Welfare and Rationality Guarantees for the Simultaneous Multiple-Round Ascending Auction

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Abstract. The simultaneous multiple-round auction (SMRA) and the combinatorial clock auction (CCA) are the two primary mechanisms used to sell bandwidth. Under truthful bidding, the SMRA is known to output a Walrasian equilibrium that maximizes social welfare provided the bidder valuation functions satisfy the gross substitutes property [20]. Recently, it was shown that the combinatorial clock auction (CCA) provides good welfare guarantees for general classes of valuation functions [7]. This motivates the question of whether similar welfare guarantees hold for the SMRA in the case of general valuation functions.

We show the answer is no. But we prove that good welfare guarantees still arise if the degree of complementarities in the bidder valuations are bounded. In particular, if bidder valuations functions are α-near-submodular then, under *truthful bidding*, the SMRA has a welfare ratio (the worst case ratio between the social welfare of the optimal allocation and the auction allocation) of at most $(1+\alpha)$. The special case of submodular valuations, namely $\alpha = 1$, was studied in [12] and produces individually rational solutions. However, for $\alpha > 1$, this is a bicriteria guarantee, to obtain good welfare under truthful bidding requires relaxing individual rationality. In particular, it necessitates a factor α loss in the degree of individual rationality provided by the auction. We prove this bicriteria guarantee is asymptotically (almost) tight.

Truthful bidding, though, is not reasonable assumption in the SMRA [10]. But, bicriteria guarantees continue to hold for natural bidding strategies that are *locally optimal*. Specifically, the welfare ratio is then at most $(1+\alpha^2)$ and the individual rationality guarantee is again at most α, for α-near submodular valuation functions. These bicriteria guarantees are also (almost) tight.

Finally, we examine what strategies are required to ensure individual rationality in the SMRA with general valuation functions. First, we provide a weak characterization, namely *secure bidding*, for individual rationality. We then show that if the bidders use a profit-maximizing secure bidding strategy the welfare ratio is at most $1+\alpha$. Consequently, by bidding securely, it is possible to obtain the same welfare guarantees as truthful bidding without the loss of individual rationality. Unfortunately, we explain why secure bidding may be incompatible with the auxiliary bidding activity rules that are typically added to the SMRA to reduce gaming.

Keywords: ascending auctions, SMRA, welfare guarantee, individual rationality, near-submodular.

1 Introduction

The question of how best to allocate spectrum dates back over a century, and the case in favour of selling bandwidth was first formalized in the academic literature as far back as 1959 by Ronald Coase [8]. Over the past twenty years there have been large number spectrum auctions world-wide and, amongst these, the Simultaneous Multi-Round Auction (SMRA) and the Combinatorial Clock Auction (CCA) have proved to be extremely successful.

Both of these multiple-item auctions are based upon the same underlying mechanism. At time t, each item j has a price p_j^t. Given the current prices, each bidder i then selects her preferred set S_i^t of items. The price of any item that has excess demand then rises in the next time period and the process is repeated. There are important differences between the two auctions however. The SMRA uses *item bidding*, that is, the auctioneer views the selection of S_i^t as a collection of bids, one bid for every item of S_i^t. It also utilizes the concept of a *standing high bid* [11]. Any item (with a positive price) has a *provisional winner*. That bidder will win the item unless a higher bid is received in a later round. If such a bid is received then the standing high bid is increased and a new provisional winner assigned (chosen at random in the case of a tie). Item bidding and standing high bids lead to a major drawback, the *exposure problem*. Namely, a large set may be desired but such a bid may result in being allocated only a smaller undesirable subset. If the bidder valuation functions satisfy the gross-substitutes property then this problem does not arise. Indeed, given truthful bidding, Milgrom [20] showed that the SMRA will terminate in a Walrasian Equilibrium that maximizes social welfare; see also [15,13] who studied a similar auction mechanism. The exposure problem is also absent when the bidder valuation functions are submodular. In that case, Fu et al. [12] show that the final allocation, whilst not necessarily a Walrasiam Equilibrium, does provide at least half of the optimal social welfare.[4] For these classes of valuation function, the SMRA is *individual rational* in every time period. That is, if bidder i is provisionally allocated set S at round t then the value of S to i is at least the price of S.

For broader classes of valuation function that permit *complementarities*, though, the exposure problem does arise under the SMRA. This is a practical issue because in spectrum auctions bidder valuation functions typically do exhibit complementarities. The CCA [22] was designed to deal with such complementarities. Specifically, the CCA uses *package bidding* rather than item bidding. A package bid is an all-or-nothing bid. Consequently, a bidder cannot be allocated a subset of her bid; in particular, a bidder cannot be allocated an undesirable subset. Unfortunately, the basic CCA mechanism cannot provide for non-trivial approximate welfare guarantees, even for auctions with additive valuation functions and a small number of bidders and items [7]. It is perhaps surprising, then, that a minor adjustment to the CCA mechanism leads to good welfare guarantees for *any* class of valuation function. Specifically, if bid increments are made proportional to excess demand the welfare of the CCA is within an $O(k^2 \cdot \log n \log^2 m)$ factor of the optimal welfare [7]. Here n is the number of items, m is the number of bidders and k is the maximum cardinality of a set desired by the bidders. The fact that the CCA can generate high welfare for general valuation functions motivates the work in this paper. Is it possible that the SMRA also performs well with general valuation functions?

1.1 Our Results

The short answer to the question posed above is NO, the SMRA cannot guarantee high social welfare for valuation functions that exhibit complementarities (see Section 2.2).

It turns out however that we can quantify precisely the welfare guarantee in terms of the magnitude of the complementarities exhibited by the valuation function. To explain this we require a few definitions. Each bidder $i \in B$ has value $v_i(S)$ for any set of items $S \subseteq \Omega$. The valuation function $v_i()$ is monotonically non-decreasing (free-disposal). Each bidder has a quasi-linear utility, that is, its *utility* for a set S is $v_i(S) - p(S)$, where $p(S)$ is the price of S. The social welfare of an allocation $\mathcal{S} = \{S_1, \ldots, S_n\}$, where the S_i are pairwise-disjoint subsets of the items, is $\omega(\mathcal{S}) = \sum_i v_i(S_i)$. Next, to quantify the extent of complementarities,

[4] Their proof is not for the SMRA, but it can be adapted to apply there.

let the *degree of submodularity* [1] of a function f be

$$\mathcal{D}(f) = \min_{x \in I} \min_{A,B: A \subset B} \frac{f(A \cup x) - f(A)}{f(B \cup x) - f(B)}$$

Note that f is submodular if and only if $\mathcal{D}(f) \geq 1$. We say that f is α *near-submodular* if $\mathcal{D}(f) \geq \frac{1}{\alpha}$. A similar concept to near-submodularity, called *bounded complementarity*, is introduced by Lehman et al. [17].

The parameter α turns out to be key in explaining the performance of the SMRA. To explain this we require one more concept. We say that a bidder $i \in B$ is λ*-individually rational* if $\lambda \cdot v_i(S_i^t) \geq p(S_i^t)$ in each round t. Note that if $\lambda = 1$ then we have individual rationality. We say that an auction mechanism is λ*-individually rational* if every bidder is λ-individually rational. We then prove in Section 3:

Theorem 1. *If bidders have α near-submodular valuations then, under (conditional) truthful bidding* [5]*,*
(i) The SMRA outputs an allocation $\mathcal{S}$ with $\omega(S) \geq \frac{1}{1+\alpha} \cdot \omega(\mathcal{S}^*)$ where $\mathcal{S}^*$ is the optimal allocation.
(ii) The auction is α-individually rational.

The bi-criteria guarantees in Theorem 1 are (almost) tight. There are examples with α near-submodular valuations where the SMRA is only α-individually rational and the welfare guarantee tends to $\frac{1}{1+\alpha}$; see Section 3.3. Despite the fact that SMRA has arbitrarily poor welfare guarantees, it seems to perform very well in practice. Theorem 1 provides an explanation for this, and confirms empirical results, since complementarities exist but are typically bounded in magnitude in most spectrum auctions. Indeed, the SMRA has been proposed for auctions where valuation functions have weak complementarities [3].

There are, however, two major drawbacks inherent in Theorem 1. The first drawback is that it relies upon *truthful bidding*, that is, in each round the bidder selects the feasible set that maximizes utility. But, as we explain in Section 2.1, there are many reasons why a bidder will not bid truthfully in the SMRA. One of these reasons is that, in a spectrum auction, a bidder may not even know its own valuation function [10]. Bidders typically can however make comparisons between similar sets. Thus, a natural method by which a bidder can select a bid is via local improvement.

We show, in Section 4, that local improvement leads to similar guarantees as truthful bidding (albeit with an additional α factor in the denominator for the welfare guarantee).

Theorem 2. *If bidders have α near-submodular valuations then, under (conditional) local improvement bidding,*
(i) The SMRA outputs an allocation $\mathcal{S}$ with $\omega(S) \geq \frac{1}{1+\alpha^2} \cdot \omega(\mathcal{S}^*)$ where $\mathcal{S}^*$ is the optimal allocation.
(ii) The auction is α-individually rational.

Again, the bounds in Theorem 2 are (almost) tight.

The second drawback is that Theorem 1 shows that the SMRA is not individually rational. That is, it may produce outcomes which give negative utility to some bidders. Consequently, in Section 5 we provide a detailed study of what bidding strategies are required to ensure the individual rationality of the SMRA, and what are the consequences for welfare when such strategies are used. Towards this end, we characterize the individual rationality of the SMRA

[5] A detailed discussion on truthful and conditional truthful biddings will follow in Section 2.1.

in terms of *secure bidding*. We then prove, in Section 5.2, that secure bidding has a good welfare guarantees, provided the bidders make profit maximizing secure bids.

Theorem 3. *If bidders have α near-submodular valuations then, under (conditional) profit maximizing secure bidding, the SMRA outputs an allocation $\mathcal{S}$ with $\omega(S) \geq \frac{1}{1+\alpha} \cdot \omega(\mathcal{S}^*)$ where $\mathcal{S}^*$ is the optimal allocation.*

Consequently, by bidding securely, it is possible to obtain the same welfare guarantees as truthful bidding without the loss of individual rationality!

2 The Simultaneous Multiple-Round Ascending Auction

The SMRA was first proposed by Milgrom, Wilson and McAfee for the 1994 FCC spectrum auction. It is an ascending price auction that simultaneously sells many items. Let B be a set of n bidders and let Ω be a collection of m items. For each item $j \in \Omega$ the auction posits an item-price p_j^t at the start of round t. Moreover, the SMRA has a unique *standing high bidder* for each item with a positive price. Specifically, at the start of round t, bidder i is the standing high bidder for a set of items S_i^t; we call S_i^t the *provisional (winning) set* for bidder i.

The SMRA mechanism: Initially $p_j^0 = 0$ for each item $j \in \Omega$, and $S_i^0 = \emptyset$ for each bidder $i \in B$ and $t = 0$. The auction then iterates over rounds as follows. In round t, bidder i bids for a set $T_i^t \subseteq \Omega \setminus S_i^t$ under the assumption that the price of each item $j \in \Omega \setminus S_i^t$ is incremented to $p_j^t + \epsilon$. We call T_i^t the *conditional bid* for i. The term conditional is used as the auction mechanism automatically assumes that bidder i also makes a bid of price p_j^t for every item $j \in S_i^t$ (recall, bidder i is the provisional winner of the items S_i^t).

The item-prices and provisional sets are then updated. Take an item j and suppose that j is in exactly k of the conditional bids. If $k = 0$ then no bidder has placed a bid on item j at the incremented price $p_j^t + \epsilon$. Thus we set $p_j^{t+1} = p_j^t$ and the standing high bidder for j remains the same, $i.e.$ if $j \in S_i^t$ then $j \in S_i^{t+1}$. On the other hand if $k > 0$ (we say that j is in *excess demand*) then at least one bidder accepted the incremented price $p_j^t + \epsilon$. Thus we set $p_j^{t+1} = p_j^t + \epsilon$. The mechanism then randomly selects a bidder i amongst these k bidders and places $j \in S_i^{t+1}$. Note that, in this case, the standing high bidder must change as the previous standing high bidder was only assumed to bid the non-increment price p_j^t.

The mechanism then proceeds to the next round. The auction terminates when the conditional bids T_i^t of all the bidders are empty, at which point each bidder i is permanently allocated her provisional set S_i^t for a price $\sum_{j \in S_i^t} p_i^t$.

An extremely important property of the SMRA is that the use of standing high bidders implies that every item with a positive price is sold.

Observation 4 *In an SMRA auction, every item with a positive price is sold.* □

2.1 Truthful Bidding in the SMRA

A key factor in determining the practical success of the auction is accurate price discovery (see, for example Cramton [9,10]). This, in turn, relies upon bidding that is truthful or, at least, approximately truthful. There are two pertinent issues here. Firstly, is the SMRA mechanism compatible with truthful bidding? Specifically, the use of conditional bidding implicitly implies that bidders are forced to rebid on their provisional sets. However, suppose that T_i^t is the optimal conditional bid, that is

$$T_i^t = \text{argmax}_{T \subseteq \Omega \setminus S_i^t} \left(v_i(T \cup S_i^t) - v_i(S_i^t) - \sum_{j \in T} (p_j^t + \epsilon) \right)$$

It need not be the case that the implicit bid $S_i^t \cup T_i^t$ is truthful. In particular, we may have

$$S_i^t \cup T_i^t \neq \text{argmax}_{T \subseteq \Omega} \left(v_i(T) - \sum_{j \in T \cap S_i^t} p_j^t - \sum_{j \in T \setminus S_i^t} (p_j^t + \epsilon) \right)$$

Recall, here, that bidder i has a personalized set of prices: $\left((\mathbf{p})_{S_i^t}, (\mathbf{p} + \epsilon \cdot \mathbf{1})_{\Omega \setminus S_i^t} \right)$. Indeed, at round t, bidder i has an ϵ discount on the prices of S_i^t.

Interestingly, truthful bidding is compatible with the SMRA (for any price trajectories) precisely if the valuation function satisfies the gross substitutes property [20]. The gross substitutes property[6] was defined by Kelso and Crawford [15] and used by them to prove the existence of Walrasian equilibrium. Moreover, with gross substitutes, the SMRA will converge to a Walrasian equilibrium; furthermore such an equilibrium will maximize social welfare (given negligible price increments) – see Milgrom [20,21].

Secondly, even if truthful bidding is compatible with the SMRA, it is unlikely that the bidders will actually bid truthfully. For example, in bandwidth auctions, firms typically have ranked bandwidth targets and budget constraints that are more important than profit-maximization. Moreover, the valuation function is often not known in advance, rather it is “learned" as the auction proceeds. Regardless, the SMRA and the CCA do both incorporate a set of bidding activity rules to encourage truthful bidding. In the CCA these include revealed preference bidding rules that are difficult to game [4,6]. However, the bidding rules in the SMRA are weaker and strategic bidding is common – examples include demand reduction, parking, and hold-up strategies [10].

Consequently, as well as examining truthful (optimal conditional) bidding, we will examine the natural strategy of local improvement bidding that consists of attempting to add one item, delete one item, or replace one item in the current proposed solution. Gul and Stacchetti [13] prove that this local improvement method finds an optimal demand set, given any set of prices, if the valuation function has the gross substitutes property. We examine the quality of outcomes, for more general valuation functions, when this local search method is used in the SMRA in Section 4.

From now on, we concentrate on conditional bidding. Thus we will omit the term conditional when we refer to conditional truthful bidding or conditional profit maximizing bidding.

2.2 A Bad Example

Unfortunately, the welfare ratio of the SMRA can be arbitrarily bad if the valuations exhibit complementarities. This is the case even for auctions with just two bidders $\{1, 2\}$ and two

[6] A valuation function satisfies the *gross substitutes property* if, given any set of prices, increasing the price of some goods does not decrease demand for another good.

items $\{a, b\}$. Suppose both bidders have value 1 for each individual, but value the pair of items at M, for some large value M. Clearly, the optimal allocation has welfare M and consists in allocating both items to one of the bidders. However, the allocation of the SMRA has welfare 2 with probability $\frac{1}{2}$. Indeed, the provisional set at round $t+1$ is the complement of the provisional set at round t since the conditional bid of each bidder will be the complement of her provisional set. So the final allocation (which occurs when both prices exceed $\frac{M}{2}$) just depends on the allocation at the end of the first round; this allocates one item to each bidder with probability $\frac{1}{2}$ since it is randomized. By further increasing the number of identical bidders, it can be shown that the probability of the low welfare outcome tends to one.

This simple example has an important implication. Note that, at any round, both bidders bid on both items until the end of the auction. Thus, excess demand is constant in each round. So, even if price increments depend on the excess demand, one cannot achieve a better welfare ratio . This contrasts sharply with the behavior of the CCA whose welfare ratio becomes polylogarithmic if the price increment is allowed to depend upon the excess demand and the size of the demand sets are bounded [7].

3 Bicriteria Guarantees for the SMRA under Truthful Bidding

We now prove Theorem 1 and show that, under truthful bidding, the worst case welfare and rationality guarantees are dependent upon the degree of submodularity in the bidder valuation functions. First, in Section 3.1 we prove the individual rationality guarantee. Then, in Section 3.2 we prove the welfare guarantee. Finally, in Section 3.3 we show that these guarantees are (almost) tight.

3.1 A Rationality Guarantee

Theorem 5. *Given α-near-submodular truthful bidders, the SMRA outputs an α-individually rational allocation.*

Proof. In order to show α-individual rationality upon termination, let us prove a stronger result. Specifically, we will show that for any time t and any bidder i, every set $S' \subseteq S_i^t$ satisfies $\alpha \cdot v_i(S') \geq p(S')$. We proceed by induction on t. The statement trivially holds for $t = 0$. For the induction hypothesis, assume that bidder i is allocated the set S_i^t in round t where

$$\alpha \cdot v_i(S') \geq p^t(S') \quad \forall S' \subseteq S_i^t \tag{1}$$

We now require the following claim:

Claim. Let $X \subseteq S_i^t \cup T_i^t$ be such that $\alpha \cdot v_i(X) \geq p^t(X \cap S_i^t) + p^{t+1}(X \setminus S_i^t)$. Then, for every $x \in T_i^t \setminus X$, we have

$$\alpha \cdot v_i(X \cup x) \geq p^t(X \cap S_i^t) + p^{t+1}(X \cup x \setminus S_i^t)$$

Proof. Take any $x \in T_i^t \setminus X$. By α near-submodularity, we have

$$\frac{v_i(X \cup x) - v_i(X)}{v_i(S_i^t \cup T_i^t) - v_i(S_i^t \cup T_i^t \setminus x)} \geq \frac{1}{\alpha}$$

Consequently,

$$\begin{aligned}\alpha \cdot v_i(X \cup x) - \alpha \cdot v_i(X) &\geq v_i(S_i^t \cup T_i^t) - v_i(S_i^t \cup T_i^t \setminus x) \\ &\geq p^{t+1}(x) \\ &= p^t(x) + \epsilon\end{aligned} \tag{2}$$

Here the second inequality follows from truthful bidding. Otherwise, $T_i^t \setminus x$ is a more profitable bid than T_i^t. The equality arises as $x \notin S_i$.

By the condition in the statement of the claim, we have $\alpha \cdot v_i(X) \geq p^t(X \cap S_i^t) + p^{t+1}(X \setminus S_i^t)$. Therefore

$$\begin{aligned}\alpha \cdot v_i(X \cup x) &\geq p^t(X \cap S_i^t) + p^{t+1}(X \setminus S_i^t) + p^{t+1}(x) \\ &= p^t(X \cap S_i^t) + p^{t+1}(X \cup x \setminus S_i^t)\end{aligned}$$

Again, the equality arises as $x \notin S_i$.

By iteratively applying the previous claim over items in a set $\hat{X} \subseteq T_i^t \setminus X$, we obtain

Claim. Let $X \subseteq S_i^t$ be such that $\alpha \cdot v_i(X) \geq p^t(X)$. Then, for every $\hat{X} \subseteq T_i^t \setminus X$, we have $\alpha \cdot v_i(X \cup \hat{X}) \geq p^t(X \cap S_i^t) + p^{t+1}(X \cup \hat{X} \setminus S_i^t)$. □

Now take any $\hat{S} \subseteq S_i^{t+1}$. To complete the proof of Theorem 5, we must show that $\alpha \cdot v_i(\hat{S}) \geq p^{t+1}(\hat{S})$. For this purpose, set $S' = \hat{S} \cap S_i^t$ and set $T' = \hat{S} \setminus S_i^t$. By the induction hypothesis, we have that $\alpha \cdot v_i(S') \geq p^t(S')$.

Thus we may apply the second claim with $X = S'$ and $\hat{X} = T'$ to obtain

$$\begin{aligned}\alpha \cdot v_i(\hat{S}) &= \alpha \cdot v_i(S' \cup T') \\ &\geq p^t(S' \cap S_i^t) + p^{t+1}((S' \cup T') \setminus S_i^t) \\ &= p^t(S') + p^{t+1}(T')\end{aligned}$$

Furthermore, note that $S' \subseteq S_i^t \cap S_i^{t+1}$. In order to be the provisional winner of an item j in both rounds t and $t+1$, it must be the case that no other bidder bid for item j at the price $p^{t+1}(j)$. Thus the price of j at time $t+1$ remains $p^t(j)$. Hence $p^{t+1}(S') = p^t(S')$, and so

$$\begin{aligned}\alpha \cdot v_i(\hat{S}) &\geq p^t(S') + p^{t+1}(T') \\ &= p^{t+1}(S') + p^{t+1}(T') \\ &= p^{t+1}(\hat{S})\end{aligned}$$

Theorem 5 follows by induction. □

We remark that this proof implies a stronger conclusion: if bidder i is truthful then she is α-individually rational *regardless* of the strategies of other bidders.

3.2 A Welfare Guarantee

Theorem 6. *Given α-near-submodular truthful bidders, the SMRA outputs an allocation $\mathcal{S} = (S_1, \ldots, S_n)$ with social welfare $\omega(\mathcal{S}) \geq \frac{1}{1+\alpha} \cdot \omega(\mathcal{S}_i^*)$ where $S^* = (S_1^*, \ldots, S_n^*)$ is an allocation of maximum welfare.*

Proof. Assume the auction terminates in round t with a set of prices $\mathbf{p}^t$. Thus $T_i^t = \emptyset$ for each bidder i. In particular, by truthfulness, we have that

$$v_i(S_i \cup (S_i^* \setminus S_i)) - p^t(S_i^* \setminus S_i) \;\leq\; v_i(S_i \cup \emptyset) - p^t(\emptyset) \;=\; v_i(S_i)$$

Thus

$$v_i(S_i^*) \;\leq\; v_i(S_i \cup S_i^*) \;\leq\; v_i(S_i) + p^t(S_i^* \setminus S_i)$$

We now obtain a $(1+\alpha)$ factor welfare guarantee.

$$\begin{aligned}
\sum v_i(S_i^*) &\leq \sum_{i=1}^{n} \big(v_i(S_i) + p^t(S_i^* \setminus S_i)\big) \\
&\leq \sum_{i=1}^{n} v_i(S_i) + \sum_{i=1}^{n} p^t(S_i^*) \\
&\leq \sum_{i=1}^{n} v_i(S_i) + \sum_{i=1}^{n} p^t(S_i) \\
&\leq \sum_{i=1}^{n} v_i(S_i) + \sum_{i=1}^{n} \alpha \cdot v_i(S_i) \\
&= (1+\alpha) \cdot \sum_{i=1}^{n} v_i(S_i)
\end{aligned}$$

Here the third inequality follows because the SMRA mechanism utilizes provisional winners. This implies that every item with a positive price is sold at the end of the auction. Consequently, $\sum_{i=1}^{n} p^t(S_i) \geq \sum_{i=1}^{n} p^t(S_i^*)$. The fourth inequality follows as the auction allocation is α-individual rational, as shown in Theorem 5. □

By combining Theorem 5 and Theorem 6 we obtain Theorem 1.

3.3 Tightness of the Bicriteria Guarantees

The bounds in Theorem 1 are almost tight. To see this, consider the following example. There are k items $X = \{x_1, x_2, \ldots, x_k\}$. Let there be a large number L of identical bidders. For any $S \subset X$, each bidder i has a valuation:

$$v_i(S) = \begin{cases} 1 & \text{if } |S| = 1 \\ (|S|-1) \cdot \alpha + 1 & \text{if } |S| \geq 2 \end{cases}$$

It is easy to verify that this function is α near-submodular.

The optimal welfare is obtained by allocating the entire set X to a single bidder achieving social welfare $(k-1) \cdot \alpha + 1$. Now let us examine the allocation produced by the SMRA. Initially, all prices are 0 and the truthful bid for each bidder is to demand the entire set X. Indeed, every bidder keeps bidding on the entire set (except for the items that she is the standing high bidder) until every item has price greater than $\frac{1}{k}\left((k-1) \cdot \alpha + 1\right)$. At this point, no profitable bids can be made and all bidders drop out.

In each round, the randomly chosen standing high bidders are all distinct with probability at least $(1 - \frac{k-1}{L-1})^k$. For $L >> k$, this probability tends to 1. So by the end of the auction, the k

items are allocated to k different bidders with probability almost 1. Since the social welfare of this allocation is only k, the expected social welfare of the SMRA is around k. When k goes to infinity, the welfare ratio tends to α.

Next consider the rationality of this allocation. Each winner was allocated exactly one item with probability almost 1, and the final price of that item is $\frac{1}{k}\left((k-1)\cdot\alpha+1\right)$. The bidder has only value 1 for the item. When k goes to infinity, this tends to α-rationality for the winners. We remark that even for $k=2$ items, the previous example ensures that the welfare guarantee cannot be improved beyond $\frac{\alpha}{2}$ since the optimal welfare is $(\alpha+1)$ and the expected welfare of the SMRA is 2.

4 Bicriteria Guarantees under Locally Optimal Bidding

As discussed in Section 2.1, the assumption of truthful bidding is unrealistic in the SMRA. Consequently, here we examine an alternate natural bidding method. Given S_i^{t-1}, a bid $T_i^t \subseteq \Omega \setminus S_i^{t-1}$ is *locally optimal* if $v_i(S_i^{t-1} \cup T_i^t) - p^t(T_i^t) \geq v_i(S_i^{t-1} \cup X) - p^t(X)$ for all $X \subseteq \Omega \setminus S_i^{t-1}$, where $|X \setminus T_i^t| \leq 1$ and $|T_i^t \setminus X| \leq 1$. Observe that a locally optimal bid can be obtained via a local improvement algorithm that, given the current solution, seeks to add one item, delete one item, or replace one item. Analysing this local improvement method is useful because local comparison is a key tool used by bidders in real bandwidth auctions. Thus, there are practical reasons to suspect that bidders will not make bids that are clearly not locally optimal. From the theoretical viewpoint, this specific local improvement method is interesting because it is guaranteed, given any set of prices, to output an optimal set if the valuations satisfy the gross substitute property [13].

Now if we assume that bidders bid on locally optimal sets, we can still obtain bicriteria guarantees on both the welfare and the rationality of the mechanism.

Theorem 2. *If bidders have α-near-submodular valuations and make locally optimal bids, then the SMRA has welfare ratio $\frac{1}{1+\alpha^2}$ and is α-individually rational.*

Proof. We first argue that the allocation of the SMRA is α-individually rational.

Lemma 1. *If bidders have α-near-submodular valuations and make locally optimal bids, then the SMRA has welfare ratio $\frac{1}{1+\alpha^2}$ and is α-individually rational.*

Proof. The proof is the same as that of Theorem 5. Truthfulness was used to prove Inequality 2 in Claim 3.1. Observe, however, that truthfulness is not necessary; locally optimality is sufficient to prove Inequality 2 since we just need that the utility of T_i is better than the utility of any subset T' of T_i such that $|T_i \setminus T'| = 1$. Moreover no condition on valuation functions is used in Claim 3.1. □

Next, we show that the social welfare of the SMRA is at least $\frac{1}{1+\alpha^2}$ of the optimal welfare.

Lemma 2. *If bidders have α-near-submodular valuations and make locally optimal bids the SMRA outputs an allocation $S=(S_1,\ldots,S_n)$ with social welfare*

$$\sum_{i=1}^{n} v_i(S_i) \geq \frac{1}{1+\alpha^2}\cdot\sum_{i=1}^{n} v_i(S_i^*)$$

where $S^*=(S_1^*,\ldots,S_n^*)$ is an allocation of maximum welfare.

Proof. Assume the auction terminates in round t with a set of prices $\mathbf{p}^t$. Thus $T_i^t = \emptyset$ for each bidder i. Let $S_i^* \setminus S_i = \{x_1, x_2, \ldots, x_\ell\}$, say. By local optimality, we have, for any $x_j \in S_i^* \setminus S_i$, that

$$\begin{aligned} v_i(S_i \cup x_j) - p^t(x_j) &\leq v_i(S_i \cup \emptyset) - p^t(\emptyset) \\ &= v_i(S_i) \end{aligned}$$

Thus

$$v_i(S_i \cup x_j) - v_i(S_i) \leq p^t(x_j) \tag{3}$$

We then have

$$\begin{aligned} v_i(S_i^*) &\leq v_i(S_i \cup S_i^*) \\ &\leq v_i(S_i) + \sum_{j=1}^{\ell} \left(v_i(S_i \cup \{x_1, \ldots, x_j\}) - v_i(S_i \cup \{x_1, \ldots, x_{j-1}\})\right) \\ &\leq v_i(S_i) + \alpha \cdot \sum_{j=1}^{\ell} \left(v_i(S_i \cup x_j) - v_i(S_i)\right) \\ &\leq v_i(S_i) + \alpha \cdot \sum_{j=1}^{\ell} p^t(x_j) \\ &= v_i(S_i) + \alpha \cdot p^t(S_i^* \setminus S_i) \end{aligned}$$

Here the third inequality follows by α near-submodularity. The fourth inequality comes from Inequality (3). We finally obtain a $(1+\alpha^2)$ factor welfare guarantee:

$$\begin{aligned} \sum v_i(S_i^*) &\leq \sum_{i=1}^{n} \big(v_i(S_i) + \alpha \cdot p^t(S_i^* \setminus S_i)\big) \\ &\leq \sum_{i=1}^{n} v_i(S_i) + \alpha \cdot \sum_{i=1}^{n} p^t(S_i^*) \\ &\leq \sum_{i=1}^{n} v_i(S_i) + \alpha \cdot \sum_{i=1}^{n} p^t(S_i) \\ &\leq \sum_{i=1}^{n} v_i(S_i) + \alpha \cdot \sum_{i=1}^{n} \alpha \cdot v_i(S_i) \\ &= (1+\alpha^2) \cdot \sum_{i=1}^{n} v_i(S_i) \end{aligned}$$

Here the third inequality follows because the SMRA mechanism utilizes provisional winners. This implies that every item with a positive price is sold at the end of the auction. Consequently, $\sum_{i=1}^n p^t(S_i) \geq \sum_{i=1}^n p^t(S_i^*)$. The fourth inequality follows as the auction allocation is α-individual rational, as shown in Lemma 1. □

By combining Lemmas 1 and 2, we obtain our theorem.

4.1 Tightness of Bicriteria Guarantees

The bounds in Theorem 2 are essentially tight. To see this, consider the following example. There are $k \cdot n + 1$ items. Specifically, there is a special item z and, for each $i \in [n]$, there is a collection of k items $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,k}\}$.

There will be two classes of bidders. First, there are n Type I bidders. Bidder $i \in [n]$ only values the items $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,k}\}$. For any $S \subseteq X_i$, she has a valuation:

$$v_i(S) = \begin{cases} \alpha & \text{if } |S| = 1 \\ (|S| - 1) \cdot \alpha^2 + \alpha & \text{if } |S| \geq 2 \end{cases}$$

Her marginal value is always zero for any item not in X_i. This function is α near-submodular.

There are $k \cdot n \cdot L$ Type II bidders, where $L >> k, n$. For each $i \in [n]$ and each $j \in [k]$, there are L identical bidders that only value the set $\{x_{i,j}, z\}$. Each such bidder ℓ has a valuation function with $v_\ell(x_{i,j}) = 1, v_\ell(z) = H$ and $v_\ell(\{x_{i,j}, z\}) = H + \alpha$, where H is an integer larger than α that we will specify later. Moreover, her marginal value is always zero for any other item. Again, these valuation functions are α near-submodular.

Together, we have $(k \cdot L + 1) \cdot n$ bidders. The optimal welfare is obtained by allocating each set X_i to the Type I bidder i and z to any Type II bidder. This allocation has social welfare $\big((k-1) \cdot \alpha^2 + \alpha)\big) \cdot n + H$.

Now consider the allocation produced by the SMRA. Initially at $\mathbf{p} = \mathbf{0}$, the unique locally optimal bid is for each bidder to bid every item in their demand set. Thus a Type I bidder demands X_i and a Type II bidder demands $\{x_{i,j}, z\}$. This bidding behavior will remain until every item has price greater than α (Type II bidders still bid since $H > \alpha$). Let us call this round time t.

After time t, the locally optimal bid for each Type I bidder i is to demand the empty set. To see this, note that $p^t(x_{i,j}) > v_i(x_{i,j}) = \alpha$ for each item in $x_{i,j} \in X_i$. Each Type I bidder drops out.

On the other hand, since $L >> k, n$, we may assume that the randomly chosen standing high bidder for *every* item in each round is Type II (this happens with probability almost 1). In particular, at time t, the standing high bidder for each item in X_i is Type II. After time t, Type II bidders only bid on item z until its price reaches H. As a result, by the end of the SMRA, every item is allocated to some Type II bidder. The total welfare of the SMRA auction is then at most $(kn - 1) \cdot 1 + (H + \alpha)$. For $n >> H$ and sufficiently large k this gives a welfare ratio that tends to α^2.

Next consider the rationality of this allocation. Amongst the "winners", each Type II bidder (except at most one) wins exactly one item. The final price of that item in some X_i is α but the bidder has a valuation for the item of one. Thus, all these bidders are only α-individually rational.

5 Individually Rational Bidding

So, as shown in Theorems 5 and 2, truthful and locally-optimal bidding can only ensure approximate individual rationality in the SMRA. Consequently, such bidding strategies are highly risky. In this section, we investigate what bidding strategies are risk-free and what are the welfare implications of such strategies.

We call a risk-free strategy *conservative*, and show in Section 5.1 that conservative bidding is (weakly) characterized by *secure bidding*. Specifically, secure bidding always produces individually rational outcomes. Conversely, if the other bidders use secure bids then the only way a bidder can ensure an individually rational solution is by also bidding securely. This result holds even with stronger assumptions on the bidding strategies of the other bidders, for example, that they make profit-maximizing secure bids.

We then examine the welfare consequences of secure bidding. Our main result, in Section 5.2, is that then the welfare ratio is at most $1 + \alpha$ provided the bidders make profit maximizing secure bids. This result is surprising in that we are able to match the welfare guarantee of truthful bidding without having to lose individual rationality.

5.1 Secure Bidding

We say that a bidding strategy is *conservative* if it cannot lead to a bidder having negative utility. Thus, conservative strategies are individually rational. To understand what strategies are conservative, we first need to understand what constitutes a bidding strategy. In the SMRA, a bidder can select a bid based upon the auction history she observed, for example, the sequence of price vectors, her sequence of conditional bids, and on her sequence of provisional sets of items. Thus, we consider a bidding strategy to be a function of these three factors.[7]

We say that a conditional bid T_i^t is *secure* for bidder i (given the provisional winning set S_i^t) if $v_i(S') \geq p(S')$ for every $S' \subseteq S_i^t \cup T_i^t$. A bidding strategy is *secure* if every conditional bid it makes is secure. It is easy to verify that any secure bidding strategy is individually rational. We now show that bidding securely in every round is essentially the only individually rational strategy.

Lemma 3. *Let t be an integer and $T_i^{\hat{t}}, S_i^{\hat{t}}, \mathbf{p}^{\hat{t}}$ be the conditional bid of bidder i, the provisional winning set of bidder i and the price vector at round $\hat{t}$ for any $\hat{t} \leq t$. If bidder i makes a non-secure bid in round $t+1$, then there exist secure bidders who can bid consistent with the history and ensure that bidder i has negative utility in the final allocation.*

Proof. Assume that the conditional bid T_i^{t+1} of bidder i at some round t is not secure, then there exists $S' \subseteq S_i^{t+1} \cup T_i^{t+1}$ such that S' satisfies $v_i(S') < p_i^{t+1}(S')$. Let us prove that there exists an auction such that, with high probability, (i) the set allocated to i is S', (ii) at any time $\hat{t} \leq t$, the provisional winning set of i is $S_i^{\hat{t}}$ and (iii) the price vector at round $\hat{t}$ is $\mathbf{p}^{\hat{t}}$.

The auction is as follows: there are many copies of the same bidder 1 whose valuation function is v_1. Let M be an integer larger than the maximum of the prices at any round $\hat{t} \leq t$ and the maximum valuation of any subset of items for bidder i. The valuation function v_1 of all the copies of bidder 1 is additive[8] and the value of each item is the following:

$$v_1(s) = \begin{cases} M + 2 \cdot \epsilon & \text{if } s \in \Omega \setminus S' \\ p^t(s) & \text{if } s \in S' \setminus S_i^t \\ p^t(s) - \epsilon & \text{if } s \in S' \cap S_i^t \end{cases}$$

Claim. Assume that i bids on $T_i^{\hat{t}}$ at any round $\hat{t} \leq t$. There is a sequence of secure bids such that, for every $\hat{t} \leq t$, with high probability

[7] In some SMRA mechanisms, bidders also know the excess demand of each item.

[8] A valuation function v is additive if $v(S) = \sum_{s \in S} v(s)$.

(i) the price vector is exactly $\mathbf{p}^{\hat{t}}$ at the end of round $\hat{t}$,
(ii) bidder i is the standing high bidder of the set $S_i^{\hat{t}+1}$.[9]

Proof. By induction on t, let us prove that if the copies of bidder 1 use the following strategy, the conclusion holds. If the price of item s does not increase from round $\hat{t}$ to round $\hat{t}+1$ then no copy of bidder 1 bids on it at round $\hat{t}$; if the price of item s increases and $s \in S_i^{\hat{t}}$ then no copy of bidder 1 bids on it at round $\hat{t}$; if the price of s increases and $s \notin S_i^{\hat{t}}$ then all the copies of bidder 1 bid on it at round $\hat{t}$.

By construction of the valuation function v_1, at any time $\hat{t} \leq t$, the value of any item $s \in \Omega \setminus (S' \cap S_i^{t+1})$ for copies of bidder 1 is at least its price. Moreover, if $s \in S_i^t$ then copies of bidders 1 do not bid on s at price $p^t(s)$ by construction. It is easy to verify that $v_1(s)$ is larger than the price of s at any round where copies of 1 bid on it. As $v_1(\cdot)$ is additive, all bids by copies of bidder 1 up to round t are secure.

Let us show that items in excess demand are those whose prices increase between rounds $\hat{t}-1$ and $\hat{t}$. If the price of an item in $\Omega \setminus S_i^{\hat{t}}$ is distinct in $\mathbf{p}^{\hat{t}-1}$ and $\mathbf{p}^{\hat{t}}$, then all the copies of bidder 1 bid on it, and it is in excess demand. Now assume $s \in S_i^{\hat{t}}$, if the price of s increases, then $s \in S_i^{\hat{t}} \setminus S_i^{\hat{t}-1}$ (the provisional winner must change when there is a price increment). Thus bidder i bids on s and then s is in excess demand.

Now let us show that with high probability, bidder i is the standing high bidder of the items in $S_i^{\hat{t}}$. Since the prices of any item s in $S_i^{\hat{t}-1} \cap S_i^{\hat{t}}$ do not increase, copies of bidder 1 do not bid on s at round $\hat{t}$. Thus s is still in $S_i^{\hat{t}}$. Moreover, bidder i is the unique bidder in excess demand for the items in $S_i^{\hat{t}} \setminus S_i^{\hat{t}-1}$. So the provisional set of bidder i contains $S_i^{\hat{t}}$. Let us prove that it does not contain any other item s with high probability. First assume that $s \in S_i^{\hat{t}-1} \setminus S_i^{\hat{t}}$. Thus the price of s increases. And since i was the standing high bidders of these items at round $\hat{t}-1$, she cannot be the standing high bidder anymore at round $\hat{t}$. Assume now that i bids on $s \notin S_i^{\hat{t}-1} \cup S_i^{\hat{t}}$. Then by construction, all the other copies of 1 also bid on s and then, with high probability (since there are many copies of bidder 1), s is not allocated to bidder i, which completes the proof of the claim. □

Now assume that at round $t+1$, bidder i decides to bid on T_i^{t+1}. Starting from round $t+1$, copies of bidder 1 securely bid on subsets in the complement of S' until the prices of all items in $\Omega \setminus S$ reach $M + 2\epsilon$. Note that since no copy of 1 bid on any item in S', all the items in S' are in the provisional set of i at the end of round $t+1$. Copies of 1 continue to perform the same bids until they drop out. On the other hand, bidder i can perform any bid.

Let us first show that the set allocated to i contains S'. At the end of round $t+1$, the price of item s in S' is $p^t(s)$ if $s \in S' \cap S_i^t$ and $p^t(s) + \epsilon$ if $s \in S' \setminus S_i^t$. Thus the price of s is above $v_1(s)$ and then copies of 1 cannot bid anymore on s since they make secure bids. Since $S' \subseteq S_i^{t+1}$, the set of items allocated to i by the SMRA contains the set S'.

Assume now that $s \notin S'$ is allocated to i at the end of the procedure. Since copies of 1 continue to bid on it until its price is at least $M + \epsilon$. This implies that bidder i bids on it at price at least $M + \epsilon$. Thus the price of the set allocated to i is at least $M + \epsilon$, which is above the value of any set for bidder i by definition of M. So i is not individually rational. Otherwise, bidder i is allocated the set S' and by definition of S', we have $p^t(S') > v_i(S')$ and then bidder i receives negative utility. □

[9] Recall that i is the standing high bidder of $S_i^{\hat{t}}$ at the beginning of round $\hat{t}$, which explains the index difference.

So, if the bids of the other bidders are secure, then performing a non-secure bid may lead to negative utility. One may ask if a similar statement still holds if stronger assumptions are made concerning the strategies of the other bidders. This is indeed the case. The following lemma states that even if we know the other bidders are truthful (or if they make profit-maximizing secure bids), making any non-secure bid is not individually rational. Bidder i performs a *profit-maximizing secure bid* T_i if the bid is secure and the utility of $S_i \cup T_i$ is maximized over all possible secure bids.

Lemma 4. *Let t be an integer and $T_i^{\hat{t}}, S_i^{\hat{t}}, \mathbf{p}^{\hat{t}}$ be the conditional bid of bidder i, the provisional winning set of bidder i and the price vector at round $\hat{t}$ for any $\hat{t} \leq t$. Assume that there is an item of value 0 for i with price $\epsilon \cdot \hat{t}$ at any round $\hat{t} \leq t$. If bidder i makes a non-secure bid in round $t+1$, then there exist truthful (or profit-maximizing secure) bidders who can bid consistent with the history and ensure that bidder i has negative utility in the final allocation.*

Proof. Let us construct an auction such that at any round $\hat{t} \leq t$, the price vector is $\mathbf{p}^{\hat{t}}$ and the provisional winning set of bidder i is $S_i^{\hat{t}}$. Assume that at some round t, the bid of i is not secure. Then there exists $S' \subseteq S_i^t \cup T_i^{t+1}$ that satisfies $v_i(S') < p(S')$.

Instance of the SMRA. Let us now construct an auction such that S' is allocated to i. Let us denote by s the item of value 0 for bidder i such that $p^{\hat{t}}(s) = \epsilon \cdot \hat{t}$ for any $\hat{t} \leq t$. Before describing formally the instance, let us give some intution. There are two main types of bidders. First we create bidders for time periods $\hat{t} \leq t$. For any item s' whose price increases at round $\hat{t}$ and such that i does not bid on s' at round $\hat{t}$, we create unit-demand bidders that bid on s in the first $\hat{t}-1$ rounds and bid on s' at round $\hat{t}$. These bidders ensure that the price vector is $\mathbf{p}^{\hat{t}}$ at any round $\hat{t}$ smaller than t. Second, we create bidder for time period $\hat{t} > t$. These bidders will ensure that that the set allocated to i is S'. Indeed, they will bid on items in the complement of S' until we are sure that, if i still bids on them, the strategy of i is not conservative. The most technical part of the proof consists in constructing the first type of bidders.

Let s' be an item distinct from s such that $p^{\hat{t}}(s') \neq p^{\hat{t}-1}(s')$. Assume moreover that $s' \notin S_i^{\hat{t}}$. Then we create three copies of a bidder b such that:

$$v_b(S) = \begin{cases} p^{\hat{t}}(s') & \text{if } S = \{s'\} \\ \epsilon \cdot \hat{t} & \text{if } S = \{s\} \end{cases}$$

Moreover, we assume that if the utility of both items is the same, bidder b prefers item s'. This preference rule can also be simulated by making a small modification to the valuation function. However, we present it using the preferences rule, as we believe it makes the proof cleaner. These three bidders are called the *initial bidders of round $\hat{t}$ for item s'*. The initial bidders are the union of the initial bidders of round $\hat{t}$ for $\hat{t} \leq t$. The initial bidders will permit to fit the price vector at any round $\hat{t} \leq t$.

Let M be the maximum of the value of a subset of items for bidder i and of $\epsilon \cdot t$. We can now create the second type of bidder that will push the prices of items in $\Omega \setminus S'$ after time t. For each item $s' \in \Omega \setminus S'$, we create three copies of the same bidder such that the value of s' is $2(M+\epsilon)$ and the value of s is $M+\epsilon$. Such bidders are called the *final bidders*. The final bidders will permit to ensure that the set S' will be the set allocated to bidder i at the end of the auction.

Let us prove the following simple facts:

Claim. Let $\hat{t} \leq t$. Assume that at round $\hat{t}-1$ of the auction, the price vector is $\mathbf{p}^{\hat{t}-1}$ then:

– All the initial bidders of round $\hat{t}' < \hat{t}$ have an empty conditional bid at round $\hat{t}$.
– If the initial bidders of round $\hat{t}' > \hat{t}$ for item s' bid on s' at round $\hat{t}$ then the price of s' must increase at any step between $\hat{t}$ and $\hat{t}'$.
– All the final bidders bid on s at round $\hat{t}$.
– If s' is an item such that $p^{\hat{t}}(s') - p^{\hat{t}-1}(s') = \epsilon$ and $s' \notin S_i^{\hat{t}}$ then there are initial bidders of round $\hat{t}$ bidding on s'.

Proof. Let b be an initial bidder of round $\hat{t}'$ for item s' with $\hat{t}' < \hat{t}$. The price of s at round $\hat{t}$ is $\epsilon\hat{t}$ which is larger than the value of s for b which is $\epsilon \cdot \hat{t}'$. Moreover, since we have created initial bidder of round $\hat{t}'$ for item s', the price of s' increases between rounds $\hat{t}'$ and $\hat{t}'+1$. And by definition initial bidders, the value of s' for b is $p^{\hat{t}'}(s')$. Thus the price of s' at time $\hat{t}$ is larger than the value of s' for b. Since b make secure bids, she does not bid on any item and then has an empty conditional bid at round $\hat{t}$.

Let $\hat{t}' > \hat{t}$ and b be an initial bidder of round $\hat{t}'$ for item s'. At round $\hat{t}$, the utility of bidder b for item s is $\epsilon \cdot \hat{t}' - \epsilon \cdot \hat{t} = \epsilon \cdot (\hat{t}' - \hat{t})$. And the utility of b for s' is $p^{\hat{t}'}(s') - p^{\hat{t}}(s')$. This difference is ϵ times the number of rounds between $\hat{t}$ and $\hat{t}'$ where the price of s' increases. Thus it at most $\epsilon \cdot (\hat{t}' - \hat{t})$ and we have equality if and only if the price of s' increases at any round between $\hat{t}$ and $\hat{t}'$. If the equality does not hold, then the utility of s is larger than the one of s'. Otherwise, the preference rule ensures that s' is prefered, which proves the second point.

The proof of the third point is straightforward. Assume that b is a final bidder for item s'. By definition of M, the utility of b on s is at least $M + 2\epsilon$ since $p^t(s) \leq M$. On the other hand, the utility of s' is at most $M + \epsilon$. Thus b bids on s'.

Let us finally prove the last point. According to the definition of the instance, we have created three initial bidders of round $\hat{t}$ for the item s'. Given $\mathbf{p}^{\hat{t}-1}$, these bidders have utility ϵ for both s and s' at round $\hat{t}$. By definition of the preference rule, if these bidders are allocated the empty set, they prefer bidding in s' rather than in s. Since they bid on at most two items s and s', one of these three bidders has an empty provisional set and then bid on s' at round $\hat{t}$.

Running the auction. Let us prove by induction that, with positive probability, at any round $\hat{t} \leq t$, the price vector is $\mathbf{p}^{\hat{t}}$ and the provisional set of bidder i is $S_i^{\hat{t}}$. For $t = 0$ the statement immediately holds. Now assume that the price vector of the auction fits $\mathbf{p}^{\hat{t}-1}$ after round $\hat{t}-1$ and that i is the standing high bidder of the items in $S_i^{\hat{t}-1}$. For each item s' such that the price of s' increases between $\hat{t}-1$ and $\hat{t}$ and such that $s' \notin S_i^{\hat{t}}$, we have created initial bidders for s' of round $\hat{t}$. Then the last point of the claim ensures that the price of s' increases between round $\hat{t}-1$ and $\hat{t}$ increases. Moreover, since these items are in excess demand and i does not bid on them, bidder i is not the standing high bidders on these items are round $\hat{t}$.

Now consider any item s' in $S_i^{\hat{t}-1}$ that is still in $S_i^{\hat{t}}$. No initial bidder of round $\hat{t}$ was created for item s'. Moreover, all the initial bidders of round $\hat{t}' > \hat{t}$ for item s' do not bid for s' by the second point of the claim. Indeed the price of s' does not increase at any step between $\hat{t}$ and $\hat{t}'$. So there is no excess demand on s' and then the price of s' and its standing high bidder remain the same. A similar argument ensures that no item x such that $p^{\hat{t}'-1}(x) = p^{\hat{t}'}(x)$ is in excess demand. Thus the price of x is not modified.

Let us finally consider the items s' in $S_i^{\hat{t}} \cap T_i^{\hat{t}-1}$. Since the condition bid of i contains s', s' is in excess demand. Moreover, since i is a candidate to be the new provisional winner of such an item, $s' \notin S_i^{\hat{t}-1}$. However other bidders may also be candidates to be provisional winner of s' (for instance initial bidders for s' of later rounds). Since the provisional winner is chosen uniformly at random amongst the candidates, bidder i is chosen with positive probability.

So, for any round $\hat{t} \leq t$, there is a positive probability that at any round $\hat{t} \leq t$ the price vector is $\mathbf{p}^{\hat{t}}$ and the provisional winning set of i is $S_i^{\hat{t}}$. Now at round $t+1$, bidder i bids on T_i^{t+1}. The behavior of the auction after this round is different (and actually simpler). Indeed, the claim ensures that no initial bidder makes any conditional bid anymore. Moreover

- Since the prices of items in S' are larger than their values for any other bidders, no bidder will bid on any item in S' after round t. Thus the set of items allocated to i contains the set S'.
- Since there exist a lot of bidders whose value on each item in $\Omega \setminus S'$ is larger than the value of any set for bidder i, these bidders will continue to bid on these items until bidder i drops out. Thus no item of $\Omega \setminus S'$ will be allocated to i at the end of the procedure.

It completes the proof of Lemma 4. □

5.2 Social Welfare under Secure Bidding

The previous results ensure that secure bidding strategies are essentially the only way to guarantee 1-individual rationality. In this section, we will assume that bidders strategies are secure. The following simple lemma ensures that any allocation where each bidder is allocated at least one item can be obtained in the SMRA with bidders only making secure bids.

Lemma 5. *Any allocation where each bidder is allocated at least one item can be obtained via the SMRA with secure bidders. In particular if each bidder is allocated at least one item then the optimal allocation can be obtained if bidders are secure.*

Proof. Let $\mathcal{S} = (S_1, \ldots, S_n)$ be an allocation of the items where $S_i \neq \emptyset$ is allocated to bidder i. Then this allocation can be obtained with secure bidders. Indeed, assume that at round $t = 0$ each bidder simply bids on the set S_i. Since all the items have price 0, all the bids are trivially secure. Then, at the end of first step, every bidder i is the standing high bidder of the set S_i and no item is in excess demand. Thus the SMRA stops and allocates to each bidder i the set S_i. □

Lemma 5 is unsatisfactory in two ways. First, if there are bidders that are allocated nothing then the situation can be far more complex. Specifically, it may then be the case, see Lemma 6, that secure bidding cannot provide a guarantee on welfare.

Lemma 6. *If super-additive bidders only make secure bids, then there is no guarantee on the social welfare of the SMRA.*

Proof. Let M be a positive integer. Consider an auction with three bidders $\{0, 1, 2\}$ and two items $\{a, b\}$. The bidders $\{0, 1\}$ are unit-demand. The value of item a for bidder 0 is equal to 1 and the value of item b for bidder 1 is equal to 1. Bidder 2 has a super-additive valuation function. Its value for each individual item is $\frac{1}{2}$ but it has value M for the set $\{a, b\}$. Observe that if bidder 2 bids securely, then she cannot bid on the items a or b once their prices rise beyond $\frac{1}{2}$. Therefore the welfare is only 1 with item a allocated to bidder 0 and item b allocated to bidder 1. Clearly, the optimal allocation has welfare M which can be arbitrarily large. □

The second unsatisfactory aspect of Lemma 5 is that the structure of the bids used there is extremely artificial, since the bidders need to know all the valuation functions in order to calculate the secure bids. Theorem 3 shows we can circumvent both of these problems if the bidders' valuation functions are α-near-submodular. Then a good welfare guarantee can be obtained if the bidders make profit-maximizing secure bids.

Theorem 3. *Given bidder valuation functions that are α near-submodular. Assume moreover that all the set values are multiple of ϵ. If each bidder bids for a profit-maximizing (conditional) secure set in every round then the SMRA outputs a solution $\mathcal{S}$ with welfare $\omega(\mathcal{S}) \geq \frac{1}{1+\alpha} \cdot \omega(\mathcal{S}^*)$.*

Proof. Let $\mathcal{S}^* = \{S_1^*, \dots, S_n^*\}$ be the optimal allocation, and let $\mathcal{S} = \{S_1, \dots, S_n\}$ be the assignment output by the SMRA. By assumption, S_i is the most profitable secure set in the final round, and the conditional bid T_i is empty in the final round and then S_i was the provisional set for bidder i in the penultimate round. Let $S_i^* \setminus S_i = \{x_1, x_2, \dots, x_k\}$ and let $X^j = \{x_1, x_2, \dots, x_j\}$, for each $j \leq k$.

For every item $x_j \in S_i$, since the conditional bid T_i is empty, there are two possibilities.

- **Case 1:** $\{x_j\}$ is a secure conditional set but not as profitable as $\emptyset$. Then $v_i\big(S_i \cup \{x_j\}\big) - p(x_j) - \epsilon < v_i(S_i)$. Let Q_{ij} be S_i, and we have $p(x_j) \geq v_i\big(Q_{ij} \cup \{x_j\}\big) - v_i(Q_{ij})$ since values are multiple of ϵ.
- **Case 2:** $\{x_j\}$ is an insecure conditional set. Then there exist a set $Q \subseteq S_i$ such that $v_i\big(Q \cup \{x_j\}\big) < p\big(Q \cup \{x_j\}\big)$. On the other hand, since S_i is a secure set, $v_i\big(Q\big) \geq p\big(Q\big)$. Let Q_{ij} be Q, and we have $p(x_j) \geq v_i\big(Q_{ij} \cup \{x_j\}\big) - v_i(Q_{ij})$.

In both case, we have $p(x_j) \geq v_i\big(Q_{ij} \cup \{x_j\}\big) - v_i(Q_{ij})$. Using these inequalities, we can bound $v_i(S_i^*)$.

$$\begin{aligned}
v_i(S_i^*) &\leq v_i(S_i \cup X^k) \\
&= v_i(S_i) + \sum_{j=1}^{k} \big(v_i(S_i \cup X^j) - v_i(S_i \cup X^{j-1})\big) \\
&\leq v_i(S_i) + \sum_{j=1}^{k} \alpha \cdot \big(v_i\big(Q_{ij} \cup \{x_j\}\big) - v_i(Q_{ij})\big) \\
&\leq v_i(S_i) + \alpha \cdot \sum_{j=1}^{k} p(x_j) \\
&\leq v_i(S_i) + \alpha \cdot p(S_i^*)
\end{aligned}$$

The second inequality is because $Q_{ij} \subseteq S_i$ and $v_i(\cdot)$ is α-near-submodular. The third inequality is derived from the case analysis above.

Finally, we are ready to bound the welfare ratio.

$$\begin{aligned}
\sum v_i(S_i^*) &\leq \sum_{i=1}^{n} v_i(S_i) + \alpha \cdot \sum_{i=1}^{n} p(S_i^*) \\
&\leq \sum_{i=1}^{n} v_i(S_i) + \alpha \cdot \sum_{i=1}^{n} p(S_i) \\
&\leq (1+\alpha) \cdot \sum_{i=1}^{n} v_i(S_i)
\end{aligned}$$

The last inequality holds because under secure bidding, the allocation is individually rational. □

The bound in Theorem 3 is almost tight. This can be seen by adapting the example in Section 3.3.

Thus, under secure bidding we are able to match the welfare guarantee of truthful bidding without having to lose individual rationality. This suggests that secure bidding might be the best strategy to use in an SMRA auction. Unfortunately, this is probably not the case. Recall, from Section 2.1, that in addition to the basic ascending price mechanism the SMRA has an associated set of bidding activity rules to encourage truthful bidding. As discussed, the rules are actually too weak to ensure truthful bidding. But the rules are strong enough to make secure bidding very risky. In particular, each bidder has an amount of eligibility points. The larger the number of points the bigger the collection of items a bidder may bid on. A bidder loses eligibility points if she bids on a small set – such bids will then hurt the bidder in later rounds. Observe that secure bidding naturally favours bidding upon small sets and is, thus, risky.

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