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A Bayesian method for linear, inequality-constrained adjustment and its application to GPS positioning

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Abstract. One of the typical approaches to linear, inequality-constrained adjustment (LICA) is to solve a least-squares (LS) problem subject to the linear inequality constraints. The main disadvantage of this approach is that the statistical properties of the estimate are not easily determined and thus no general conclusions about the superiority of the estimate can be made. A new approach to solving the LICA problem is proposed. The linear inequality constraints are converted into prior information on the parameters with a uniform distribution, and consequently the LICA problem is reformulated into a Bayesian estimation problem. It is shown that the LS estimate of the LICA problem is identical to the Bayesian estimate based on the mode of the posterior distribution. Finally, the Bayesian method is applied to GPS positioning. Results for four field tests show that, when height information is used, the GPS phase ambiguity resolution can be improved significantly and the new approach is feasible.

Key words: Linear, inequality-constrained adjustment (LICA) – Least-squares estimation – Bayesian estimation – Uniform prior distribution – GPS positioning

1 Introduction

In survey data processing, after observations are obtained, we usually establish observation equations and then use a least-squares (LS) method to find the estimate of the unknown parameter vector (see e.g. Mikhail 1976). However, sometimes there is prior information about the parameters that can be used. The prior information may come from: (1) the nature of the problem itself, (2) the requirements that have to be met, or (3) partial knowledge of the parameters (see e.g. Lu et al. 1993). The usual method of using this prior information is to establish some constraints on unknowns. For example, on seas or rivers, some prior information on height may be known, so we can establish a constraint for height and use this constraint to improve GPS positioning (Remondi 1992; Ueno et al. 2000; Zhu and Santerre 2002). If the established constraints are in the form of inequalities (in this paper, we consider only linear inequalities), we have the problem of inequality-constrained adjustment.

A lot of research has been undertaken on the problem of linear, inequality-constrained adjustment (LICA). One of the typical approaches is the LS approach, which solves an LS problem subject to the linear inequality constraints. The main disadvantage of this approach is that the statistical properties of the estimate are not easily determined and no general conclusions about the superiority of the estimate can be made. In this paper, we will propose a new approach to the LICA problem. The idea is to convert the linear inequality constraints into prior information on the parameter vector with a uniform distribution, and then use a Bayesian method to find the solution. The new approach not only allows us to find the mean square error (MSE) matrix of the estimate of the parameter vector, but also gives some insight into the LS approach.

The rest of this paper is organized as follows. In Sect. 2, we introduce the LICA problem and the typical approaches to solving this problem. In particular, we give a detailed analysis of the LS approach. In Sect. 3, we apply the Bayesian estimation theory to the LICA problem. Solutions of the LICA problem based on Bayes's approach are suggested. In Sect. 4, we apply the suggested methods to GPS positioning and show that, when height information is used, the carrier-phase ambiguity resolution can be improved significantly. Finally, some conclusions are drawn in Sect. 5.

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2 The LICA problem and typical approaches

Suppose that the mathematical model underlying the LICA is

$$\mathbf{l} = A\mathbf{x} + \mathbf{e} \tag{1a}$$

$$G\mathbf{x} \geq \mathbf{w} \tag{1b}$$

where $\mathbf{l}$ is the vector of observations, $\mathbf{e}$ is the vector of observational noise with zero mean and covariance matrix $D(\mathbf{e}) = \sigma_0^2 Q_e$, $\mathbf{x}$ is the deterministic vector of unknowns to be estimated, A is the coefficient matrix with full column rank, G is the coefficient matrix of the constraints with full row rank, and $\mathbf{w}$ is a constant vector. A typical approach to estimating $\mathbf{x}$ from Eq. (1) is to solve the following LS problem with inequality constraints (LSI):

$$\min \Phi(\mathbf{x}) \equiv (A\mathbf{x} - \mathbf{l})^T P (A\mathbf{x} - \mathbf{l}) \tag{2a}$$

$$\text{subject to } G\mathbf{x} \geq \mathbf{w} \tag{2b}$$

where $\equiv$ is used to mean 'is defined to be' and the weight matrix $P = Q_e^{-1}$. This LSI problem can be solved by the active set algorithms (see e.g. Björck 1996, Sect. 5.2). Alternatively, it can be transformed to an equivalent linear complementarity problem (LCP), and then a method based on LCP can be applied to obtain the solution (Schaffrin 1981; Björck 1996, Sect. 7.7.1). Notice that this LS approach does not require the assumption of a full distribution for the noise vector $\mathbf{e}$.

In order to understand the LSI problem [Eq. (2)], in the following we transform it to an equivalent problem. Write

$$\hat{\mathbf{x}} = (A^T P A)^{-1} A^T P \mathbf{l}, \quad Q_{\hat{\mathbf{x}}} = (A^T P A)^{-1} \tag{3}$$

$\hat{\mathbf{x}}$ is the solution of the LS problem [Eq. (2a)] without the constraints [Eq. (2b)] and $Q_{\hat{\mathbf{x}}}$ is its cofactor matrix. Then for the objective function $\Phi(\mathbf{x})$ in Eq. (2a), we have

$$\begin{aligned}
\Phi(\mathbf{x}) &= (A\mathbf{x} - \mathbf{l})^T P (A\mathbf{x} - \mathbf{l}) \\
&= (A\mathbf{x} - A\hat{\mathbf{x}} + A\hat{\mathbf{x}} - \mathbf{l})^T P (A\mathbf{x} - A\hat{\mathbf{x}} + A\hat{\mathbf{x}} - \mathbf{l}) \\
&= (A\hat{\mathbf{x}} - \mathbf{l})^T P (A\hat{\mathbf{x}} - \mathbf{l}) + (A\mathbf{x} - A\hat{\mathbf{x}})^T P (A\mathbf{x} - A\hat{\mathbf{x}}) \\
&= (A\hat{\mathbf{x}} - \mathbf{l})^T P (A\hat{\mathbf{x}} - \mathbf{l}) + (\mathbf{x} - \hat{\mathbf{x}})^T Q_{\hat{\mathbf{x}}}^{-1} (\mathbf{x} - \hat{\mathbf{x}})
\end{aligned} \tag{4}$$

Notice that the first term in Eq. (4) is a constant, so the LSI problem [Eq.(2)] is equivalent to

$$\min \Phi_{\hat{x}}(\mathbf{x}) \equiv (\mathbf{x} - \hat{\mathbf{x}})^T Q_{\hat{\mathbf{x}}}^{-1} (\mathbf{x} - \hat{\mathbf{x}}) \tag{5a}$$

$$\text{subject to } G\mathbf{x} \geq \mathbf{w} \tag{5b}$$

This is called the least-distance problem, and Lu et al. (1993) took its solution as an estimate of $\mathbf{x}$ in Eq. (1). Now we use a simple two-dimensional (2-D) case as an example to illustrate the solution. Assume $\mathbf{x} = (x, y)^{\mathrm{T}}$, where x and y satisfy the following constraints:

$$w_{x2} \geq x \geq w_{x1} \tag{6a}$$

$$w_{y2} \geq y \geq w_{y1} \tag{6b}$$

Let $\tilde{\mathbf{x}}$ be the solution of the LSI problem [Eq.(2) or (5)]. The following two situations can occur: First, the LS estimate $\hat{\mathbf{x}}$ [see Eq. (3)] satisfies all constraints, i.e. geometrically it lies in the region defined by the inequality constraints (Fig. 1a). In this situation, $\tilde{\mathbf{x}} = \hat{\mathbf{x}}$. Second, the LS estimate $\hat{\mathbf{x}}$ does not satisfy all constraints. Geometrically it lies outside the region defined by the constraints (Fig. 1b and c). In Fig. 1b, $\hat{\mathbf{x}}$ satisfies the constraint in Eq. (6b) and $w_{x2} \geq x$, but does not satisfy $x \geq w_{x1}$. Note that the contours of the function $\Phi_{\hat{x}}(\mathbf{x})$ are ellipses with $\hat{\mathbf{x}}$ as their center and their shape defined by $Q_{\hat{\mathbf{x}}}$. Therefore, in this situation, the LSI solution $\tilde{\mathbf{x}}$ should be the *tangent* point of the ellipse for which $x = w_{x1}$ is its tangent line (Fig. 2). In Fig. 1c, $\hat{\mathbf{x}}$ satisfies $w_{x2} \geq x, w_{y1} \leq y$, but does not satisfy $x \geq w_{x1}$ and $y \leq w_{y2}$. Since $\hat{\mathbf{x}}$ is the center of those ellipses, it is easy to observe that the LSI solution $\tilde{\mathbf{x}}$ is (w_{x1}, w_{y2}), the intersection point of lines $x = w_{x1}$ and $y = w_{y2}$ (Fig. 3).

Although for this 2-D case it is easy to understand and find the solution of the LSI problem, numerically it is usually quite involved to find a solution for a general case. Since it is difficult to find an explicit expression for the solution for a general case, the statistical properties of the estimate are not easily determined and no general conclusions about the superiority of the estimate can be made (Rao and Toutenburg 1999, p. 72).

There is another approach to estimating $\mathbf{x}$ in the LICA model [Eq. (1)]. When the general linear inequality constraints in Eq. (1a) become interval constraints for the components of $\mathbf{x}$, they can be transformed into an ellipsoidal constraint (the ellipsoid encloses the cuboid), and then an estimate of $\mathbf{x}$ can be found by solving a

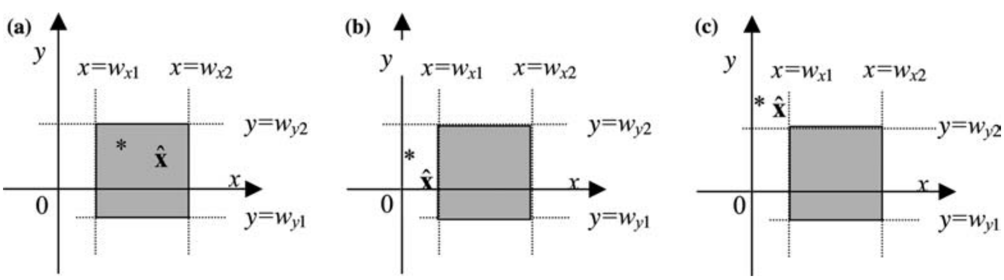

Fig. 1. (**a**) $\hat{\mathbf{x}}$ satisfies both Eqs. (6a) and (6b); (**b**) $\hat{\mathbf{x}}$ satisfies only Eq. (6b); (**c**) $\hat{\mathbf{x}}$ satisfies neither Eq. (6a) nor Eq. (6b)

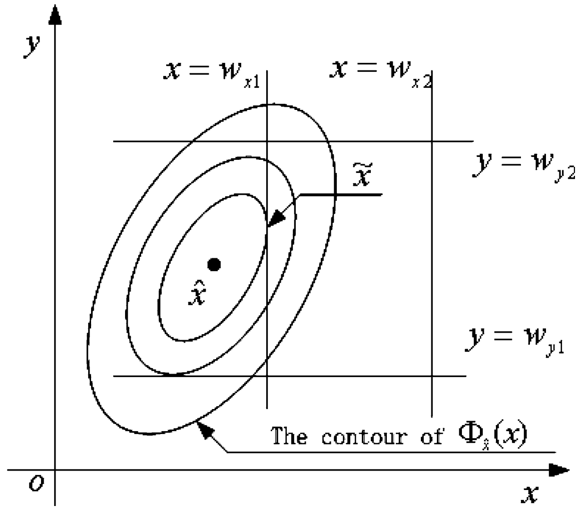

Fig. 2. Relationship between contours of the function $\Phi_{\hat{x}}$ (x) and inequality constraints $x > w_{x1}$

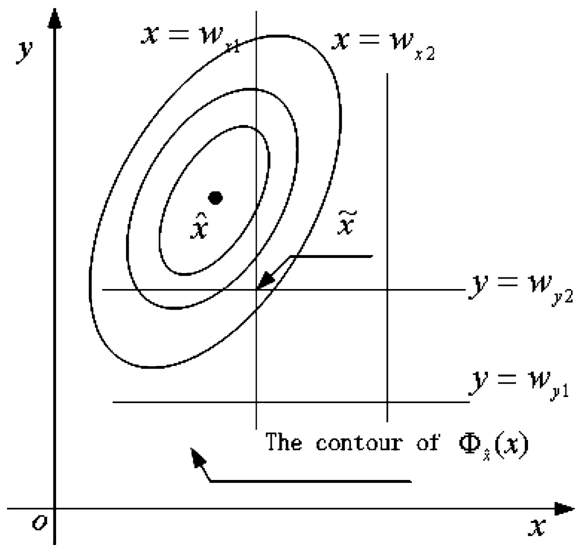

Fig. 3. Relationship between contours of the function $\Phi_{\hat{x}}(x)$ and inequality constraints $x > w_{x1}$ and y w_{y2}

minimax problem subject to this ellipsoidal constraint (Rao and Toutenburg 1999, Sect. 3.13). The advantage of this approach is that an explicit expression for the solution can be obtained and some statistical properties of the estimate can be determined. However, the estimate is usually biased and may not satisfy the linear inequality constraints. Also, it appears difficult to transform general linear inequality constraints into an ellipsoidal constraint, and we are not aware of any effective algorithms for such transformations.

There are other important research works concerning Eq. (1). Riesmeier (1984) and Koch and Riesmeier (1985) studied the posterior distribution of the LS estimate $\hat{\mathbf{x}}$ [Eq. (3)] for inequality hypothesis tests for Eq. (1). Their work is relevant to (but different from) ours to be given in Sect. 3, in which we will give the posterior distribution of $\mathbf{x}$, and use it to find the estimates of $\mathbf{x}$.

3 The Bayesian solutions of LICA

Bayesian statistics is widely used in geodesy (see e.g. Koch 1990; Betti et al. 1993; Xu and Rummel 1994b; Zhu 1995, 1997; Zhu and Wang 2000; Zhu et al. 2001). The advantage of the Bayes estimation is that it can make use of not only the information contained in the observations, but also information from historical data or other sources. The most crucial requirement for applying the Bayesian estimation techniques is that the prior information should be properly described (see e.g. Xu 1991; Zhu and Wang 2000). Nevertheless, the Bayesian approach may not be suitable in some cases, for example in designing higher-order non-linear filters (see e.g. Xu 1999, 2003). If prior information is not objective, different interpretations and practical consequences are expected (Xu 1992; Xu and Rummel 1994a, b).

The inequality constraints in Eq. (1) confine the value of the parameter vector $\mathbf{x}$, so this is a very special kind of prior information. According to maximum entropy, this prior information can be described by the uniform distribution (see e.g. Koch 1990, p. 17)

$$f(\mathbf{x}) = \begin{cases} \frac{1}{s} & \text{if } G\mathbf{x} \geq \mathbf{w} \\ 0 & \text{else} \end{cases} \tag{7}$$

where $\mathbf{x}$ is now assumed to be a stochastic vector (note that the parameter vector $\mathbf{x}$ in Sec. 1 is assumed to be deterministic) and s is the volume of the region defined by $G\mathbf{x} \geq \mathbf{w}$ (for simplicity, we assume that s is finite here, although our results to be given later are still valid even if s is infinite). We then reformulate Eq. (1) into the following model:

$$\begin{aligned} &\mathbf{l} = A\mathbf{x} + \mathbf{e}, \quad \mathbf{e} \sim N(0, \sigma_0^2 Q_e) \\ &\mathbf{x} \sim f(\mathbf{x}), \quad f(\mathbf{x}) \text{ is defined by Eq. (7)} \end{aligned} \tag{8}$$

Note that the inequality constraints in Eq. (1) have been converted into a prior distribution in Eq. (8) and the assumption of a normal distribution for $\mathbf{e}$ has been added, which is not required for the LS approach [Eq. (5)]. Based on Eq. (8), we will present the Bayesian approach to estimating $\mathbf{x}$.

From Bayesian statistics, the posterior distribution of $\mathbf{x}$ satisfies (see e.g. Koch 1990, pp. 4–5) $f(\mathbf{x}|\mathbf{l}) \propto f(\mathbf{l}|\mathbf{x})f(\mathbf{x})$.

It then follows from $f(\mathbf{l}|\mathbf{x}) = f(\mathbf{e})$ and Eqs. (7) and (8) that

$$\begin{aligned} f(\mathbf{x}|\mathbf{l}) &\propto \begin{cases} \exp\left\{-\frac{1}{2\sigma_0^2}(\mathbf{l}-A\mathbf{x})^T Q_e^{-1}(\mathbf{l}-A\mathbf{x})\right\}\frac{1}{s} & \text{if } G\mathbf{x} \geq \mathbf{w} \\ 0 & \text{else} \end{cases} \\ &\propto \begin{cases} \exp\{-\frac{1}{2\sigma_0^2}(\mathbf{l}-A\mathbf{x})^T Q_e^{-1}(\mathbf{l}-A\mathbf{x})\} & \text{if } G\mathbf{x} \geq \mathbf{w} \\ 0 & \text{else} \end{cases} \end{aligned} \tag{9}$$

However, from Eq. (4) we have

$$\begin{aligned} &\exp\{-(\mathbf{l} - A\mathbf{x})^T Q_e^{-1}(\mathbf{l} - A\mathbf{x})/2\sigma_0^2\} \\ &\quad \propto \exp\{-(\hat{\mathbf{x}} - \mathbf{x})^T Q_{\hat{\mathbf{x}}}^{-1}(\hat{\mathbf{x}} - \mathbf{x})/2\sigma_0^2\} \end{aligned}$$

thus Eq. (9) can be rewritten as

$$f(\mathbf{x}|\mathbf{l}) \propto \begin{cases} \exp\{-(\hat{\mathbf{x}}-\mathbf{x})^T Q_{\hat{\mathbf{x}}}^{-1}(\hat{\mathbf{x}}-\mathbf{x})/2\sigma_0^2\} & \text{if } G\mathbf{x} \geq w \\ 0 & \text{else} \end{cases} \tag{10}$$

Note that for the model without constraints (here we assume $\mathbf{x}$ is deterministic)

$$\mathbf{l} = A\mathbf{x} + \mathbf{e}, \quad \mathbf{e} \sim N(0, \sigma_0^2 Q_e)$$

the LS estimate is $\hat{\mathbf{x}} = (A^T PA)^{-1} A^T P\mathbf{l}$ [see Eq. (3)]. For the normally distributed observations $\mathbf{l}$, its linear function $\hat{\mathbf{x}}$ should be normally distributed too, and

$$f(\hat{\mathbf{x}}) \propto \exp\{-(\hat{\mathbf{x}}-\mathbf{x})^T Q_{\mathbf{x}}^{-1}(\hat{\mathbf{x}}-\mathbf{x})/2\sigma_0^2\} \tag{11}$$

Therefore, the difference between the posterior distributions of $\mathbf{x}$ from the model with inequality constraints and those from the model without constraints is the domain of the parameter vector $\mathbf{x}$.

The Bayesian theory allows at least four different estimates of $\mathbf{x}$ to be determined based on the posterior distribution, i.e. the mean, the median, the mode, and the point that minimizes the loss function (see e.g. Carlin and Louis 1996). For symmetric posterior density functions, the mean and the median are identical. For a symmetric, unimodal posterior distribution, the mean, the median, and the mode all coincide. For convenience, we will study only two estimates: the one based on the mean and the one based on the mode.

The Bayesian estimate based on the mode is actually the point at which the maximal value of the density is reached. In Fig. 4 (a 1-D case), the solid lines correspond to the density function $f(\mathbf{x}|\mathbf{l})$ [Eq. (10)], and the complete lines (including the dashed part and the solid part) correspond to $f(\hat{\mathbf{x}})$ [Eq. (11)]. In Fig. 4a, the maximum value of $f(\mathbf{x}|\mathbf{l})$ is reached at point $\hat{x}$, so $\hat{x}$ will be taken as the Bayesian estimate of x. In Fig. 4b, the maximum value of $f(\mathbf{x}|\mathbf{l})$ is reached at point $x = w_1$, so w_1 is the Bayesian estimate, and for the same reason in Fig. 4c, w_2 is the Bayesian estimate. Thus, for a general case, the Bayesian estimate based on the mode is the optimal solution of

$$\begin{aligned} &\max f(\mathbf{x}|\mathbf{l}) \equiv \exp\{-(\hat{\mathbf{x}}-\mathbf{x})^T Q_{\hat{\mathbf{x}}}^{-1}(\hat{\mathbf{x}}-\mathbf{x})/2\sigma_0^2\} \\ &\text{subject to} \quad G\mathbf{x} \geq \mathbf{w} \end{aligned} \tag{12}$$

Let $\tilde{\mathbf{x}}$ denote the solution of Eq. (12). The MSE matrix of the Bayesian estimate can be expressed as

$$\mathrm{MSE}(\tilde{\mathbf{x}}) = \int (\mathbf{x}-\tilde{\mathbf{x}})(\mathbf{x}-\tilde{\mathbf{x}})^T \cdot f(\mathbf{x}|\mathbf{l})\mathrm{d}\mathbf{x} \tag{13}$$

Comparing Eqs. (12) and (5), we observe that the Bayesian estimate based on the mode is identical to the solution of the LSI problem. The case shown in Fig. 4a actually belongs to the situation shown in Fig. 1a, and the cases shown in Fig. 4b and c belong to the situations shown in Fig. 1b and c, respectively. The difference between this Bayesian approach, based on the mode, and the LS approach, is that the MSE matrix of the Bayesian estimate can be obtained by using Eq. (13), although it may not be easy to compute it for high-dimensional cases.

The other method is to take the posterior mean as a Bayesian estimate. The mean can be expressed as

$$\bar{\mathbf{x}} = \int \mathbf{x} \cdot f(\mathbf{x}|\mathbf{l})\mathrm{d}\mathbf{x} \tag{14}$$

Its MSE matrix is

$$\mathrm{MSE}(\bar{\mathbf{x}}) = D(\tilde{\mathbf{x}}) = \int (\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T f(\mathbf{x}|\mathbf{l})\mathrm{d}\mathbf{x} \tag{15}$$

The integration in Eq. (14) can be found analytically for a low-dimensional $\mathbf{x}$ or numerically for a high-dimensional $\mathbf{x}$, but the integration in Eq. (15) has to be found by a numerical method.

4 Application of the Bayesian solutions of LICA to GPS positioning

The Laurentian Region of the Canadian Hydrographic Service (CHS) and the Canadian Coast Guard (CCG) conduct bathymetric surveys on the St. Lawrence River every year to check if its 300-km channel maintains the required nominal depth for navigation. Usually, this requires about 60–70 personnel to work together, and is costly and limited (Marceau et al. 1996). In order to improve the bathymetric surveys, the CHS implemented digital tide gauges and developed a network of 15 tide-gauge stations along the St. Lawrence River in 1991, and the CCG started to establish a GPS reference station network in 1996. The network of tide gauges is called the Coastal and Oceanic Water Level Information System (COWLIS) and can provide regular tidal readings in real time (CHS 1997). The CCG–GPS network can provide GPS carrier-phase observations to a bathymetric survey ship in St. Lawrence River in real time. These two systems can make the bathymetric surveying much more efficient by eliminating costly deployment (support and maintenance) of personnel, and in particular the GPS technique can automatically and accurately determine the water level where the bathymetric survey ship is working. In this application, the GPS carrier-phase

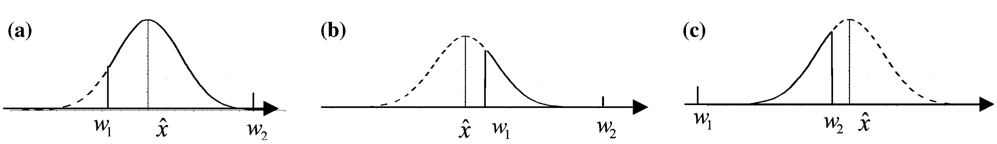

Fig. 4. The posterior distribution of x

ambiguities have to be resolved. For these St. Lawrence River bathymetric surveys, the distances between the closest GPS reference stations can reach 50 km. Due to the (reasonably) long baselines, finding a reliable integer ambiguity solution is a challenging problem for these bathymetric surveys.

Because COWLIS can provide the tide-gauge readings in real time, we can also use the information from the tide gauges to interpolate the height of the ship in real time. The height information [once reduced from the Chart Datum to geodetic (ellipsoidal) height] can then be used for GPS positioning. For any GPS epoch, if we use h_0 to denote the height from COWLIS, the inequality constraint can be expressed as

$$|h - h_0| \leq w \tag{16}$$

where h is the element in the vector $\mathbf{x}$ corresponding to the height. With this inequality constraint, the GPS positioning model becomes a LICA problem underlain by the model

$$\begin{aligned} \mathbf{l} &= A\mathbf{x} + \mathbf{x}_N + \mathbf{e} \\ h - h_0 &\geq -w \\ h - h_0 &\leq w \end{aligned} \tag{17}$$

where $\mathbf{x}_N$ is the integer ambiguity vector, $\mathbf{x}$ denotes the vector of position parameters, and w is a threshold depending on specific applications. For the determination of w, see Ueno et al. (2000).

In our project, the LS ambiguity search technique developed by Hatch (1990) is used. In this 'on-the-fly' technique, we must compute the position vector $\bar{\mathbf{x}}$ and residual vector $\bar{\mathbf{e}}$ for every possible solution in the search space. We can easily obtain $\bar{\mathbf{x}}$ by Eq. (14) and then $\bar{\mathbf{e}}$, and perform the tests for searching the ambiguity solution. If this solution is correct, it means that any ambiguity has been resolved successfully. In our tests, we first perform the ambiguity resolution by using all the test data (more than 7000 epochs). The results of the resolution are then used as correct ambiguities. In order to study the suggested method, we use only five epochs to fix the ambiguity vector (i.e. the initialization period is five epochs), so every five epochs we can obtain a solution for the ambiguity vector. Comparing these solutions with the correct one (determined from all of the test data), we can obtain the success rate, which is used to evaluate this method of ambiguity resolution.

Four field tests (1015, 1017, D11, and D14) are studied in this research. Tests 1015 and 1017 were conducted on 15 and 17 October, 1999, in the region of Quebec City. The distance from the hydrographic survey vessel to the COWLIS station in the port of Quebec is about 5 km; the distance to the GPS reference station at Lauzon is about 7 km. Tests D11 and D14 were conducted in October 1998 at Lake St. Pierre. The distance from the hydrographic survey vessel in test D11 to the GPS reference station at Trois-Rivières is about 45 km; the distance to the COWLIS gauge station in the lake is about 8 km. In test D14, the distance from the hydrographic survey vessel to the GPS reference station at Trois-Rivières is about 35 km; the distance to the COWLIS gauge station in the lake is about 5 km. All data sets contain dual-frequency phase and P(Y) code observations, and the observation interval is 1 sec.

For comparison, four methods have been used in our research. The first method fixes ambiguities without using any prior information (the last column in Table 1); the second fixes ambiguities with the mode-based Bayesian method (LSI method); the third fixes ambiguities by viewing height information as quasi-observations (Zhu and Santerre 2002); and the fourth is the mean-based Bayesian method presented in Sect. 3. The success rates of the GPS L1 ambiguity resolution are shown in Table 1.

Before making comments on Table 1, we remark on the third method, which views the height information as quasi-observations. This method can actually be considered as another variant of the Bayesian method. If the quasi-observation is assumed to be normally distributed, according to the Bayesian theory the posterior distribution will also be normally distributed (see e.g. Koch 1990, pp. 6–7). The solution based on the mode is actually the one with posterior maximum likelihood. For a normal distribution, the solutions based on the mode and on the mean are the same (note that we do not have inequality constraints there). This means that if the prior distribution is a normal distribution, the solution based on the mean is actually the one with posterior maximum likelihood. From mathematical statistics, we know that for a linear model with a normally distributed noise vector, the LS estimate and the maximum likelihood estimate are the same. Therefore, the estimate by the method that views the height information as a quasi-observation is actually the same as that obtained by the mean-based Bayesian method with a prior normal distribution.

Table 1. Success rates of GPS L1 ambiguity resolution for a 5-sec. initialization period. Values in %

Cases	w = 10 cm	20 cm	30 cm	40 cm	50 cm	60 cm	Quasi-observation (σ_h = 20 cm)	LS solution of LICA (w = 20 cm)	No prior height
1015	67	93	93	91	90	90	95	90	69
1017	95	95	95	95	95	95	96	92	61
D14	86	91	90	90	90	90	92	86	63
D11	65	84	80	71	64	62	77	71	50

In Table 1, the results given in the columns ‘w = 10 cm’, … , ‘60 cm’ were obtained by the mean-based Bayesian method, and ‘10 cm’, … , ‘60 cm’ are the values of w in Eq. (17). The results given in the column ‘Quasi-observation’ correspond to the method when viewing the prior height information as a quasi-observation. Zhu and Santerre (2002) suggested that (20 cm)^2 may be taken as the variance of quasi-observations for the bathymetric survey in the St. Lawrence River, so Table 1 shows only the results with this variance. Note that the variance of quasi-observations can be determined by a robust estimation method (Yang 1991). Results in the column ‘LS solution of LICA’ were obtained by the LS mode-based methods (Lu et al. 1993). For height constraints in the bathymetric survey in the St. Lawrence River, Ueno et al. (2000) suggested that $w = 20$ cm should be used as a threshold of height validation, so only the results for this threshold are given in Table 1 for the mode-based LICA solution. From Table 1, we can make the following comments.

1. When the mean-based Bayesian method is used, the best threshold value for the height is 20–30 cm. If 40 cm is used as a threshold, good results can still be obtained, but if 10 cm is used, the result becomes worse. This means that the threshold value of 10 cm is too small, and in many situations the region determined by this threshold value does not contain the real value of the ellipsoidal height in the bathymetric survey in the St. Lawrence River.
2. Comparing the column ‘20 cm’ and column ‘Quasi-observation’ in Table 1, we observe that the results obtained by the Bayesian method are close to those obtained by the method viewing the height information as quasi-observation. This means that, for the bathymetric survey in the St. Lawrence River, either method can be used.
3. When the prior height information is used, results can be improved significantly regarding ambiguity fixing.

5 Conclusions

1. The LICA problem can be solved based on the Bayesian principle. The LS solution of the LICA is proved to be identical to the Bayesian solution based on the mode.
2. The Bayesian solution of LICA allows us to evaluate the statistical properties of the solutions, i.e. to find the MSE matrix of the estimate of the parameter vector.
3. The Bayesian solution of LICA can be applied to GPS positioning. When there is prior height information available to be used, the GPS ambiguity resolution can be improved significantly.

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