

Perturbation Analyses for the Cholesky Factorization with Backward Rounding Errors

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Abstract. This paper gives perturbation analyses of the Cholesky factorization with the form of perturbations we could expect from the equivalent backward error in A resulting from numerically stable computations. The analyses more accurately reflect the sensitivity of the problem than previous such results. Both numerical results and an analysis show the standard method of symmetric pivoting usually improves the condition of the problem. It follows that the computed R will usually have more accuracy when we use the standard symmetric pivoting strategy.

1 Introduction

Let $A \in \mathcal{R}^{n \times n}$ be a symmetric positive definite matrix. Then A has a unique Cholesky factorization of the form $A = R^T R$, where R is upper triangular with positive diagonal elements and is called the Cholesky factor of A .

Suppose there is a perturbation ΔA in A and $A + \Delta A$ is also symmetric positive definite, then $A + \Delta A$ has a unique Cholesky factorization

$$A + \Delta A = (R + \Delta R)^T (R + \Delta R).$$

The goal of the perturbation analysis is to give a bound on $\|\Delta R\|$ (or $|\Delta R|$) in terms of $\|\Delta A\|$ (or $|\Delta A|$). There have been several papers dealing with the perturbation analysis for the Cholesky factor for a general perturbation ΔA , see [2], [8], and [10] for normwise analyses, and [11, 12] for componentwise analyses. Recently perturbation results of a different flavor were presented in [4]. For ΔA corresponding to the backward rounding error in A resulting from standard numerical stable computations, a nice norm-based perturbation result was presented in [4]. Note [11, 12] also included component perturbation bounds for a different and somewhat complicated form of the backward rounding error in A . The purpose of this paper is to establish new first-order perturbation bounds for ΔA corresponding to the backward rounding error, which are sharper than the corresponding first-order result in [4].

Now we introduce notation. To simplify the presentation, for any $n \times n$ matrix X , we define the upper triangular matrix

$$\text{up}(X) \equiv \begin{bmatrix} \frac{1}{2}x_{11} & x_{12} & \cdots & x_{1n} \\ 0 & \frac{1}{2}x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2}x_{nn} \end{bmatrix}, \quad (1)$$

and the diagonal matrix

$$\text{diag}(X) \equiv \text{diag}(x_{11}, x_{22}, \dots, x_{nn}) \equiv \text{diag}(x_{ii}).$$

For any matrix $C \equiv (c_{ij}) \equiv [c_1, \dots, c_n] \in \mathcal{R}^{n \times n}$, denote by $c_j^{(i)}$ the vector of the leading i elements of c_j . Using ‘u’ to denote ‘upper’, we define

$$\text{uvec}(C) \equiv [c_1^{(1)T}, c_2^{(2)T}, \dots, c_n^{(n)T}]^T, \quad (2)$$

the vector formed by stacking the columns of the upper triangular part of C into one long vector. Note that for any upper triangular X , $\|\text{uvec}(X)\|_2 = \|X\|_F$. To help our norm analysis, for any matrix $C \in \mathcal{R}^{n \times n}$ we define

$$\text{duvec}(C) \equiv \text{diag}(\underbrace{1, \sqrt{2}, 1, \dots}_2, \underbrace{\sqrt{2}, \sqrt{2}, \dots}_n, \sqrt{2}, 1) \text{uvec}(C). \quad (3)$$

Thus for any symmetric matrix X we have $\|\text{duvec}(X)\|_2 = \|X\|_F$. Finally let $\mathcal{D}_n$ be the set of all $n \times n$ real positive-definite diagonal matrices.

At the end of this section we summarize the strongest norm-based results for a general perturbation ΔA presented in [2], as those results will be compared with the new results for ΔA having the form of backward rounding error.

Theorem 1. (see [2]) *Let $A \in \mathcal{R}^{n \times n}$ be symmetric positive definite with the Cholesky factorization $A = R^T R$. Let ΔA be symmetric. If $\epsilon \equiv \|\Delta A\|_F / \|A\|_2$ satisfies $\kappa_2(A)\epsilon < 1$, where $\kappa_2(A) \equiv \|A\|_2 \|A^{-1}\|_2$, then $A + \Delta A$ has the Cholesky factorization*

$$A + \Delta A = (R + \Delta R)^T (R + \Delta R),$$

such that

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leq \kappa_C(A)\epsilon + O(\epsilon^2), \quad (4)$$

$$\frac{1}{2}\kappa_2^{1/2}(A) \leq \kappa_C(A) \leq \kappa'_C(A) \leq \kappa_2(A), \quad (5)$$

where

$$\kappa_C(A) \equiv \|W_R^{-1}\|_2 \|A\|_2^{1/2}, \quad (6)$$

$$\kappa'_C(A) \equiv \inf_{D \in \mathcal{D}_n} \kappa'_C(A, D), \quad \kappa'_C(A, D) \equiv \kappa_2(R)\kappa_2(D^{-1}R), \quad (7)$$

with $W_R \in \mathcal{R}^{\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}}$ defined by

$$W_R = \begin{bmatrix} 2r_{11} & & & & & \\ \sqrt{2}r_{12} & \sqrt{2}r_{11} & & & & \\ & 2r_{12} & 2r_{22} & & & \\ & & & \ddots & & \\ \sqrt{2}r_{1n} & & & & \sqrt{2}r_{11} & \\ & \sqrt{2}r_{1n} & \sqrt{2}r_{2n} & & \sqrt{2}r_{12} & \sqrt{2}r_{22} \\ & & & & & \ddots \\ & & & & 2r_{1n} & 2r_{2n} \cdots 2r_{nn} \end{bmatrix}. \quad (8)$$

If standard symmetric pivoting is used in computing the Cholesky factorization $PAP^T = R^T R$, we have

$$\frac{1}{2}\kappa_2^{1/2}(A) \leq \kappa_c(PAP^T) \leq \kappa'_c(PAP^T) \leq \kappa_2^{1/2}(A)\sqrt{2n(n+1)(4^n + 6n - 1)}/6.$$

In [2] we showed the bound (4) is attainable to first order in ϵ . So $\kappa_c(A)$ is the condition number for the Cholesky factor R . It is expensive to estimate or evaluate $\kappa_c(A)$ directly by using the usual methods. Fortunately we can use $\kappa'_c(A)$ as an approximation of $\kappa_c(A)$. According to the well-known result of van de Sluis [13], if D is chosen to equilibrate the rows of R , i.e., all rows of $D^{-1}R$ have unit length, then $\kappa'_c(A, D)$ with such a D is at most a factor of $\sqrt{n}$ off the infimum $\kappa'_c(A)$. Notice that $\kappa'_c(A, D)$ can be estimated by a standard condition estimator in $O(n^2)$ flops.

2 Main results

In this section we first derive a tight perturbation bound, leading to the condition number $\chi_c(A)$ for perturbation having the form of equivalent backward rounding error. Also we derive a practical estimate $\chi'_c(A)$ of $\chi_c(A)$. Then we compare $\chi_c(A)$ with $\kappa_c(A)$, and $\chi'_c(A)$ with $\kappa'_c(A)$. Finally we show how standard pivoting improves the condition number $\chi_c(A)$.

Before proceeding we introduce the following result presented in [3], see also [6, Theorems 10.5 and 10.7].

Lemma 2. (see [3]) *Let $A \equiv D_c H D_c \in \mathcal{R}^{n \times n}$ be a symmetric positive definite floating point matrix, where $D_c \equiv \text{diag}(A)^{1/2}$. If*

$$n\epsilon \|H^{-1}\|_2 < 1, \quad (9)$$

where $\epsilon \equiv (n+1)u/(1-2(n+1)u)$ with u being the unit round-off, then Cholesky factorization applied to A succeeds (barring underflow and overflow) and produces a nonsingular $\tilde{R}$, which satisfies

$$A + \Delta A = \tilde{R}^T \tilde{R}, \quad |\Delta A| \leq \epsilon d d^T, \quad (10)$$

where $d_i = a_{ii}^{1/2}$.

This lemma is applicable to any standard algorithms for the Cholesky factorization (see, for example, [5] for the standard algorithms). Based on this, we establish the following result.

Theorem 3. *Suppose all of the assumptions in Lemma 2 hold. Let $A = R^T R$ be the Cholesky factorization of A . Set $\hat{R} \equiv R D_c^{-1}$. Then for the perturbation ΔA and result $\tilde{R}$ in (10) we have with $\Delta R \equiv \tilde{R} - R$ that*

$$\frac{\|\Delta R\|_F}{\|R\|_2} \leq \chi_c(A) \epsilon + O(\epsilon^2), \quad (11)$$

$$\chi_c(A) \leq \chi'_c(A) \leq n \|H^{-1}\|_2, \quad (12)$$

$$\chi_c(A) \leq \frac{1}{\sqrt{2}} \|H^{-1}\|_2, \quad (13)$$

where

$$\begin{aligned}\chi_C(A) &\equiv n \|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2 / \|R\|_2, \\ \chi'_C(A) &\equiv \inf_{D \in \mathcal{D}_n} \chi'_C(A, D), \\ \chi'_C(A, D) &\equiv n \|\hat{R}^{-1}\|_2 \|\hat{R}^{-1}D\|_2 \|D^{-1}R\|_2 / \|R\|_2,\end{aligned}$$

with

$$\mathcal{D}_c \equiv \text{diag}(a_{11}^{1/2}, \underbrace{a_{22}^{1/2}, a_{22}^{1/2}}_2, \dots, \underbrace{a_{nn}^{1/2}, a_{nn}^{1/2}}_n, \dots, a_{nn}^{1/2}) \in \mathcal{R}^{\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}}, \quad (14)$$

and $W_{\hat{R}}$ being just W_R in (8) with each entry r_{ij} replaced by $\hat{r}_{ij}$. The bound in (11) is approximately attainable to first-order in ϵ .

Proof. Let $G \equiv \Delta A / \epsilon$. By (10) and (9) it is easy to show $A + tG$ is symmetric positive definite for all $t \in [0, \epsilon]$, and so it has the Cholesky factorization

$$A + tG = R^T(t)R(t), \quad |t| \leq \epsilon, \quad (15)$$

with $R(0) = R$ and $R(\epsilon) = \tilde{R} \equiv R + \Delta R$. Differentiating (15) and setting $t = 0$ in the result gives

$$G = R^T \dot{R}(0) + \dot{R}(0)^T R, \quad (16)$$

which, combined with $R = \hat{R}D_c$, gives

$$\hat{R}^T \dot{R}(0) D_c^{-1} + D_c^{-1} \dot{R}(0) \hat{R} = D_c^{-1} G D_c^{-1}, \quad (17)$$

The upper and lower triangular parts of (17) contain identical information, and it is easy to show that by using (2) and (3) the upper triangular part of (17) can be rewritten as the following matrix-vector equation form

$$W_{\hat{R}} \text{uvec}(\dot{R}(0) D_c^{-1}) = \text{duvec}(D_c^{-1} G D_c^{-1}). \quad (18)$$

Notice from (2) that $\text{uvec}(\dot{R}(0) D_c^{-1}) = \mathcal{D}_c^{-1} \text{uvec}(\dot{R}(0))$ with $\mathcal{D}_c$ as in (14), then from (18) we have

$$\text{uvec}(\dot{R}(0)) = \mathcal{D}_c W_{\hat{R}}^{-1} \text{duvec}(D_c^{-1} G D_c^{-1}), \quad (19)$$

which with $G = \Delta A / \epsilon$ and (10) gives

$$\|\dot{R}(0)\|_F \leq \|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2 \|D_c^{-1} d d^T D_c^{-1}\|_F = n \|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2. \quad (20)$$

The Taylor expansion of $R(t)$ about $t = 0$ gives at $t = \epsilon$

$$R + \Delta R \equiv \tilde{R} = R(\epsilon) = R(0) + \epsilon \dot{R}(0) + O(\epsilon^2), \quad (21)$$

which, combined with (20), gives (11).

Obviously there exists a symmetric matrix $F \in \mathcal{R}^{n \times n}$ such that

$$\|\mathcal{D}_c W_{\hat{R}}^{-1} \text{duvec}(D_c^{-1} F D_c^{-1})\|_2 = \|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2 \|D_c^{-1} F D_c^{-1}\|_F.$$

Then by taking $G = (\min_{f_{ij} \neq 0} d_i d_j / |f_{ij}|)F$, we have $|\Delta A| \leq \epsilon d d^T$ and from (19) that

$$\|\dot{R}(0)\|_F = (\min_{f_{ij} \neq 0} d_i d_j / |f_{ij}|) \|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2 \|D_c^{-1} F D_c^{-1}\|_F \geq \|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2,$$

which shows the first-order bound in (11) is approximately attained for such G .

It remains to prove (12) and (13). From (16) with $R = \hat{R} D_c$, it follows that

$$\dot{R}(0)R^{-1} + R^{-T}\dot{R}^T(0) = \hat{R}^{-T}D_c^{-1}GD_c^{-1}\hat{R}^{-1}.$$

From this we see with notation ‘up’ (see (1)) that

$$\dot{R}(0) = \text{up}(\hat{R}^{-T}D_c^{-1}GD_c^{-1}\hat{R}^{-1})R, \quad (22)$$

so with (19) and the fact that $\text{up}(XD) = \text{up}(X)D$ and $\|\text{up}(X)\|_F \leq \|X\|_F$ for any $X \in \mathcal{R}^{n \times n}$ and $D \in \mathcal{D}_n$ we have

$$\begin{aligned} \|\mathcal{D}_c W_{\hat{R}}^{-1} \text{duvec}(D_c^{-1}GD_c^{-1})\|_2 &= \|\text{up}(\hat{R}^{-T}D_c^{-1}GD_c^{-1}\hat{R}^{-1})R\|_F \\ &\leq \|\hat{R}^{-1}\|_2 \|D_c^{-1}GD_c^{-1}\|_F \|\hat{R}^{-1}D\|_2 \|D^{-1}R\|_2. \end{aligned}$$

Actually this holds for any symmetric $G \in \mathcal{R}^{n \times n}$ since it was essentially obtained from the matrix equation $R^T X + X^T R = G$ with X triangular. Notice $\|\text{duvec}(D_c^{-1}GD_c^{-1})\|_2 = \|D_c^{-1}GD_c^{-1}\|_F$, thus we must have

$$\|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2 \leq \|\hat{R}^{-1}\|_2 \|\hat{R}^{-1}D\|_2 \|D^{-1}R\|_2.$$

Since this is true for any $D \in \mathcal{D}_n$, we see that the first inequality in (12) holds. The second inequality in (12) follows from $\chi'_c(A, I) = n\|\hat{R}^{-1}\|_2^2 = n\|H^{-1}\|_2$.

Similarly we can prove (13) by using the fact that $\|\text{up}(X)\|_F \leq \frac{1}{\sqrt{2}}\|X\|_F$ for any symmetric $X \in \mathcal{R}^{n \times n}$. $\square$

Since the first-order bound (11) is approximately attainable, $\chi_c(A)$ can be regarded as the *condition number* for the Cholesky factorization with the form of perturbation error satisfying (10).

Recently Drmač et al. [4] gave a perturbation analysis with the same form of perturbation error. Their asymptotic perturbation bound can be presented as follows:

$$\frac{\|\Delta R\|_2}{\|R\|_2} \leq (\tfrac{1}{2} + \lceil \log n \rceil)n\|H^{-1}\|_2\epsilon + O(\epsilon^2). \quad (23)$$

This suggests that $\|H^{-1}\|_2$ can be regarded as a conditioning measure. From (12) we see our bound (11) is sharper than (23). In fact it can be much sharper.

As an illustration suppose $A = \begin{bmatrix} 1 & \delta \\ \delta & \delta^2 + \delta^4 \end{bmatrix}$, $R = \begin{bmatrix} 1 & \delta \\ 0 & \delta^2 \end{bmatrix}$. Then it is easy to show for small $\delta > 0$, $\chi_c(A) \leq \chi'_c(A) = O(1/\delta)$, $\|H^{-1}\|_2 = O(1/\delta^2)$. Thus (23) may overestimate the true relative error of the computed Cholesky factor, and the approximation $\chi'_c(A)$ to the condition number $\chi_c(A)$ is a significant improvement to $\|H^{-1}\|_2$. Furthermore it is easy to see $\|H^{-1}\|_2$ is invariant if pivoting is used in computing the Cholesky factorization of A , whereas $\chi_c(A)$

and $\chi'_C(A)$ depend on any pivoting. Thus $\chi_C(A)$ and $\chi'_C(A)$ more closely reflects the true sensitivity of the Cholesky factorization than $\|H^{-1}\|_2$.

As far as we can see, it is expensive to compute or approximate $\chi_C(A)$ directly by the usual approach. Fortunately $\chi'_C(A)$ is quite easy to estimate. By the result of van der Sluis [13], $\|\tilde{R}^{-1}D\|_2 \|D^{-1}R\|_2$ will be nearly minimum when the rows of $D^{-1}R$ are equilibrated. Then a procedure for obtaining an $O(n^2)$ condition estimator for the Cholesky factorization with backward rounding errors is to choose $D = D_r \equiv \text{diag}(\|R(i, :)\|_2)$, and then use a standard norm estimator to estimate all factors in $\chi'_C(A, D)$. Numerical experiments suggest usually $\chi'_C(A)$ is a reasonable approximation of $\chi_C(A)$. But $\chi'_C(A)$ can still be very larger than $\chi_C(A)$, even though it can be much smaller than $n\|H^{-1}\|_2$. For the example above we have $\chi_C(A) = O(1)$.

Numerical tests suggest that usually $\chi_C(A)$ is smaller or much smaller than $\kappa_C(A)$, defined by (6). This is not surprising since (10) provides more information about the perturbation in the data. In [1] we proved the following results.

Theorem 4.

$$\begin{aligned} \frac{1}{n} \chi_C(A) &\leq \kappa_C(A) \leq \frac{\max_i a_{ii}}{\min_i a_{ii}} \chi_C(A), \\ \frac{1}{n} \chi'_C(A) &\leq \kappa'_C(A) \leq \frac{\max_i a_{ii}}{\min_i a_{ii}} \chi'_C(A). \end{aligned} \quad (24)$$

The first inequality in (24) is attainable, since equality will hold by taking $A = cI$ with $c > 0$. The second inequality is at least nearly attainable. In fact taking $A = R^T R$, where $R = \text{diag}(\delta^{n-1}, \delta^{n-2}, \dots, \delta, 1) + e_1 e_n^T$ with small $\delta > 0$, we easily obtain

$$\kappa_2(A) = O\left(\frac{1}{\delta^{2n-2}}\right), \quad \frac{\max_i a_{ii}}{\min_i a_{ii}} \chi_C(A) = \frac{2}{\delta^{2n-2}} O(1) = O\left(\frac{1}{\delta^{2n-2}}\right).$$

This example also suggests that possibly $\kappa_C(A)$ is much larger than $\chi_C(A)$ if the maximum element is much larger than the minimum one on the diagonal of A .

The standard symmetric pivoting strategy can usually improve $\chi_C(A)$, just as it can usually improve $\kappa_C(A)$. In [1] we showed the following theorem.

Theorem 5. *Let $A \in \mathcal{R}^{n \times n}$ be symmetric positive definite with the Cholesky factorization $PAP^T = R^T R$ when the standard pivoting strategy is used. Then*

$$\chi_C(PAP^T) \leq \chi'_C(PAP^T) \leq \|H^{-1}\|_2^{1/2} n \sqrt{2n(n+1)(4^n + 6n - 1)/6}, \quad (25)$$

where $PAP^T \equiv D_c H D_c$ with $D_c \equiv \text{diag}(PAP^T)^{1/2}$.

One may not be impressed by the 4^n factor in the upper bound, and may wonder if it can significantly be improved. In fact we can prove for any n the upper bound can nearly be approximated by a parametrized family of matrices (cf. [1, pp.25–26]). But such examples are rare in practice.

Our numerical experiments confirms that $\chi_C(PAP^T)$ is usually (much) smaller than $\chi_C(A)$. Thus Cholesky factorization with standard symmetric pivoting will usually give more accurate $\tilde{R}$.

By following the approach of [7], it is straightforward, but detailed and lengthy, to extend the first-order results to provide rigorous perturbation bounds. For such results, see [1].

Table 1. Results for Pascal matrices without pivoting.

n	$\chi_C(A)$	$\chi_{C1}(A, D_r)$	$\kappa_C(A)$	$\kappa_{C1}(A, D_r)$	$\frac{\sqrt{2}}{2}n\ H^{-1}\ _2$
1	5.0e-01	1.0e+00	5.0e-01	1.0e+00	7.1e-01
2	2.2e+00	6.0e+00	2.1e+00	6.3e+00	4.8e+00
3	8.9e+00	3.6e+01	9.7e+00	5.0e+01	3.6e+01
4	3.9e+01	2.3e+02	5.5e+01	4.8e+02	3.0e+02
5	1.7e+02	1.4e+03	3.5e+02	4.9e+03	2.5e+03
6	7.8e+02	9.2e+03	2.5e+03	5.2e+04	2.2e+04
7	3.5e+03	5.8e+04	1.9e+04	5.7e+05	2.0e+05
8	1.6e+04	3.7e+05	1.5e+05	6.3e+06	1.7e+06
9	7.7e+04	2.3e+06	1.3e+06	7.0e+07	1.5e+07
10	3.6e+05	1.5e+07	1.1e+07	7.9e+08	1.4e+08
11	1.7e+06	9.1e+07	9.8e+07	9.0e+09	1.2e+09
12	8.4e+06	5.6e+08	8.7e+08	1.0e+11	1.1e+10
13	4.1e+07	3.5e+09	7.8e+09	1.2e+12	9.8e+10
14	2.0e+08	2.2e+10	7.1e+10	1.4e+13	8.8e+11
15	9.7e+08	1.3e+11	6.5e+11	1.6e+14	7.9e+12

Table 2. Results for Pascal matrices with pivoting, $\tilde{A} \equiv PAP^T$.

n	$\chi_C(\tilde{A})$	$\chi_{C1}(\tilde{A}, D_r)$	$\kappa_C(\tilde{A})$	$\kappa_{C1}(\tilde{A}, D_r)$	$\frac{\sqrt{2}}{2}n\ H^{-1}\ _2$
1	5.0e-01	1.0e+00	5.0e-01	1.0e+00	7.1e-01
2	1.6e+00	4.9e+00	1.5e+00	4.2e+00	4.8e+00
3	4.1e+00	1.8e+01	5.1e+00	1.6e+01	3.6e+01
4	1.3e+01	6.1e+01	2.2e+01	8.0e+01	3.0e+02
5	3.6e+01	2.1e+02	8.3e+01	3.3e+02	2.5e+03
6	7.7e+01	6.7e+02	2.5e+02	1.3e+03	2.2e+04
7	1.8e+02	2.2e+03	9.4e+02	5.1e+03	2.0e+05
8	4.8e+02	7.8e+03	4.0e+03	2.4e+04	1.7e+06
9	1.2e+03	2.7e+04	1.6e+04	1.0e+05	1.5e+07
10	3.6e+03	9.0e+04	7.6e+04	4.7e+05	1.4e+08
11	7.5e+03	2.7e+05	2.4e+05	1.8e+06	1.2e+09
12	1.8e+04	9.2e+05	8.3e+05	8.2e+06	1.1e+10
13	3.9e+04	2.9e+06	3.2e+06	3.1e+07	9.8e+10
14	9.4e+04	8.5e+06	1.3e+07	1.2e+08	8.8e+11
15	2.2e+05	2.8e+07	5.4e+07	4.9e+08	7.9e+12

3 Numerical experiments

In section 2 we presented a new first-order perturbation bound for the Cholesky factor with the change caused by backward rounding errors, defined $\chi_C(A) \equiv n\|\mathcal{D}_c W_{\hat{R}}^{-1}\|_2/\|A\|_2^{1/2}$ as the condition number, and suggested $\chi_C(A)$ could be estimated in practice by $\chi'_C(A, D_r) \equiv n\|\hat{R}^{-1}\|_2\|\hat{R}^{-1}D_r\|_2\|D_r^{-1}R\|_2/\|R\|_2$ with $D_r = \text{diag}(\|R(i, :)\|_2)$, which can be estimated by standard norm estimators

in $O(n^2)$ flops. Our new first-order result is (much) better than the previous corresponding result. Also we compare $\chi_c(A)$ with $\kappa_c(A)$, the condition number for general perturbation ΔA , and compare the corresponding estimates $\chi'_c(A)$ with $\kappa'_c(A)$ as well.

Now we give a set of examples to show our findings. The matrices are $n \times n$ Pascal matrices, (with elements $a_{1j} = a_{i1} = 1$, $a_{ij} = a_{i,j-1} + a_{i-1,j}$), $n = 1, 2, \dots, n$. The results are shown in Table 1 without pivoting and in Table 2 with pivoting.

Note in Tables 1 and 2 how $\frac{1}{\sqrt{2}}n\|H^{-1}\|_2$ can be worse than $\chi_c(A)$. In Table 2 pivoting is seen to give a significant improvement to $\chi_c(A)$. Also we observe from both the tables that $\chi'_c(A)$ is a reasonable approximation of $\chi_c(A)$. We see $\chi_c(A)$ is smaller than $\kappa_c(A)$ for $n > 2$.

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