On the Success Probability of the $L_0$-regularized Box-constrained Babai Point

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Abstract—We consider the success probability of the $L_0$-regularized box-constrained Babai point, which is a suboptimal solution to the $L_0$-regularized box-constrained integer least squares problem and can be used for MIMO detection. First, we derive formulas for the success probability of both $L_0$-regularized and unregularized box-constrained Babai points. Then we investigate the properties of the $L_0$-regularized box-constrained Babai point, including the optimality of the regularization parameter and the monotonicity of the ratio of the two success probabilities. Finally a bound on the success probability of the $L_0$-regularized Babai point is derived.

I. INTRODUCTION

In many applications, the unknown parameter vector $x^*$ and the observation vector $y$ satisfy the following linear relation:

$$y = Ax^* + v, \quad v \sim N(0, \sigma^2I),$$

(1)

where $A \in \mathbb{R}^{m \times n}$ is a model matrix and $v$ is an $m$ dimensional noise vector. In this paper, we consider the case that $x^*$ is random, and given integer $M \geq 1$, the entries of $x^*$ are independently distributed over the set

$$\mathcal{X} = \{0\} \cup \{ \pm 1, \pm 3, \ldots, \pm (2M - 1) \},$$

(2)

and the elements in $\mathcal{X}$ are chosen independently with

$$\Pr(x_k^* = i) = \left\{ \begin{array}{ll} p/(2M), & i \in \mathcal{X}, i \neq 0 \\ 1 - p, & i = 0, \end{array} \right.$$ \hspace{1cm}

(3)

where $p$ is a positive constant satisfying

$$p/(2M) \leq 1 - p.$$ \hspace{1cm}

(4)

If $p$ is small, $x^*$ is sparse. This model has applications in many applications, including multiple user detection in CDMA communications \cite{1}, in which $\mathcal{X}$ is a 2M-ary constellation, and when $x_k^*$ takes 0, the $k$-th user is said to be inactive.

To detect $x^*$ from the observation $y$, one can apply the maximum a posteriori (MAP) estimation method, which leads to the following regularized optimization problem \cite{1}:

$$\min_{x \in \mathcal{X}^n} \frac{1}{2} ||y - Ax||_2^2 + \lambda ||x||_0,$$

(5)

where the $L_0$-"norm" $||x||_0$ is the number of nonzero elements of $x$, and the regularization parameter $\lambda = \lambda^* > 0$ with

$$\lambda^* := \sigma^2 \ln \frac{1 - p}{p/(2M)} \geq 0.$$ \hspace{1cm}

(6)

In \cite{1}, the inequality \cite{4} is strict, then $\lambda^* > 0$. We allow \cite{4} to be an equality so that $\lambda^*$ can be zero. This gives us more flexibility. To make some of our results more applicable, in this paper we assume that the $\lambda$ in \cite{5} is only a given nonnegative parameter, unless we state explicitly that it is defined by \cite{6}.

This unifies some results for $\lambda = \lambda^*$ and $\lambda = 0$. We refer to \cite{5} as the $L_0$-"norm" regularized "box" constrained integer least squares ($L_0$-RBILS) problem.

If no a priori information about $x^*$ is available, one applies the maximum likelihood (ML) estimation method, leading to the box constrained integer least squares (BILS) problem:

$$\min_{x \in \mathbb{Z}^n} \frac{1}{2} ||y - Ax||_2^2.$$ \hspace{1cm}

(7)

We can regard \cite{7} as a special case of \cite{5}. If we replace the constraint box $\mathcal{X}$ by $\mathbb{Z}^n$, \cite{7} becomes the ordinary integer least squares (OILS) problem

$$\min_{x \in \mathbb{Z}^n} \frac{1}{2} ||y - Ax||_2^2.$$ \hspace{1cm}

(8)

Since an ILS problem is NP-hard \cite{2}, \cite{3}, often a suboptimal solution instead of the optimal one is found in some applications. One often used one is the Babai point \cite{4}, which was originally introduced for the OILS problem \cite{8} but has been extended to the BILS problem \cite{5}, \cite{6} and to the $L_0$-RBILS problem \cite{1}. It is the first integer point found by the Schnorr-Euchner type search methods when the radius of the initial search ellipsoid is set to $\infty$ \cite{7}. To see how good the Babai point is as a detector, one can use success probability (SP) as a measure, which is the probability that the detector is equal to the true parameter vector. If the SP of the Babai point is high, one will not need to try to find the optimal detector. A formula of the SP is useful not only in theoretical analysis, but also in practice, such as the design of a good permutation strategy, which can be used to enhance the SP of the Babai point \cite{8}.

In this paper we study the SP of the $L_0$-regularized Babai point $x^{RB}$ for the $L_0$-RBILS problem \cite{5}. A concise formula for the SP of the $L_0$-regularized Babai point $x^{RB}$ is established, some important properties of the SP are presented and a bound on the SP is derived.

Here we would like to point out that the discrete sparse signal detection or recovery has recently attracted much attention \cite{9}, and various methods have been proposed, see, e.g., \cite{1}, \cite{10}–\cite{24}. Comparisons of the $L_0$-RBILS detector with other discrete sparse signal detectors are beyond the scope of this paper. However, it should be noted that unlike most
of the existing detectors, the $L_0$-RBILS detector can be very efficiently computed and some solid theory can be developed.

II. BABAI POINTS

In this section, we first transform the $L_0$-RBILS problem (5) to an equivalent one by the QR factorization of $A$ and define the corresponding Babai point. Two Babai points corresponding to two special cases are also introduced.

Let the QR factorization of $A$ be

$$A = [Q_1, Q_2] \begin{bmatrix} R \end{bmatrix} = Q_1R,$$

where $[Q_1, Q_2] \in \mathbb{R}^{m \times m}$ is orthogonal and $R \in \mathbb{R}^{n \times n}$ is upper triangular with positive diagonal entries. With $\hat{y} := Q_1^T y$ and $\tilde{v} := Q_1^T v$, the model (1) can be transformed to

$$\hat{y} = Rx^* + \tilde{v}, \quad \tilde{v} \sim N(0, \sigma^2 I),$$

and the $L_0$-RBILS problem (1) is equivalent to

$$\min_{x \in X} \frac{1}{2} \|\hat{y} - Rx\|^2_2 + \lambda \|x\|_0.$$

Here we define the $L_0$-regularized Babai point $x^{RB}_n = [x_1^{RB}, \ldots, x_n^{RB}]^T$ corresponding to (11). Rewrite (11) as

$$\min_{x \in X} \sum_{k=1}^{n} \left( \frac{1}{2} \left( \hat{y}_k - r_{kk} x_k - \sum_{j=k+1}^{n} r_{kj} x_j \right)^2 + \lambda \|x_k\|_0 \right),$$

where $\|x_k\|$ is either 0 or 1. Suppose that $x_1^{RB}, \ldots, x_{k-1}^{RB}$ have been defined and we would like to define $x_k^{RB}$ as an estimator of $x_k$. The idea is to choose $x_k^{RB}$ to be the solution of the optimization problem:

$$\min_{x \in X} \left\{ f_k(x_k) := \frac{1}{2} \left( \hat{y}_k - r_{kk} x_k - \sum_{j=k+1}^{n} r_{kj} x_j^{RB} \right)^2 + \lambda \|x_k\|_0 \right\},$$

To simplify the objective function in (12), write

$$c_k := \left( \hat{y}_k - \sum_{j=k+1}^{n} r_{kj} x_j^{RB} \right) / r_{kk}.$$

Then we have

$$f_k(x_k) = \frac{1}{2} r_{kk}^2 (x_k - c_k)^2 + \lambda \|x_k\|_0.$$

Obviously, the solution to (12) is either 0 or $\lfloor c_k \rfloor x$, which denotes the integer in $X$ nearest to $c_k$. To see which one is the solution, we define

$$g_k := \frac{1}{2} r_{kk}^2 \lfloor c_k \rfloor^2 x - r_{kk} \lfloor c_k \rfloor x c_k + \lambda.$$

Observe that $g_k = f_k(\lfloor c_k \rfloor x) - f_k(0)$ when $\lfloor c_k \rfloor x \neq 0$, and $g_k \geq 0$ when $\lfloor c_k \rfloor x = 0$. Thus, the solution to (12) is

$$x_k^{RB} = \begin{cases} \lfloor c_k \rfloor x, & \text{if } g_k \geq 0, \\ 0, & \text{if } g_k < 0. \end{cases}$$

In (1) a detector equivalent to $x_k^{RB}$ is defined and it is referred to as the decision-directed detector. If we set $\lambda = 0$ in (5), $x_k^{RB}$ just becomes the (unregularized) box-constrained Babai point $x_k^{RB}$ of the BILS problem (7):

$$x_k^{RB} = \left[ \left( \hat{y}_k - \sum_{j=k+1}^{n} r_{kj} x_j^{RB} \right) / r_{kk} \right] x,$$

which is an extension of the ordinary Babai point for the OILS problem (3):

$$x_k^{OB} = \left[ \left( \hat{y}_k - \sum_{j=k+1}^{n} r_{kj} x_j^{OB} \right) / r_{kk} \right] x,$$

III. SP OF THE $L_0$-REGULARIZED BABAI POINT $x_k^{RB}$

One measure of the goodness of $x_k^{RB}$ is the SP $Pr(x_k^{RB} = x^*)$. In this section, we derive a formula for it. In our probability analysis, we need to use the well-known error function:

$$\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-t^2) dt.$$

From (10), we have

$$\hat{y}_k = r_{kk} x_k^* + \sum_{j=k+1}^{n} r_{kj} x_j^* + \tilde{v}_k, \quad \tilde{v}_k \sim N(0, \sigma^2).$$

Substituting it into (13), we obtain

$$c_k = x_k^* + \sum_{j=k+1}^{n} r_{kj} (x_j^* - x_j^{RB}) + \tilde{v}_k / r_{kk}.$$

If $x_j^* = x_j^*$ for $j = k+1, k+2, \ldots, n$, then

$$c_k - x_k^* \sim N(0, \sigma^2 / \sigma^2_{kk}).$$

The key to finding $Pr(x_k^{RB} = x^*)$ is to find the range of $c_k - x_k^*$ for which $x_k^{RB} = x_k^*$ for $k = n, n-1, \ldots, 1$. When $x_k^{RB} = x_k^*$, $c_k - x_k^* = c_k - x_k^{RB}$. Thus we will find the range of $c_k - x_k^*$ for which $x_k^{RB} = x_k^*$. Since the elements of the constraint set $X$ are symmetric with respect to 0 and the distribution of $x_k^*$ is symmetric with respect to 0, we only focus on the nonnegative case that $x_k^* \geq 0$ and $c_k \geq 0$ in our following analysis, and we can simply double the relevant probabilities when we also take the nonpositive case into account.

If $|c_k| x = 0$, i.e., $0 \leq c_k \leq 1/2$, obviously in this case from (16) $x_k^{RB} = 0$. In the rest of this paragraph we assume that $|c_k| x > 0$. From (15),

$$g_k = - r_{kk}^2 [c_k] x \left( c_k - [c_k] x - \frac{\lambda}{r_{kk}^2 [c_k] x} + \frac{1}{2} [c_k] x \right).$$

Then, from (16) and (18),

$$x_k^{RB} = 0 \Leftrightarrow c_k - [c_k] x \leq \frac{\lambda}{r_{kk}^2 [c_k] x} - \frac{1}{2} [c_k] x,$$

$$x_k^{RB} = [c_k] x \Leftrightarrow c_k - [c_k] x > \frac{\lambda}{r_{kk}^2 [c_k] x} - \frac{1}{2} [c_k] x.$$
In addition, if \( M \geq 2 \), \( c_k - |c_k|^\infty \) has to satisfy
\[
\begin{cases}
-1/2 < c_k - |c_k|^\infty \leq 1, & |c_k|^\infty = 1, \\
-1 < c_k - |c_k|^\infty \leq 1, & |c_k|^\infty = 2j - 1, j = 2, \ldots, M-1, \\
-1 < c_k - |c_k|^\infty \leq 2M-1, & |c_k|^\infty = 2M-1,
\end{cases}
\]
(21)
and if \( M = 1 \), it has to satisfy
\[
-1/2 < c_k - |c_k|^\infty.
\]
(22)
To derive the formula for the SP of \( x^{\text{RB}} \), we define
\[
\rho_k^{\text{RB}} := \frac{p}{2M} \cdot \frac{(M-1)p}{M} \operatorname{erf}(\tilde{r}_{kk})
\]
\[
\int \left( 1 - \frac{2M-1}{2M} \right) \operatorname{erf} \left( \frac{1}{2} \tilde{r}_{kk} \right).
\]
(33)
\qed
Remark 1. For \( i = 1, 2, \ldots, n \), denote the events
\[
E_{i}^{\text{RB}} := (x_{i}^{R} = x_{i}^{*}, x_{i+1}^{\text{RB}} = x_{i+1}^{*}, \ldots, x_{n}^{\text{RB}} = x_{n}^{*}).
\]
In [1] a formula for the symbol error rate (SER) \( 1 - \Pr(x_{i}^{\text{RB}} = x_{i}^{*}) \) is derived for the special case \( M = 1 \). It is stated in [7] that for a general 2M-ary constellation the SER can be approximated using the union bound. Here our formula for \( \Pr(x_{i}^{\text{RB}} = x_{i}^{*}|E_{i+1}^{\text{RB}}) \) is a rigorous one and it leads to the formula for \( \Pr(x_{i}^{\text{RB}} = x_{i}^{*}) \).

Remark 2. An earlier version of Theorem 1 was presented in [25]. Here the formula for \( \Pr(x_{i}^{\text{RB}} = x_{i}^{*}) \) is much more concise (although the derivation is longer), and it facilitates the development of new theoretical results to be given in this paper and [8].

Remark 3. In [26] a formula for \( \Pr(x_{i}^{\text{RB}} = x_{i}^{*}) \) is derived for the case that \( x_{i}^{*} \) is uniformly distributed over a box \( X^{n} \) which includes all consecutive integer points, so [32] cannot be obtained from that formula.

Remark 4. From [23], we see \( j_{k} \) is not continuous with respect to \( \tilde{r}_{kk} \in (0, \infty) \) at some discrete points. This makes our later analyses of \( \rho_{k}^{\text{RB}} \) in [30] much complicated. For later uses, we define the points of discontinuity here. Let \( \gamma^{(i)} \) satisfy
\[
M - i + 1 = \frac{1}{2} \left( 1 + \frac{\lambda}{\gamma^{(i)}} \right)^{2} + 1, \quad i = 1, \ldots, M - 1,
\]
(34)
\[
\gamma^{(0)} = 0, \quad \gamma^{(M)} = \infty.
\]
(35)
Then, from [23],
\[
r_{kk} \in (\gamma^{(i-1)}, \gamma^{(i)}) \Rightarrow j_{k} = M - i + 1, \quad i = 1, \ldots, M.
\]
(36)
Corollary 1. Let the regularization parameter \( \lambda \) be any positive number. We have
\[
\lim_{r_{kk} \to 0} \rho_{k}^{\text{RB}} = 1 - p, \quad \lim_{r_{kk} \to \infty} \rho_{k}^{\text{RB}} = 1,
\]
\[
\lim_{r_{kk} \to 0} \rho_{k}^{\text{RB}} = \frac{p}{2M}, \quad \lim_{r_{kk} \to \infty} \rho_{k}^{\text{RB}} = 1.
\]
We make a remark about this corollary.

Remark 5. Under the condition [4], from Corollary 7, we observe
\[
\lim_{r_{kk} \to 0} \rho_{k}^{\text{RB}} \geq \lim_{r_{kk} \to 0} \rho_{k}^{\text{RB}}.
\]
If [4] is a strict inequality, then when \( r_{kk} \) is small enough,
\[
\rho_{k}^{\text{RB}} > \rho_{k}^{\text{RB}}.
\]
For a general \( r_{kk} \), this inequality may not hold. But the proof of Theorem 2 shows it still holds when \( \lambda = \lambda^{*} \) (see [60]).
IV. SOME PROPERTIES OF $x_{\lambda^*}^{RB}$

In this section we present some interesting properties of the $L_0$-regularized Babai point $x_{\lambda^*}^{RB}$.

A. Some basic properties of $\rho_k^{RB}$

We present some results about the function $\rho_k^{RB}$ (see (33)), which will be used in some later proofs.

**Lemma 2.** Let the regularization parameter $\lambda = \lambda^*$ (see (6)). With the same notation as in Theorem [7],

$$
\frac{\partial \rho_k^{RB}}{\partial r_{kk}} = \frac{2}{\sqrt{\pi}} \left[ \frac{M-j_k}{M} e^{-\frac{r_{kk}}{M}} + (1-p)(2j_k-1)e^{-\left(\frac{1}{4(2j_k-1)r_{kk}} + \frac{1}{4(2j_k-1)}\right)^2} \right],
$$

(37)

and it is continuous.

**Proof.** See [8] Lemma 2. □

**Lemma 3.** Let $p$ satisfy (4) and the regularization parameter $\lambda$ be positive. Then:

1) $\rho_k^{RB}$ is continuous with respect to $\lambda$ for any fixed $p$.
2) $\rho_k^{RB}$ is continuous with respect to $p$ when $\lambda = \lambda^*$.

**Proof.** See [8] Lemma 2. □

B. The superiority of $x_{\lambda^*}^{RB}$ with $\lambda = \lambda^*$

Note that the $L_0$-regularized Babai point $x_{\lambda^*}^{RB}$ is a function of $\lambda$. For clarity, here we write $x_{\lambda^*}^{RB}$ as $x_{\lambda^*}^{RB}(\lambda)$. As we know the regularization parameter $\lambda^*$ defined in (6) is optimal in the sense that the solution of (5) with $\lambda = \lambda^*$, which is the MAP estimator of $\gamma$ in [1], has higher SP than the solution of (5) with $\lambda$ chosen differently. The following theorem shows that $\lambda^*$ is also optimal in maximizing the SP of $x_{\lambda^*}^{RB}(\lambda)$.

**Theorem 2.** When $\lambda = \lambda^*$, the SP of $x_{\lambda^*}^{RB}(\lambda)$ reaches the global maximum with respect to $\lambda$, i.e., for any $\lambda \in [0, \infty)$,

$$
\Pr(x_{\lambda^*}^{RB}(\lambda^*) = x^*) \geq \Pr(x_{\lambda^*}^{RB}(\lambda) = x^*),
$$

(38)

and in particular, for the box-constrained Babai point $x_{\lambda^*}^{RB}$,

$$
\Pr(x_{\lambda^*}^{RB} = x^*) \geq \Pr(x_{0}^{RB} = x^*),
$$

(39)

with equality if and only $\lambda^* = 0$, i.e., $p = 2M/(2M + 1)$.

**Proof.** See [8] Theorem 2. □

**Remark 6.** When $\lambda = 0$, $x_{\lambda^*}^{RB}(\lambda) = x_{\lambda^*}^{RB}$. Thus, (39) is a special case of (38). The inequality (39) was given in [25], but the proof there was completely different.

**Remark 7.** The inequality (39) shows the advantage of applying the $L_0$-regularization. Numerical experiments in [8] indicate that the $L_0$-regularization can give significant SP improvement, especially when $p$ is small, i.e., $x^*$ is sparse.

**Remark 8.** For an arbitrary $\lambda \in [0, \infty)$, the inequality $\Pr(x_{\lambda^*}^{RB}(\lambda) = x^*) \geq \Pr(x_{\lambda^*}^{RB} = x^*)$ may not hold. In practice, if $p$ is unknown, i.e., $\lambda^*$ is unknown, one may try to maximize $\Pr(x_{\lambda^*}^{RB}(\lambda) = x^*)$ with respect to $\lambda$.

C. Monotonicity property

The following result shows how the ratio of the left hand side to the right hand side of changes when $p$ in (4) changes from $0^+$ to $2M/(2M + 1)$.

**Theorem 3.** The ratio $\Pr(x_{\lambda^*}^{RB}(\lambda) = x^*)/\Pr(x_{\lambda^*}^{RB} = x^*)$ is strictly decreasing for $p \in (0, 2M/(2M + 1))$.


**Remark 9.** Although the ratio $\Pr(x_{\lambda^*}^{RB} = x^*)/\Pr(x_{\lambda^*}^{RB} = x^*)$ is decreasing for $p \in (0, 2M/(2M + 1))$, it is easy to find an example for which $\Pr(x_{\lambda^*}^{RB}(\lambda^*) = x^*)$ is not always decreasing for $p \in (0, 2M/(2M + 1))$.

**Corollary 2.** Under the same condition as Theorem (3), we have

$$
1 \leq \frac{\Pr(x_{\lambda^*}^{RB}(\lambda^*) = x^*)}{\Pr(x_{\lambda^*}^{RB} = x^*)} < \frac{1}{\prod_{k=1}^{M} \text{erf} \left( \frac{r_{kk}}{2\sqrt{\gamma}^2} \right)},
$$

(40)

where the bounds are (nearly) attainable with respect to $p$.

**Proof.** See [8] Corollary 2. □

V. A BOUND ON THE SP OF $x_{\lambda^*}^{RB}$ WITH $\lambda = \lambda^*$

In this section, we give a bound on the SP of $x_{\lambda^*}^{RB}$ when the regularization parameter $\lambda = \lambda^*$ (see (9)). This bound is an upper bound and a lower bound under different conditions respectively. The bound is invariant with respect to the column permutations of $A$ and it will help us to understand the limit on SP that column permutations can achieve. The approach we use in our proof can be regarded as an extension of that used in [26], which deals with an un-regularized problem. Here the proof is more complicated due to the discontinuity issue with $j_k$ in (28).

To prepare for the proof of our theorem, we first define our target function associated with $\rho_k^{RB}$ in (30) and introduce a lemma. We use $\gamma$ to replace $r_{kk}$ and use $j_k$ to replace $j_k$ in (30) and rewrite $\rho_k^{RB}$ as $\rho(\gamma)$:

$$
\rho(\gamma) = \frac{p}{2M} + \frac{M - j_k}{M} \text{erf} (\gamma) - \frac{p}{2M} \text{erf} \left( \frac{1}{2(2j_k - 1)} \gamma - \frac{1}{2} (2j_k - 1) \gamma \right) + (1-p) \text{erf} \left( \frac{1}{2(2j_k - 1)} \gamma + \frac{1}{2} (2j_k - 1) \gamma \right),
$$

(41)

where $\lambda = \lambda^*/2^2$ (see (27)). Note that $j_\gamma$ (cf. (30)) in (41) is a function of $\gamma$ and $\lambda$.

Our target function associated with $\rho(\gamma)$ is defined as

$$
F(\zeta) := \ln \left( \rho(\eps^C) \right).
$$

(42)

It is easy to verify that when $\zeta \neq \ln \gamma^{(i)}$ for $i = 0, 1, \ldots, M$ (see (34) and (35) for the definition of $\gamma^{(i)}$),

$$
F'(\zeta) = \frac{\rho'(\eps^C)}{\rho(\eps^C)} e^{C},
$$

(43)

$$
F''(\zeta) = \frac{\rho''(\eps^C)}{\rho(\eps^C)} e^{C} + \frac{\rho'(\eps^C)}{\rho(\eps^C)} e^{C} - \left( \frac{\rho'(\eps^C)}{\rho(\eps^C)} e^{C} \right)^2.
$$

(44)
Lemma 4. Let the regularization parameter $\lambda = \lambda^*$ (so $\lambda = \lambda^*/\sigma^2$), and let $\gamma^{(i)}$ be defined by $\gamma^{(i)} = \frac{p}{i }$ and $F$ be given by (41) and (42), respectively. Then

1) $F'(\zeta)$ is continuous on $(-\infty, \infty)$.
2) $F''(\zeta) \to 0^+$ as $\zeta \to -\infty$, and $F''(\zeta) \to 0^-$ as $\zeta \to +\infty$.
3) When $M = 1$, $F''(\zeta)$ is continuous on $(-\infty, \infty)$.
4) When $M \geq 2$, for each $i = 1, \ldots , M - 1$, $F''(\zeta)$ is continuous on the interval $[\ln \gamma^{(i-1)}, \ln \gamma^{(i)}]$, where $\ln \gamma^{(0)} := -\infty$ and $\ln \gamma^{(M)} := \infty$, and $\lim_{\zeta \to +\infty} F''(\zeta) < \lim_{\zeta \to -\infty} \gamma^{(i)} + F''(\zeta)$.
5) Let

$$
\mu_1 = \min \{ \mu : F''(\ln \mu) = 0 \}, \\
\mu_2 = \max \{ \mu : F''(\ln \mu) = 0 \}
$$

i.e. the exponential of the smallest and largest roots of $F''$. $F'(\zeta)$ is strictly increasing on $(-\infty, \ln \mu_1)$, and strictly decreasing on $(\ln \mu_2, \infty)$.


Using Lemma 4 we can show

Theorem 4. Let the regularization parameter $\lambda = \lambda^*$ and let $\mu_1, \mu_2$ be defined by (45). Denote

$$
\omega = \left( \prod_{k=1}^n r_{kk} \right)^{1/n}, \quad \bar{\omega} = \frac{\omega}{\sqrt{2}\sigma}.
$$

1) If $\max_{1 \leq k \leq n} r_{kk} \leq \sqrt{2}\sigma \mu_1$, then

$$
P_{RB}(R) := \Pr(x_{RB} = x^*) \geq (\rho(\bar{\omega}))^n.
$$

2) If $\min_{1 \leq k \leq n} r_{kk} \geq \sqrt{2}\sigma \mu_2$, then

$$
P_{RB}(R) := \Pr(x_{RB} = x^*) \leq (\rho(\bar{\omega}))^n.
$$

In (46) and (47) the equality holds if and only if $r_{kk} = \omega$ for all $k = 1, 2, \ldots , n$.


Remark 10. In practice, we can obtain $\ln \mu_1$ and $\ln \mu_2$ using a numerical method, e.g., Newton’s method, for solving a nonlinear equation. Although $F''$ is not globally differentiable as it is not continuous at $\ln \gamma^{(i)}$, $i = 1, \ldots , M - 1$, it is piecewise differentiable, and we can apply a numerical method on each interval to find the roots. To avoid a possible numerical problem caused by the discontinuities we suggest first plotting $F''(\zeta)$ and then choosing two initial points near to the two roots $\ln \mu_1$ and $\ln \mu_2$, respectively, for the corresponding iterations. Although theoretically the values $\mu_1$ and $\mu_2$ can be anywhere on $(0, \infty)$, in practice they usually do not have a large magnitude. Some examples are presented in Table 1. We used the MATLAB built-in function fzero to find the two roots. Note that from (41) and (28), we can observe that $F''(\zeta)$ depends on only $\lambda = \lambda^*/\sigma^2$, which is determined by $M$ and $p$ (see (6)).

Remark 11. The SP of $x_{RB}$ denoted by $P_{RB}(R)$ depends on the R-factor of the QR factorization of $A$ (see (9) and (29)).

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</tbody>
</table>

If we incorporate a column permutation strategy in computing the QR factorization of $A : AP = Q_1 R$, where $P$ is a permutation matrix, then we can define the corresponding $L_0$-regularized Babai point $x_{RB}$ and its SP is $P_{RB}(R)$. Then in Theorem 4 $\omega = \left( \prod_{k=1}^n r_{kk} \right)^{1/n}$. This quantity $\omega$ is invariant with respect to $P$. In fact,

$$
\omega = \det(R)^{1/n} = \det(A^T A)^{1/(2n)},
$$

which is independent of $P$.

When $\sigma$ is large, it is likely the condition in Theorem 4 holds, then we have the lower bound. When $\sigma$ is small, it is likely the condition in Theorem 4 holds, then we have the upper bound. In applications with small $\sigma$, we would like to find good column permutations so that the SP of $x_{RB}$ is as close to the upper bound as possible. For the study of the effect of column permutations on $P_{RB}(R)$, see [8].

VI. SUMMARY

We have presented a formula for the SP of the $L_0$-regularized Babai point $x_{RB}$ under an $L_0$-RBILS problem setting to detect $x^*$ in the linear model (1). We have shown that with the regularization parameter $\lambda$ given by (6), $x_{RB}$ is more preferable than the unregularized box-constrained Babai point $x_{RB}$ in terms of success probability. Furthermore, we have shown that the ratio of the SP of $x_{RB}$ to that of $x_{RB}$ monotonically decreases as $p$ increases within its domain. Additionally, a bound on the SP of $x_{RB}$ has been derived, which acts as a lower or upper bound depending on the conditions satisfied.

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REFERENCES


