COMP/MATH 553 Algorithmic Game Theory

Lecture 19& 20: Revenue Maximization in Multi-item Settings

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Menu

- Recap: Challenges for Revenue Maximization in Multi-item Settings
- Duality and Upper Bound of the Optimal Revenue
- SREV and BREV

Optimal Multi-Item Auctions

- Large body of work in the literature :
  - e.g. [Laffont-Maskin-Rochet'87], [McAfee-McMillan'88], [Wilson'93], [Armstrong'96], [Rochet-Chone'98], [Armstrong'99],[Zheng'00], [Basov'01], [Kazumori'01], [Thanassoulis'04],[Vincent-Manelli '06,'07], [Figalli-Kim-McCann'10], [Pavlov'11], [Hart-Nisan'12], …

- No general approach.

- Challenge already with selling 2 items to 1 bidder:
  - Simple and closed-form solution seems unlikely to exist in general.

- **Simple and Approximately Optimal Auctions**.

Selling Separately and Grand Bundling

- **Theorem:** For a single additive bidder, either selling separately or grand bundling is a 6-approximation [Babaioff et. al. '14].
  - Selling separately: post a price for each item and let the bidder choose whatever he wants. Let SREV be the optimal revenue one can generate from this mechanism.
  - Grand bundling: bundle all the items together and sell the bundle. Let BREV be the optimal revenue one can generate from this mechanism.
  - We will show that Optimal Revenue $\leq$ 2BREV + 4SREV.

Upper Bound for the Optimal Revenue

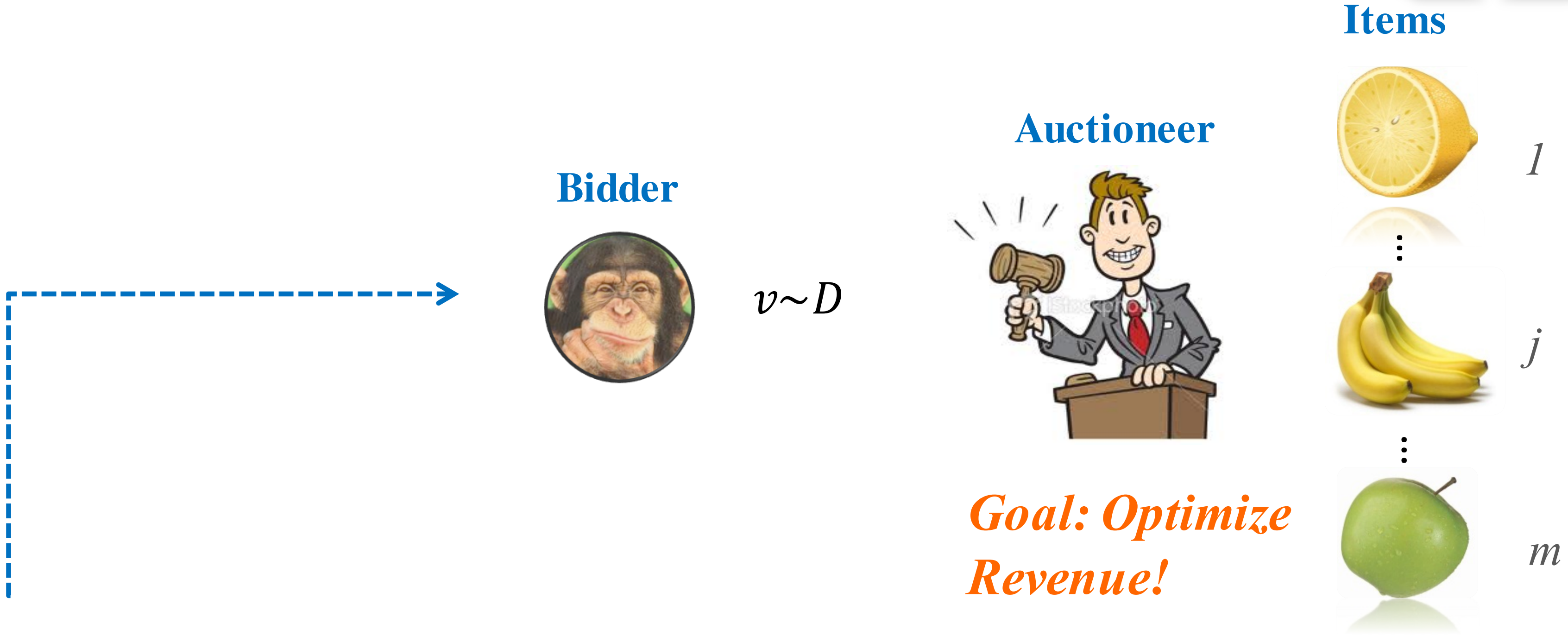
- ☐ Social Welfare is an upper bound for revenue.
- ☐ Unfortunately, could be arbitrarily bad.
- ☐ Consider the following 1 item 1 bidder case, and suppose the bidder's value is drawn from the equal revenue distribution, e.g., v $\in [1, +\infty), f(v) = \frac{1}{v^2}$ and $F(v) = 1 - \frac{1}{v}$.
- ☐ The optimal revenue = 1.
- ☐ What is the optimal social welfare?

Upper Bound for the Optimal Revenue

- ❑ Suppose we have 2 items for sale. r_1 is the optimal revenue for selling the first item and r_2 is the optimal revenue for selling the second item.
- ❑ Is the optimal revenue upper bounded by $r_1 + r_2$?
  - NO… We have seen an example.
- ❑ What is a good upper bound for the optimal revenue, i.e., within a constant factor?

Upper Bound of the Optimal Revenue via Duality

Multi-item Auction: Set Up



Goal: Optimize Revenue!

Bidder:

- ***Valuation*** aka ***type*** $v \sim D$. Let $\boldsymbol{V}$ be the support of D.
- **Additive and quasi-linear utility:**
  - $\boldsymbol{v} = (v_1, v_2, \dots, v_m)$ and $v(S) = \sum_{j \in S} v_j$ for any set S.
- **Independent items:** $\boldsymbol{v} = (v_1, v_2, \dots, v_m)$ is sampled from $D = \times_j D_j$.

Our Duality (Single Bidder)

Primal LP (Revenue Maximization for 1 bidder)

Variables:

$\boldsymbol{x_j(v)}$: the prob. for receiving item j when reporting v.

$\boldsymbol{p(v)}$: the price to pay when reporting v.

Constraints:

$\boldsymbol{v} \cdot \boldsymbol{x}(\boldsymbol{v}) - p(\boldsymbol{v}) \geq \boldsymbol{v} \cdot \boldsymbol{x}(\boldsymbol{v}') - p(\boldsymbol{v}'),\ \forall \boldsymbol{v} \in \boldsymbol{V}, \boldsymbol{v}' \in \boldsymbol{V} \cup \{\emptyset\}$ (BIC & IRConstraints)

$\boldsymbol{x}(\boldsymbol{v}) \in P = [0,1]^m, \forall \boldsymbol{v} \in \boldsymbol{V}$ (Feasibility Constraints)

Objective:

$$\max \sum_{\boldsymbol{v}} f(\boldsymbol{v}) p(\boldsymbol{v})$$

Partial Lagrangian

Primal LP:

$$\max \sum_{\boldsymbol{v}} f(v) p(v)$$

s.t. $v \cdot x(\boldsymbol{v}) - p(v) \geq v \cdot x(\boldsymbol{v}') - p(v'),\ \forall v \in \boldsymbol{V}, v' \in \boldsymbol{V} \cup \{\emptyset\}$ (BIC & IR Constraints)

$x(v) \in P = [0,1]^m, \forall v \in \boldsymbol{V}$ (Feasibility Constraints)

Partial Lagrangian (Lagrangify only the truthfulness constraints):

$$\min_{\lambda > 0} \max_{x \in P, p} L(\lambda, x, p)$$

where

$$L(\lambda, x, p) = \sum_v f(v) p(v) + \sum_{v, v'} \lambda(v, v') \cdot (v \cdot (x(v) - x(v')) - (p(v) - p(v'))$$

Strong Duality: Opt Rev $= \max_{x \in P, p} \min_{\lambda \geq 0} L(\lambda, x, p) = \min_{\lambda \geq 0} \max_{x \in P, p} L(\lambda, x, p)$.

Weak Duality: Opt Rev $\leq \max_{x \in P, p} L(\lambda, x, p)$ for all $\lambda \geq 0$.

Proof: On the board.

Partial Lagrangian

Primal LP:

$$\max \sum_{\boldsymbol{v}} f(v)p(v)$$

s.t. $v \cdot x(\boldsymbol{v}) - p(v) \geq v \cdot x(\boldsymbol{v'}) - p(v'),\ \forall v \in \boldsymbol{V}, v' \in \boldsymbol{V} \cup \{\emptyset\}$ (BIC & IR Constraints)

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$$= \sum_v p(v) \cdot \left(f(v) + \sum_{v'} \lambda(v', v) - \sum_v \lambda(v, v') \right)$$

$$+ \sum_v x(v) \cdot \left(v \cdot \sum_{v'} \lambda(v, v') - \left(\sum_{v'} v' \cdot \lambda(v', v) \right) \right)$$

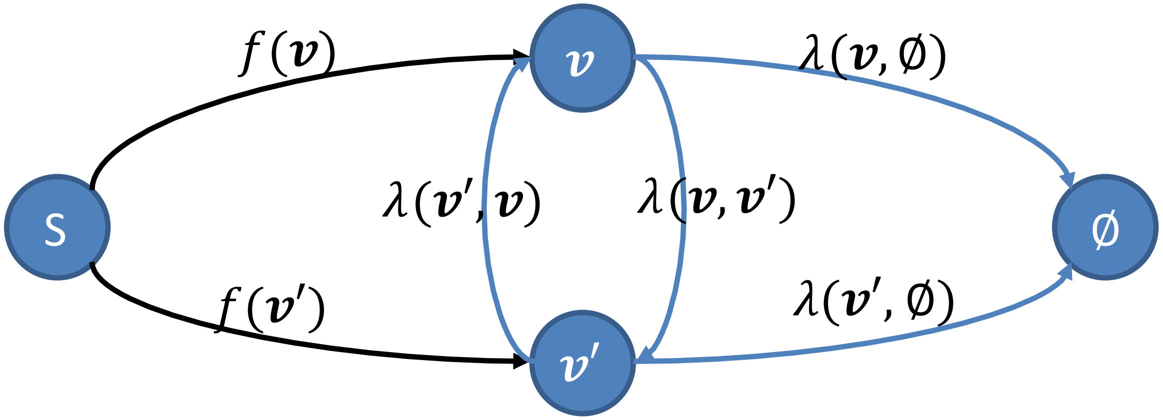
Better be 0, o.w. dual = +∞

The Dual Variables as a Flow

- Observation: If the dual is finite, for every $\boldsymbol{v} \in \boldsymbol{V}$

$$\boxed{\boldsymbol{f(v) + \sum_{v'} \lambda(v',v) - \sum_{v'} \lambda(v,v')=0}}$$

- This means λ is a flow on the following graph:
  - There is a super source s, a super sink $\emptyset$ (IR type) and a node for each $\boldsymbol{v} \in \boldsymbol{V}$.
  - $f(\boldsymbol{v})$ flow from s to $\boldsymbol{v}$ for all $\boldsymbol{v} \in \boldsymbol{V}$.
  - $\lambda(\boldsymbol{v}, \boldsymbol{v}')$ flow from $\boldsymbol{v}$ to $\boldsymbol{v}'$, for all $\boldsymbol{v} \in \boldsymbol{V}$ and $\boldsymbol{v}' \in \boldsymbol{V} \cup \{\emptyset\}$.



- Suffice to only consider λ that corresponds to a **flow**!

Duality: Interpretation

- Partial Lagrangian Dual (after simplification)

$$\min_{\boldsymbol{flow}\ \boldsymbol{\lambda}}\ \max_{\boldsymbol{x}\in P} L(\lambda, x, p)$$

where

$$L(\lambda, x, p) = \sum_v f(v)\cdot x(v)\left(v - \frac{1}{f(v)}\sum_{v'}\lambda(v', v)(v' - v)\right)$$

$$= \sum_v f(v)\cdot\sum_j x_j(v)\cdot\Phi_j^{(\lambda)}(v)$$

virtual welfare of allocation $\boldsymbol{x}$ w.r.t. $\Phi^{(\lambda)}(\cdot)$

virtual valuation of $\boldsymbol{v}$ (m-dimensional vector) w.r.t. λ

$$\boldsymbol{\Phi}^{(\lambda)}(\boldsymbol{v}) = \boldsymbol{v} - \frac{1}{f(\boldsymbol{v})}\sum_{\boldsymbol{v}'}\lambda(\boldsymbol{v}', \boldsymbol{v})(\boldsymbol{v}' - \boldsymbol{v})$$

where $\Phi_{\mathrm{j}}^{(\lambda)}(v) = v_j - \frac{1}{f(\boldsymbol{v})}\sum_{\boldsymbol{v}'}\lambda(\boldsymbol{v}', \boldsymbol{v})\left(v_j' - v_j\right)$

Note: every flow λ corresponds to a virtual value function $\Phi^{(\lambda)}(\cdot)$

$\overbrace{\text{Optimal Revenue}}^{\text{Primal}} \leq \overbrace{\text{Optimal Virtual Welfare w.r.t. any } \lambda}^{\text{Dual}}$ (Weak Duality)

Optimal Revenue = Optimal Virtual Welfare w.r.t. to optimal λ^* (Strong Duality)

Duality: Implication

- ☐ Strong duality implies Myerson's result in single-item setting.
  - $\Phi^{(\lambda^*)}(v_i) =$ Myerson's virtual value.

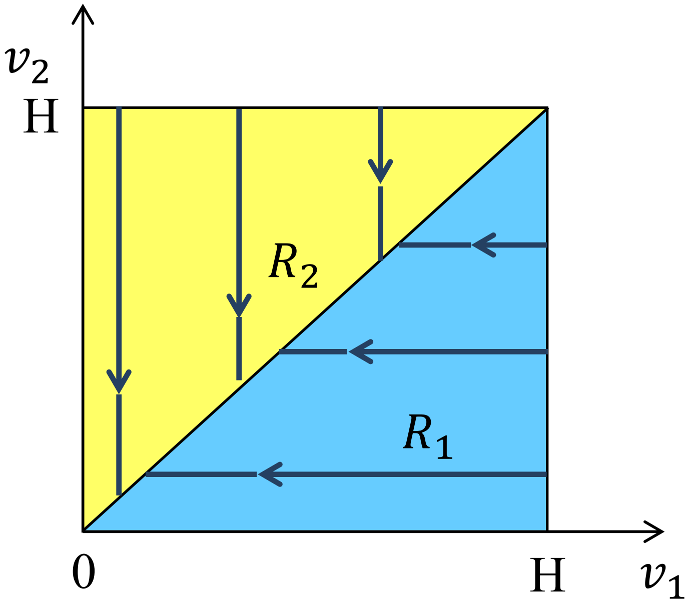
- ☐ Weak duality:

[Cai-Devanur-Weinberg '16]: A **canonical way** for deriving approximately tight upper bounds for the optimal revenue.

Single Bidder Flow

Single Bidder: Flow

- For simplicity, assume $\boldsymbol{V} = [H]^m \subseteq \mathbb{Z}^m$ for some integer H.
- Divide the bidder's type set into m regions
  - R_j contains all types that have j as the favorite item.
- **Our Flow**:
  - No cross-region flow ($\lambda(v', v) = 0$ if v, v' are not in the same region).
  - for any $v', v \in R_j$, $\lambda(v', v) > 0$ only if $v'_{-j} = v_{-j}$ and $v'_j = v_j + 1$.
- Our flow λ has the following two properties: for all j and $\boldsymbol{v} \in R_j$
  - $\Phi_{-j}^{(\lambda)}(\boldsymbol{v}) = v_{-j}$.
  - $\Phi_j^{(\lambda)}(v) = \varphi_j(v_j)$, where $\varphi_j(\cdot)$ is the Myerson's Virtual Value function for D_j.

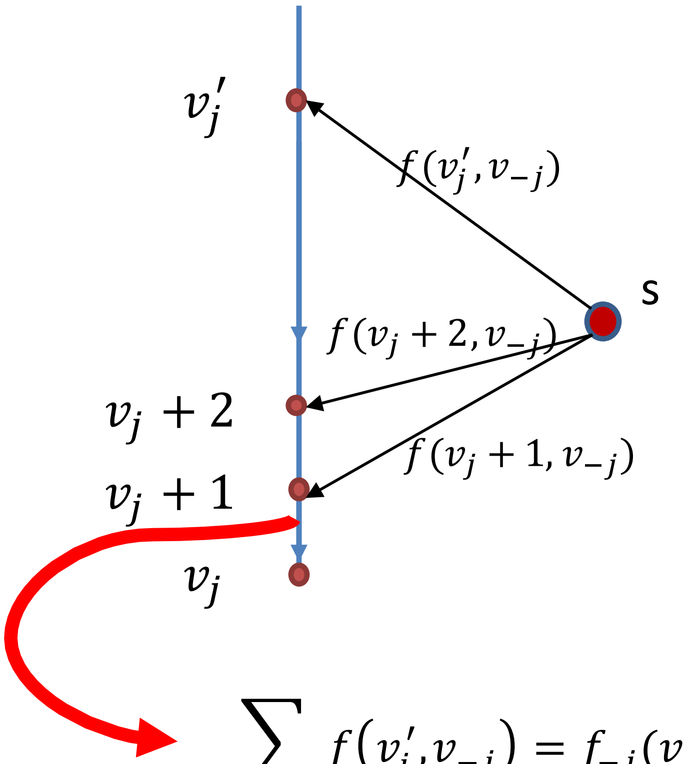


Virtual Valuation:

$$\Phi_j^{(\lambda)}(\boldsymbol{v}) = \boldsymbol{v_j} - \frac{1}{f(\boldsymbol{v})} \sum_{\boldsymbol{v'}} \lambda(\boldsymbol{v'}, \boldsymbol{v})(\boldsymbol{v'_j} - \boldsymbol{v_j})$$

Single Bidder: Flow (cont.)

For item j:



$$\sum_{v_j' > v_j} f(v_j', v_{-j}) = f_{-j}(v_{-j}) \cdot (1 - F_j(v_j))$$

$$\Phi_j^{(\lambda)}(v) = v_j - \frac{1}{f(v)} \sum_{v_j' > v_j} f(v_j', v_{-j}) = v_j - \frac{1 - F_j(v_j)}{f_j(v_j)}$$

Myerson virtual value function for D_j.

Intuition behind Our Flow

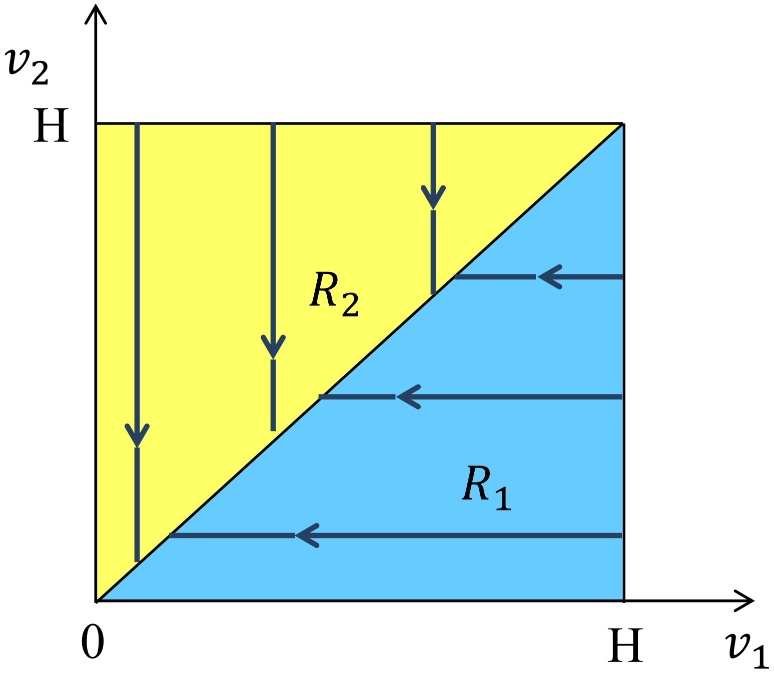
❑ **Virtual Valuation:**

$$\Phi_{\mathrm{j}}^{(\lambda)}(\boldsymbol{v})$$

$$= \boldsymbol{v_j} - \frac{1}{f(\boldsymbol{v})}\sum_{\boldsymbol{v}'}\lambda(\boldsymbol{v}',\boldsymbol{v})\left(\boldsymbol{v}'_j - \boldsymbol{v_j}\right)$$

❑ **Intuition:**

- Empty flow ➔ social welfare.
- Replace the terms that contribute the most to the social welfare with Myerson's virutal value.



❑ Our flow λ has the following two properties: for all j and $\boldsymbol{v} \in R_j$

- $\Phi_{-j}^{(\lambda)}(\boldsymbol{v}) = v_{-j}$.
- $\Phi_{j}^{(\lambda)}(v) = \varphi_j(v_j)$, where $\varphi_j(\cdot)$ is the Myerson's Virtual Value function for D_j.

Upper Bound for a Single Bidder

Corollary: $\Phi_j^{(\lambda)}(\boldsymbol{v}) = v_j \cdot \mathbb{I}\left[\boldsymbol{v} \notin R_j\right] + \varphi_j(v_j) \cdot \mathbb{I}[\boldsymbol{v} \in R_j]$.

Upper Bound for Revenue (single-bidder):

$$\mathrm{REV} \leq \max_{x \in P} L(\lambda, x, p) = \sum_{v} \sum_{j} f(\boldsymbol{v}) x_j(\boldsymbol{v}) \cdot \left(v_j \cdot \mathbb{I}\left[\boldsymbol{v} \notin R_j\right] + \varphi_j(v_j) \cdot \mathbb{I}[\boldsymbol{v} \in R_j]\right)$$

Interpretaion: the optimal attainable revenue is no more than the welfare of all non-favorite items plus some term related to the Myerson's single item virtual values.

Theorem: Selling separately or grand bundling achieves at least **1/6** of the upper bound above. This recovers the result by Babaioff et. al. [BILW '14].

Remark: the same upper bound can be easily extended to unit-demand valuations.

Theorem: Posted price mechanism achieves **1/4** of the upper bound above. This recovers the result by Chawla et. al. [CMS '10, '15].

SREV and BREV

Single Additive Bidder

- **[BILW ’14]** The optimal revenue of selling m independent items to an additive bidder, whose valuation $\boldsymbol{v}$ is drawn from $D = \times_j D_j$ is no more than $6\max\{\mathrm{SREV(D)}, \mathrm{BREV(D)}\}$.
  - $\mathrm{SREV(D)}$ is the optimal revenue for selling the items separately.
  - Formally, $\mathrm{SREV(D)} = \sum_j r_j = r$, where $r_j = \max_x x \cdot \Pr_{v_j}[v_j \geq x]$.
  - $\mathrm{BREV(D)}$ is the optimal revenue for selling the grand bundle.
  - Formally, $\mathrm{BREV(D)} = \max_x x \cdot \Pr_{\boldsymbol{v}}[\sum_j v_j \geq x]$.

Single Additive Bidder

Corollary: $\Phi_j^{(\lambda)}(\boldsymbol{v}) \leq v_j \cdot \mathbb{I}[\boldsymbol{v} \notin R_j] + \varphi_j(v_j) \cdot \mathbb{I}[\boldsymbol{v} \in R_j]$.

Goal: upper bound $L(\lambda, x, p)$ for any $x \in P$ using SREV and BREV.

$$L(\lambda, x, p) = \sum_{v} \sum_{j} f(\boldsymbol{v}) x_j(\boldsymbol{v}) \cdot \left(v_j \cdot \mathbb{I}[\boldsymbol{v} \notin R_j] + \varphi_j(v_j) \cdot \mathbb{I}[\boldsymbol{v} \in R_j]\right)$$

$$= \underbrace{\sum_{\boldsymbol{v}} \sum_{j} f(\boldsymbol{v}) x_j(\boldsymbol{v}) \cdot v_j \cdot \mathbb{I}[\boldsymbol{v} \notin R_j]}_{\text{NON-FAVORITE}} + \underbrace{\sum_{\boldsymbol{v}} \sum_{j} f(\boldsymbol{v}) x_j(\boldsymbol{v}) \cdot \varphi_j(v_j) \cdot \mathbb{I}[\boldsymbol{v} \in R_j]}_{\text{SINGLE}}$$

Bounding SINGLE

- SINGLE = $\sum_{\boldsymbol{v}} \sum_j f(\boldsymbol{v}) x_j(\boldsymbol{v}) \cdot \varphi_j(v_j) \cdot \mathbb{I}[\boldsymbol{v} \in R_j]$

$$= \sum_j \sum_{v_j} f_j(v_j) \cdot \varphi_j(v_j) \cdot \underbrace{\left(\sum_{v_{-j}} f_{-j}(v_{-j}) \cdot x_j(\boldsymbol{v}) \cdot \mathbb{I}[\boldsymbol{v} \in R_j]\right)}_{\substack{\text{view as the probability of} \\ \text{allocating item } j \text{ to the bidder} \\ \text{when her value for } j \text{ is } v_j.}}$$

- For each item j, this is Myerson’s virtual welfare $\leq r_j$.

- SINGLE $\leq r$

NON-FAVORITE: Core-Tail Decomposition

- NON-FAVORITE $= \sum_v \sum_j f(\boldsymbol{v}) x_j(\boldsymbol{v}) \cdot v_j \cdot \mathbb{I}[\boldsymbol{v} \notin R_j]$

$$\le \sum_{v} \sum_{j} f(\boldsymbol{v}) \cdot v_j \cdot \mathbb{I}[\boldsymbol{v} \notin R_j] = \sum_{j} \sum_{v_j} f_j(v_j) \cdot v_j \cdot \Pr_{v_{-j}}[\boldsymbol{v} \notin R_j]$$

$$\le \sum_{j} \sum_{v_j \ge r} f_j(v_j) \cdot v_j \cdot \Pr_{v_{-j}}[\exists k \ne j, v_k \ge v_j] \quad + \quad \sum_{j} \sum_{v_j < r} f_j(v_j) \cdot v_j$$

TAIL

CORE

NON-FAVORITE: Bounding the TAIL

- TAIL = $\sum_j \sum_{v_j \geq r} f_j(v_j) \cdot v_j \cdot \Pr_{v_{-j}}[\exists k \neq j, v_k \geq v_j]$

- Sell each item separately at price v_j:

$$v_j \cdot \Pr_{v_{-j}}[\exists k \neq j, v_k \geq v_j] \leq \sum_{k \neq j} v_j \cdot \Pr_{v_k}[v_k \geq v_j] \leq \sum_{k \neq j} r_k \leq r, \forall v_j$$

- Sell each item separately at price r:

$$\text{TAIL} \leq \sum_j \sum_{v_j \geq r} f_j(v_j) \cdot r = \sum_j r \cdot \Pr_{v_j}[v_j \geq r] \leq \sum_j r_j \leq r$$

NON-FAVORITE: Bounding the CORE

$$v_j' = v_j \cdot \mathbb{I}\left[v_j \le r\right]$$
$$v' = \sum_j v_j{}'$$

- CORE $= \sum_j \sum_{v_j \le r} f_j(v_j) \cdot v_j = \mathrm{E}[v']$
- Lemma: $\mathrm{Var}\left[v_j'\right] \le 2r_j \cdot r$
- Corollary: $\mathrm{Var}[v'] = \sum_j \mathrm{Var}\left[v_j'\right] \le 2r^2$
- Chebyshev Inequality: for any random variable X, $\Pr[|X - E[X]| \ge a] \le \frac{\mathrm{Var}[X]}{a^2}$.
- By Chebyshev Inequality,

$$\Pr[v' < \mathrm{CORE} - 2r] \le \frac{\mathrm{Var}[v']}{4r^2} \le \frac{1}{2}$$

- $\Pr\left[\sum_j v_j \ge \mathrm{CORE} - 2r\right] \ge 1/2$. If selling the grand bundle at price CORE$-2r$, the bidder will buy it with prob. $\ge 1/2$.
- 2BREV+2r ≥ CORE

Putting Everything Together

- $\text{REV} \le \max_{x \in P} L(\lambda, x, p) \le \text{SINGLE} + \text{TAIL} + \text{CORE}$
  - $\text{SINGLE} \le r$
  - $\text{TAIL} \le r$
  - $\text{CORE} \le 2\text{BREV} + 2r$
- **[BILW '14]** Optimal Revenue $\le 2\text{BREV} + 4\text{SREV}$.