textit{-antiperiodic if and only if } \mathbf{Y} = -\mathbf{I}; \tag{22}$$

$$\mathbf{L}_{F_Y}(t,0) \quad \textit{is } 2T\textit{-periodic if and only if } \mathbf{\Phi}^2 = (\mathbf{Y}\mathbf{\Phi})^2, \tag{23}$$

where a $\mathbf{Y}$ *satisfying* $\mathbf{\Phi}^2 = (\mathbf{Y}\mathbf{\Phi})^2$ *always exists. Finally this last condition on* $\mathbf{Y}$ *has some useful equivalences:*

$$\mathbf{\Phi}^2 = (\mathbf{Y}\mathbf{\Phi})^2 \Leftrightarrow \mathbf{\Phi}^2 = (\mathbf{\Phi}\mathbf{Y})^2 \Leftrightarrow \mathbf{\Phi} = \mathbf{Y}\mathbf{\Phi}\mathbf{Y}. \tag{24}$$

Proof: The expression for $\mathbf{\Phi}(t,0)$ with (6) shows that

$$\begin{aligned}\mathbf{L}_{F_Y}(t+T,0) &= \mathbf{\Phi}(t+T,0)\cdot \mathrm{e}^{-(t+T)\mathbf{F}_Y} = \mathbf{\Phi}(t,0)\cdot\mathbf{\Phi}\cdot \mathrm{e}^{-(t+T)\mathbf{F}_Y}\\ &= \mathbf{L}_{F_Y}(t,0)\cdot \mathrm{e}^{t\mathbf{F}_Y}\cdot \mathbf{L}_{F_Y}(T,0)\cdot \mathrm{e}^{T\mathbf{F}_Y}\cdot \mathrm{e}^{-(t+T)\mathbf{F}_Y},\end{aligned}$$

proving (19). Next $\mathbf{L}_{F_Y}(T,0) = \mathbf{\Phi}\cdot \mathrm{e}^{-T\mathbf{F}_Y} = \mathbf{Y}^{-1}$ from (17), proving (20). But from (20), (19) is equal to $\mathbf{L}_{F_Y}(t,0)$ if and only if $\mathbf{Y} = \mathbf{I}$, proving (21), and equal to $-\mathbf{L}_{F_Y}(t,0)$ if and only if $\mathbf{Y} = -\mathbf{I}$, proving (22). The equivalences in (24) are obvious. Repeated use of (19) gives

$$\begin{aligned}\mathbf{L}_{F_Y}(t+2T,0) &= \mathbf{L}_{F_Y}(t+T,0)\cdot \mathrm{e}^{(t+T)\mathbf{F}_Y}\cdot \mathbf{L}_{F_Y}(T,0)\cdot \mathrm{e}^{-(t+T)\mathbf{F}_Y}\\ &= \mathbf{L}_{F_Y}(t,0)\cdot \mathrm{e}^{t\mathbf{F}_Y}\cdot \mathbf{L}_{F_Y}(T,0)\cdot \mathrm{e}^{T\mathbf{F}_Y}\cdot \mathbf{L}_{F_Y}(T,0)\cdot \mathrm{e}^{-(t+T)\mathbf{F}_Y},\end{aligned}$$

which is equal to $\mathbf{L}_{F_Y}(t,0)$ if and only if $\mathbf{L}_{F_Y}(T,0)\cdot \mathrm{e}^{T\mathbf{F}_Y}\cdot \mathbf{L}_{F_Y}(T,0) = \mathrm{e}^{T\mathbf{F}_Y}$, or from (20) and (17), if and only if $\mathbf{Y}^{-1}\mathbf{Y}\mathbf{\Phi}\mathbf{Y}^{-1} = \mathbf{Y}\mathbf{\Phi}$. This with (24) proves (23). That such a $\mathbf{Y}$ exists follows from Lemma 2.4 with $\mathbf{X} = \mathbf{\Phi}$ because $\mathbf{\Phi}\mathbf{Y} = \mathbf{Y}\mathbf{\Phi}$ and $\mathbf{Y}^2 = \mathbf{I}$ imply $(\mathbf{Y}\mathbf{\Phi})^2 = \mathbf{\Phi}\mathbf{Y}\mathbf{Y}\mathbf{\Phi} = \mathbf{\Phi}^2$. ■

This theorem shows that a *real* Floquet factorization exists with T-periodic $\mathbf{L}_{F_Y}(t,0)$ if and *only if* $\mathbf{\Phi} \triangleq \mathbf{\Phi}(T,0)$ has a real logarithm; see (21). A real Floquet factorization exists with T-antiperiodic $\mathbf{L}_{F_Y}(t,0)$ if and only if $-\mathbf{\Phi}$ has a real logarithm; see (22). Whether either of these exists or not, a real Floquet factorization necessarily exists with $2T$-periodic $\mathbf{L}_{F_Y}(t,0)$, and the only conditions on $\mathbf{Y}$, which we call the Yakubovich matrix, are that it is real, that $\mathbf{Y\Phi}$ has a real logarithm, and that $\mathbf{\Phi}^2 = (\mathbf{Y\Phi})^2$. The construction of a Yakubovich matrix $\mathbf{Y}$ can be based on the stem function [9, p.411] (as was done via $\mathbf{K}$ in the proof of Lemma 2.4)

$$f(x) = \begin{cases} -1, & x \in \mathbb{R}^- \\ 1, & \text{otherwise.} \end{cases}$$

We note that the choice of $\mathbf{Y}$ gives an *a priori* knowledge of the periodicity (T-periodic, T-antiperiodic, or primarily $2T$-periodic) of the factor $\mathbf{L}(t,0)$, something which could not be determined *a priori* from previous theory. We also note that not all matrices $\mathbf{F}_Y$ in (17) will contain the classical stability information of the original system; i.e., the system (1) with (4) is stable if and only if the eigenvalues of $\mathbf{F}$ satisfying (7) have negative real parts. Even so, a full generalization of the converse in the last paragraph of Theorem 2.3 is useful for designing feedback systems [15,16], so we give this here.

Corollary 3.2 *Let* $\mathbf{L}(t,0)$ *and* $\mathbf{F}$ *be arbitrary real matrices satisfying (see (19)),*

$$\mathbf{L}(t+T,0) = \mathbf{L}(t,0) \cdot \mathrm{e}^{t\mathbf{F}} \cdot \mathbf{L}(T,0) \cdot \mathrm{e}^{-t\mathbf{F}} \tag{25}$$

with $\det \mathbf{L}(t,0) \neq 0$*, and let* $\mathbf{L}(t,0)$ *have a continuous derivative, then*

$$\mathbf{X}(t,0) \triangleq \mathbf{L}(t,0) \cdot \mathrm{e}^{t\mathbf{F}} \tag{26}$$

is a fundamental matrix for some equation of the form (1) with a real T*-periodic matrix* $\mathbf{A}(t)$*.*

Proof: Putting $t = 0$ in (25) shows $\mathbf{L}(0,0) = \mathbf{I}$, and (26) shows $\mathbf{X}(t,0)$ is nonsingular with a continuous derivative and $\mathbf{X}(0,0) = \mathbf{I}$. Define

$$\mathbf{A}(t) \triangleq \dot{\mathbf{X}}(t,0)\mathbf{X}^{-1}(t,0) = [\mathbf{L}(t,0)\mathbf{F} + \dot{\mathbf{L}}(t,0)]\mathbf{L}^{-1}(t,0).$$

Replacing t by $t+T$ in this and using (25) shows, after some cancellation, that $\mathbf{A}(t+T) = \mathbf{A}(t)$. Since $\dot{\mathbf{X}}(t,0) = \mathbf{A}(t)\mathbf{X}(t,0)$ the result is proven. ∎

We now show how Corollary 2.2 fits in with the general result of Theorem 3.1 by showing the equivalence of the set $\mathcal{R}Log\mathbf{\Phi}^2$ with the set $\mathcal{R}Log(\mathbf{Y\Phi})$ for $\mathbf{Y}$ in (23). The use of T is unnecessary in this — it is included for consistency.

Corollary 3.3 *For any nonsingular* $\mathbf{\Phi} \in \mathbb{R}^{n\times n}$ *and* $0 < T \in \mathbb{R}$*,*

$$\mathcal{R}Log\mathbf{\Phi}^2 \equiv \{2T\mathbf{F}_Y : \exists \text{ real } \mathbf{Y} \text{ with } T\mathbf{F}_Y \in \mathcal{R}Log(\mathbf{Y\Phi}) \text{ and } \mathbf{\Phi}^2 = (\mathbf{Y\Phi})^2\}. \tag{27}$$

Proof: Any element $2T\mathbf{F}_Y$ of the set on the right side of the equivalence is real and satisfies $\mathbf{Y\Phi} = \mathrm{e}^{T\mathbf{F}_Y}$, $\mathrm{e}^{2T\mathbf{F}_Y} = \mathbf{Y\Phi Y\Phi} = \mathbf{\Phi}^2$, showing it belongs to the left set. Now consider any $2T\mathbf{F} \in \mathcal{R}Log\mathbf{\Phi}^2$, then $2T\mathbf{F}$ is real and $\mathbf{\Phi}^2 = \mathrm{e}^{2T\mathbf{F}}$. Define $\mathbf{Y}_F \triangleq \mathrm{e}^{T\mathbf{F}}\mathbf{\Phi}^{-1}$, so $\mathbf{Y}_F\mathbf{\Phi} = \mathrm{e}^{T\mathbf{F}}$ is real and $(\mathbf{Y}_F\mathbf{\Phi})^2 = \mathrm{e}^{2T\mathbf{F}} = \mathbf{\Phi}^2$, showing $2T\mathbf{F}$ belongs to the right set. ∎

This shows that the set of $\mathbf{F}_{2T}$ in (12) of Corollary 2.2 is identical to the set of $\mathbf{F}_Y$ satisfying (17) with $\mathbf{\Phi}^2 = (\mathbf{Y\Phi})^2$ in Theorem 3.1. That is, Corollary 2.2 provides *all* possible $2T$-periodic $\mathbf{L}(t, 0)$, just by choosing the different possible real logarithms. However Corollary 2.2 still has a few shortcomings. First, it does not provide the corresponding $\mathbf{Y}$. We now see from (17) and (20), or the proof of Corollary 3.3, that this is

$$\mathbf{Y} = \mathbf{L}_{F_{2T}}(T, 0)^{-1} = \mathrm{e}^{T\mathbf{F}_{2T}} \cdot \mathbf{\Phi}(T, 0)^{-1}. \tag{28}$$

Another way of viewing this is that Corollary 2.2 does not give the periodicity of $\mathbf{L}(t, 0)$ *a priori*. Hence there is no means of determining beforehand if in fact a real T-periodic factor exists. Second, if it is possible to specify $\mathbf{Y}$, then the periodicity of $\mathbf{L}(t, 0)$ can in fact be *assigned*. This can be useful if a specific periodicity is required, for example in the design of a stabilizing feedback [15,16]. Finally, the use of the matrix $\mathbf{Y}$ fits nicely with the general approach of analysing LTP systems by focusing on the state transition matrix after one period $\mathbf{\Phi}(T, 0)$, rather than after some other number of periods.

4 Near T-periodic Floquet Factorizations

The results in Theorem 2.3 here require (15), but our new results have not insisted on this so far. The near T-periodicity of the form (15) is both elegant and important, so we examine exactly when it occurs.

Corollary 4.1 *With the conditions and notation of Theorem 3.1, if for some real* $\mathbf{Y}$*,* $\mathbf{Y\Phi}$ *has a real logarithm* $T\mathbf{F}_Y$ *and* $\mathbf{\Phi}^2 = (\mathbf{Y\Phi})^2$ *(which are the necessary and sufficient conditions for* $\mathbf{L}_{F_Y}(t, 0)$ *in (18) to be* $2T$*-periodic), then the following are equivalent:*

$$\mathbf{L}_{F_Y}(t+T,0) = \mathbf{L}_{F_Y}(t,0)\cdot\mathbf{C} \text{ for all } t \text{ and some constant matrix } \mathbf{C}, \quad (29)$$
$$\mathbf{L}_{F_Y}(t+T,0) = \mathbf{L}_{F_Y}(t,0)\cdot\mathbf{L}_{F_Y}(T,0) \quad \text{for all } t, \quad (30)$$
$$\mathrm{e}^{t\mathbf{F}_Y}\cdot\mathbf{L}_{F_Y}(T,0) = \mathbf{L}_{F_Y}(T,0)\cdot\mathrm{e}^{t\mathbf{F}_Y} \quad \text{for all } t, \quad (31)$$
$$\mathrm{e}^{t\mathbf{F}_Y}\cdot\mathbf{Y} = \mathbf{Y}\cdot\mathrm{e}^{t\mathbf{F}_Y} \quad \text{for all } t, \quad (32)$$
$$\mathbf{F}_Y\mathbf{Y} = \mathbf{Y}\mathbf{F}_Y. \quad (33)$$

Here the following are also equivalent, and hold if the above hold:

$$\mathbf{\Phi}\mathbf{Y} = \mathbf{Y}\mathbf{\Phi}; \quad \mathbf{Y}^2 = \mathbf{I}; \quad \mathbf{L}_{F_Y}(T,0) = \mathbf{Y}. \quad (34)$$

Note how (30) parallels (6). If (29) holds then we also have

$$\mathbf{C} = \mathbf{L}_{F_Y}(T,0) = \mathbf{Y}^{-1} = \mathbf{Y}. \quad (35)$$

Proof: Since $\mathbf{Y\Phi} = \mathrm{e}^{T\mathbf{F}_Y}$, $\mathbf{Y}$ is nonsingular. Clearly (30) implies (29). If (29) holds, taking $t=0$ and using (18) and (20) shows $\mathbf{C} = \mathbf{L}_{F_Y}(T,0) = \mathbf{Y}^{-1}$, so that (30) and all but the last equality in (35) must hold. We see from (19) that (30) holds if and only if (31) holds, which is true if and only if (32) holds, see (20). Taking derivatives of (32) with respect to t and setting $t=0$ shows it implies (33). Conversely (33) clearly implies (32).

Next if (32) holds, then taking $t=T$ and using (17) shows $\mathbf{\Phi Y} = \mathbf{Y\Phi}$, while $\mathbf{\Phi}^2 = (\mathbf{Y\Phi})^2$ shows the equivalence of $\mathbf{\Phi Y} = \mathbf{Y\Phi}$ and $\mathbf{Y}^2 = \mathbf{I}$. Finally (20) shows the equivalence of $\mathbf{Y}^2 = \mathbf{I}$ and $\mathbf{L}_{F_Y}(T,0) = \mathbf{Y}$, where if either is true we see the last equality in (35) is true. ■

An important consequence of this is that, for T-periodic $\mathbf{A}(t)$, the near T-periodicity (19) for general $2T$-periodic $\mathbf{L}(t,0)$ specializes to our variant (30) of Yakubovich and Starzhinskii's (15) if and only if $\mathbf{FY} = \mathbf{YF}$.

Corollary 4.1 has shown that Yakubovich and Starzhinskii have characterized *exactly* that set of real Floquet factorizations with $2T$-periodic $\mathbf{L}(t,0)$ satisfying the near T-periodicity of the form (15), which we now see is (30). This is both elegant and useful because only calculations on $[0,T]$ are required. For example, $\mathbf{L}(t,0)$ in the second half of the $2T$-period can be formed simply from the first half: $\mathbf{L}(t+T,0) = \mathbf{L}(t,0)\cdot\mathbf{L}(T,0)$.

We would like to obtain similar benefits for the more general factorizations of Theorem 3.1. But instead of (30), we only have (19) in general:

$$\mathbf{L}(t+T,0) = \mathbf{L}(t,0)\cdot\mathrm{e}^{t\mathbf{F}}\cdot\mathbf{L}(T,0)\cdot\mathrm{e}^{-t\mathbf{F}}.$$

In principle this gives $\mathbf{L}(t,0)$ over its whole period of $2T$ from calculations on only the first half, but it is not in general computationally simple. However we can give a simple extension of the Lyapunov reducibility theorem (see (10)–(11)) to obtain $\mathbf{x}(t)$ for any t. Suppose we only know $\mathbf{L}(t,0)$ for $t\in[0,T]$. For any integer k, the solution $\mathbf{x}(t)$ for $t \in [2kT, (2k+1)T]$ may be found via (10) and (11) as before:

$$\dot{\mathbf{z}}(t) = \mathbf{F}\mathbf{z}(t), \quad \mathbf{z}(0) = \mathbf{x}(0), \quad \mathbf{x}(t) = \mathbf{L}(t,0)\mathbf{z}(t) = \mathbf{L}(t,0) \cdot \mathrm{e}^{t\mathbf{F}}\mathbf{x}(0), \tag{36}$$

since $\mathbf{L}(t,0) = \mathbf{L}(t-2kT,0)$. For the second half of this $2T$-period we see

$$\mathbf{x}(t+T) = \mathbf{\Phi}(t+T,0)\mathbf{x}(0) = \mathbf{\Phi}(t,0)\mathbf{\Phi}(T,0)\mathbf{x}(0) = \mathbf{L}(t,0)\mathrm{e}^{t\mathbf{F}}\mathbf{x}(T).$$

This can be found efficiently by a different solution, but with the same transformation:

$$\dot{\mathbf{w}}(t) = \mathbf{F}\mathbf{w}(t), \quad \mathbf{w}(0) = \mathbf{x}(T), \quad \mathbf{x}(t+T) = \mathbf{L}(t,0)\mathbf{w}(t), \tag{37}$$

once $\mathbf{x}(T)$ is known from (36). Thus for finding $\mathbf{x}(t)$ for any t we can still work with calculations from only one period no matter which real $2T$-periodic Floquet factorization we choose.

We see from Corollary 4.1 that the Yakubovich and Starzhinskii factorizations with $2T$-periodic $\mathbf{L}(t,0)$ in Theorem 2.3 are the subset of those defined by Theorem 3.1 that are obtained by insisting on any of the equivalent constraints (29) to (33), which imply (34). The question arises as to whether there are other meaningful factorizations than those in Theorem 2.3. The answer is yes. Section 5 gives a case where (33) does not hold. In that particular case, $\mathbf{\Phi}^2 = (\mathbf{Y}_1\mathbf{\Phi})^2$, $\mathbf{\Phi}\mathbf{Y}_1 = \mathbf{Y}_1\mathbf{\Phi}$, $\mathbf{Y}_1^2 = \mathbf{I}$, but $\mathbf{F}_1\mathbf{Y}_1 \neq \mathbf{Y}_1\mathbf{F}_1$. Such new factorizations may be as useful in practice as the Yakubovich and Starzhinskii factorizations in Theorem 2.3, see the comment following (37).

The condition $\mathbf{\Phi}^2 = (\mathbf{Y}\mathbf{\Phi})^2$ in Theorem 3.1 was weaker than expected, so here we examine it more closely.

Lemma 4.2 *For nonsingular* $\mathbf{\Phi}$ *and* $\mathbf{Y}$*, consider the three equations*

$$\mathbf{Y}^2 = \mathbf{I}, \quad \mathbf{\Phi}\mathbf{Y} = \mathbf{Y}\mathbf{\Phi}, \quad \mathbf{\Phi}^2 = (\mathbf{Y}\mathbf{\Phi})^2. \tag{38}$$

Any two of these equations imply the third, but we can have any one without either of the other two.

Proof:

$$\begin{aligned}
\mathbf{Y}^2 = \mathbf{I} \ \text{ and } \ \mathbf{\Phi}\mathbf{Y} = \mathbf{Y}\mathbf{\Phi} \ &\Rightarrow (\mathbf{Y}\mathbf{\Phi})^2 = \mathbf{\Phi}\mathbf{Y}\mathbf{Y}\mathbf{\Phi} = \mathbf{\Phi}^2,\\
\mathbf{Y}^2 = \mathbf{I} \ \text{ and } \ \mathbf{\Phi}^2 = (\mathbf{Y}\mathbf{\Phi})^2 &\Rightarrow \mathbf{Y}\mathbf{\Phi}^2 = \mathbf{\Phi}\mathbf{Y}\mathbf{\Phi} \Rightarrow \mathbf{Y}\mathbf{\Phi} = \mathbf{\Phi}\mathbf{Y},\\
\mathbf{\Phi}\mathbf{Y} = \mathbf{Y}\mathbf{\Phi} \ \text{ and } \ \mathbf{\Phi}^2 = (\mathbf{Y}\mathbf{\Phi})^2 &\Rightarrow \mathbf{\Phi}^2 = \mathbf{\Phi}\mathbf{Y}^2\mathbf{\Phi} \Rightarrow \mathbf{Y}^2 = \mathbf{I}.
\end{aligned}$$

However, $\mathbf{Y} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ gives $\mathbf{Y}^2 = \mathbf{I}$ only. $\mathbf{Y} = 2\mathbf{I}$ gives $\boldsymbol{\Phi}\mathbf{Y} = \mathbf{Y}\boldsymbol{\Phi}$ only. $\mathbf{Y} = \begin{bmatrix} 2^{-1} & 0 \\ 0 & 2 \end{bmatrix}$, $\boldsymbol{\Phi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ gives $\boldsymbol{\Phi}^2 = (\mathbf{Y}\boldsymbol{\Phi})^2$ only. ■

Now consider the matrices $\boldsymbol{\Phi}$ and $\mathbf{Y}$ given by

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} \alpha & \alpha - 1 & 0 & 0 \\ \alpha + 1 & \alpha & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then $\boldsymbol{\Phi}$ does not have a real logarithm, while for every $\alpha \in \mathbb{R}$, $\mathbf{Y}$ is real and such that $\mathbf{Y}\boldsymbol{\Phi}$ *does* have a real logarithm with $\boldsymbol{\Phi}^2 = (\mathbf{Y}\boldsymbol{\Phi})^2$, but $\mathbf{Y}^2 \neq \mathbf{I}$ and $\boldsymbol{\Phi}\mathbf{Y} \neq \mathbf{Y}\boldsymbol{\Phi}$. Thus once again we see it is possible for only one of the conditions (38) to hold.

5 Examples of Real Floquet Factors

Not all real Floquet factors of the state transition matrix of a real system satisfy the hypotheses of Theorem 2.3. The following gives an example where $\mathbf{Y}_1\mathbf{F}_1 \neq \mathbf{F}_1\mathbf{Y}_1$. Consider the T-periodic matrix with $\alpha \neq 0$ so $T = 1/2$:

$$\mathbf{A}(t) = 2\pi \begin{bmatrix} -1 + \alpha\cos^2(2\pi t) & 1 - \alpha\sin(2\pi t)\cos(2\pi t) & 0 \\ -1 - \alpha\sin(2\pi t)\cos(2\pi t) & -1 + \alpha\sin^2(2\pi t) & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

If we define the rotation matrix

$$\mathbf{R}(\theta) \triangleq \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

then it can be verified that the state transition matrix of this system is

$$\boldsymbol{\Phi}(t,0) = \begin{bmatrix} \mathbf{R}(2\pi t) & 0 \\ 0 & 1 \end{bmatrix} \operatorname{diag}\{e^{2\pi(\alpha-1)t}, e^{-2\pi t}, e^{-2\pi t}\}.$$

Note that $\mathbf{\Phi} \triangleq \mathbf{\Phi}(T,0) = \mathrm{diag}\left\{-\mathrm{e}^{\pi(\alpha-1)}, -\mathrm{e}^{-\pi}, \mathrm{e}^{-\pi}\right\}$ does not have a real logarithm. One suitable choice for $\mathbf{Y}$ is $\mathbf{Y} = \mathrm{diag}\,\{-1,-1,1\}$, giving a logarithm such that

$$\mathbf{F} = \frac{1}{T}\log(\mathbf{Y\Phi}) = \mathrm{diag}\,\{2\pi(\alpha-1), -2\pi, -2\pi\}\,,$$

where since $\mathrm{e}^{t\mathbf{F}} = \mathrm{diag}\,\{\mathrm{e}^{2\pi\,(\alpha-1)t}, \mathrm{e}^{-2\pi\,t}, \mathrm{e}^{-2\pi t}\}$,

$$\mathbf{L}(t,0) \triangleq \mathbf{\Phi}(t,0)\mathrm{e}^{-t\mathbf{F}} = \begin{bmatrix} \mathbf{R}(2\pi t) & 0 \\ 0 & 1 \end{bmatrix}.$$

In this case, it is easy to check that all the aspects of Theorem 2.3 are satisfied, notably $\mathbf{FY} = \mathbf{YF}$ because both $\mathbf{F}$ and $\mathbf{Y}$ are diagonal.

If we now take $\mathbf{Y}_1 = \mathrm{diag}\,\{-1,1,-1\}$, then

$$\begin{aligned}
\mathbf{F}_1 &= \frac{1}{T}\log(\mathbf{Y}_1\mathbf{\Phi}) = \frac{1}{T}\log(\mathrm{diag}\left\{\mathrm{e}^{\pi(\alpha-1)}, -\mathrm{e}^{-\pi}, -\mathrm{e}^{-\pi}\right\}) \\
&= \mathbf{F} + \mathbf{F}_2, \quad \mathbf{F}_2 \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\,\pi \\ 0 & -2\,\pi & 0 \end{bmatrix}, \quad \mathrm{e}^{t\mathbf{F}_2} = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{R}(2\pi t) \end{bmatrix},
\end{aligned}$$

so $\mathbf{L}_1(t,0) \triangleq \mathbf{\Phi}(t,0)\mathrm{e}^{-t\mathbf{F}_1} = \mathbf{\Phi}(t,0)\mathrm{e}^{-t\mathbf{F}}\mathrm{e}^{-t\mathbf{F}_2} = \mathbf{L}(t,0)\mathrm{e}^{-t\mathbf{F}_2}$. Then

$$\mathbf{L}_1(t,0) = \begin{bmatrix} \cos(2\pi t) & \sin(2\pi t)\cos(2\pi t) & -\sin^2(2\pi t) \\ -\sin(2\pi t) & \cos^2(2\pi t) & -\sin(2\pi t)\cos(2\pi t) \\ 0 & \sin(2\pi t) & \cos(2\pi t) \end{bmatrix},$$

which again has period $T = 1/2$. We see that $\mathbf{L}_1(t,0)$ and $\mathbf{F}_1$ satisfy (19), and also that $\mathbf{Y}_1 = \mathbf{Y}_1^{-1} = \mathbf{L}_1(T,0)$. However, for t not an integer multiple of T, unlike the Yakubovich and Starzhinskii construction leading to (15) (see also (30) and (35)), we have

$$\mathbf{L}_1(t+T,0) \neq \mathbf{L}_1(t,0)\cdot\mathbf{Y}_1.$$

Here $\mathbf{L}_1(t,0)$ and $\mathbf{F}_1$ provide another real decomposition, with $2T$-periodic $\mathbf{L}_1(t,0)$, of the same state transition matrix $\mathbf{\Phi}(t,0)$ that, in turn, corresponds to the real $\{T = 1/2\}$-periodic system matrix $\mathbf{A}(t)$. However it is clear that $\mathbf{Y}_1\mathbf{F}_1 \neq \mathbf{F}_1\mathbf{Y}_1$, showing that the conditions given in Theorem 2.3 are only sufficient and not necessary.

6 Conclusions

Floquet theory guarantees the existence of (possibly complex) factors for the state transition matrix of a linear T-periodic system. It is common practice to appeal to Corollary 2.2 in order to determine real Floquet factorizations. A major disadvantage of this is that the system is treated as having $2T$-periodic coefficients; hence any more efficient factorization with a real T-periodic (or T-antiperiodic) factor $\mathbf{L}(t,0)$ will generally be lost. Yakubovich proved that a real factorization is always possible using calculations based on only $[0,T]$. This requires the construction of a non-singular matrix $\mathbf{Y}$ such that $\mathbf{Y\Phi}(T,0)$ has a real logarithm, $\mathbf{Y}^2 = \mathbf{I}$, and $\mathbf{YF} = \mathbf{FY}$. In this paper we proved a more general result requiring the construction of a non-singular matrix $\mathbf{Y}$ such that $\mathbf{Y\Phi}(T,0)$ has a real logarithm, but which then need only satisfy $\mathbf{\Phi}^2(T,0) = (\mathbf{\Phi}(T,0)\mathbf{Y})^2$. Yakubovich's result is a special case of this result. Thus, there are useful factorizations besides those given by Theorem 2.3. We have also shown there are no other real factorizations besides those given by Corollary 2.2, and in particular that the factorizations in Theorem 2.3 form a subset of those in Corollary 2.2. These results have direct applications to control engineering, where it is possible to use this knowledge to construct a continuous periodic stabilizing feedback for LTP systems using full-state or observer-based information [15,16]. In particular, the results presented here allow the control engineer to assign the stability of the closed-loop periodic system (via the matrix $\mathbf{F}$), to take advantage of working on the transformed system (11) using the knowledge of the matrix $\mathbf{L}(t,0)$, and to synthesize a controller with a specific periodicity (T, $2T$, $3T$, etc.) by means of assigning the Yakubovich matrix $\mathbf{Y}$. We report on these findings elsewhere.

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