COMP/MATH 553 Algorithmic Game Theory
Lecture 19 & 20: Revenue Maximization in Multi-item Settings

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Menu

- Recap: Challenges for Revenue Maximization in Multi-item Settings
- Duality and Upper Bound of the Optimal Revenue
- SREV and BREV
Large body of work in the literature:

- e.g. [Laffont-Maskin-Rochet’87], [McAfee-McMillan’88], [Wilson’93], [Armstrong’96], [Rochet-Chone’98], [Armstrong’99],[Zheng’00], [Basov’01], [Kazumori’01], [Thanassouli’s’04],[Vincent-Manelli ’06,’07], [Figalli-Kim-McCann’10], [Pavlov’11], [Hart-Nisan’12], ...

No general approach.

Challenge already with selling 2 items to 1 bidder:

- Simple and closed-form solution seems unlikely to exist in general.

Simple and Approximately Optimal Auctions.
Theorem: For a single additive bidder, either selling separately or grand bundling is a 6-approximation [Babaioff et. al. ’14].

- **Selling separately:** post a price for each item and let the bidder choose whatever he wants. Let $SREV$ be the optimal revenue one can generate from this mechanism.

- **Grand bundling:** bundle all the items together and sell the bundle. Let $BREV$ be the optimal revenue one can generate from this mechanism.

We will show that $\text{Optimal Revenue} \leq 2BREV + 4SREV$. 
Social Welfare is an upper bound for revenue.

Unfortunately, could be arbitrarily bad.

Consider the following 1 item 1 bidder case, and suppose the bidder’s value is drawn from the equal revenue distribution, e.g., \( v \in [1, +\infty) \), \( f(v) = \frac{1}{v^2} \) and \( F(v) = 1 - \frac{1}{v} \).

The optimal revenue = 1.

What is the optimal social welfare?
Suppose we have 2 items for sale. \( r_1 \) is the optimal revenue for selling the first item and \( r_2 \) is the optimal revenue for selling the second item.

Is the optimal revenue upper bounded by \( r_1 + r_2 \)?

- NO… We have seen an example.

What is a good upper bound for the optimal revenue, i.e., within a constant factor?
Upper Bound of the Optimal Revenue via Duality
Bidder:

- **Valuation** aka type \( v \sim D \). Let \( V \) be the support of \( D \).
- **Additive and quasi-linear utility:**
  - \( v = (v_1, v_2, ..., v_m) \) and \( v(S) = \sum_{j \in S} v_j \) for any set \( S \).
- **Independent items:** \( v = (v_1, v_2, ..., v_m) \) is sampled from \( D = \times_j D_j \).
**Primal LP** (Revenue Maximization for 1 bidder)

**Variables:**

\( x_j(v) \): the prob. for receiving item \( j \) when reporting \( v \).

\( p(v) \): the price to pay when reporting \( v \).

**Constraints:**

\[ v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v'), \quad \forall v \in V, v' \in V \cup \{\emptyset\} \] (BIC & IR Constraints)

\[ x(v) \in P = [0,1]^m, \quad \forall v \in V \] (Feasibility Constraints)

**Objective:**

\[ \max \sum_v f(v)p(v) \]
Primal LP:

$$\max \sum_v f(v)p(v)$$

s.t. $v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$, $\forall v \in V, v' \in V \cup \{\emptyset\}$ (BIC & IR Constraints)

$x(v) \in P = [0,1]^m, \forall v \in V$ (Feasibility Constraints)

Partial Lagrangian (Lagrangify only the truthfulness constraints):

$$\min_{\lambda > 0} \max_{x \in P, p} L(\lambda, x, p)$$

where

$$L(\lambda, x, p) = \sum_v f(v)p(v) + \sum_{v, v'} \lambda(v, v') \cdot (v \cdot (x(v) - x(v')) - (p(v) - p(v')))$$

Strong Duality: Opt Rev $= \max_{x \in P, p} \min_{\lambda \geq 0} L(\lambda, x, p) = \min_{\lambda \geq 0} \max_{x \in P, p} L(\lambda, x, p)$.

Weak Duality: Opt Rev $\leq \max_{x \in P, p} L(\lambda, x, p)$ for all $\lambda \geq 0$.

Proof: On the board.
Partial Lagrangian

**Primal LP:**

$$\max \sum_v f(v)p(v)$$

s.t. $v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$, $\forall v \in V, v' \in V \cup \{\emptyset\}$ (BIC & IR Constraints)

$$x(v) \in P = [0,1]^m, \forall v \in V$$ (Feasibility Constraints)

**Partial Lagrangian** (Lagrangify only the truthfulness constraints):

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$$= \sum_v p(v) \cdot \left( f(v) + \sum_{v'} \lambda(v', v) - \sum_v \lambda(v, v') \right)$$

$$+ \sum_v x(v) \cdot \left( v \cdot \sum_{v'} \lambda(v', v) - \left( \sum_{v'} \cdot \lambda(v', v) \right) \right)$$

Better be 0, o.w. dual = $+\infty$
Observation: If the dual is finite, for every \( v \in V \)

\[
f(v) + \sum_{v'} \lambda(v', v) - \sum_{v} \lambda(v, v') = 0
\]

This means \( \lambda \) is a flow on the following graph:

- There is a super source \( s \), a super sink \( \emptyset \) (IR type) and a node for each \( v \in V \).
- \( f(v) \) flow from \( s \) to \( v \) for all \( v \in V \).
- \( \lambda(v, v') \) flow from \( v \) to \( v' \), for all \( v \in V \) and \( v' \in V \cup \{\emptyset\} \).

Suffice to only consider \( \lambda \) that corresponds to a flow!
Partial Lagrangian Dual (after simplification)

\[
\min_{\text{flow } \lambda} \max_{x \in P} L(\lambda, x, p)
\]

where

\[
L(\lambda, x, p) = \sum_v f(v) \cdot x(v) \left( v - \frac{1}{f(v)} \sum_{v'} \lambda(v', v)(v' - v) \right)
\]

\[
= \sum_v f(v) \cdot \sum_j x_j(v) \cdot \Phi_j^{(\lambda)}(v)
\]

virtual welfare of allocation $x$ w.r.t. $\Phi^{(\lambda)}(\cdot)$

virtual valuation of $v$ (m-dimensional vector) w.r.t. $\lambda$

Note: every flow $\lambda$ corresponds to a virtual value function $\Phi^{(\lambda)}(\cdot)$

Primal

Optimal Revenue \leq Optimal Virtual Welfare w.r.t. any $\lambda$ (Weak Duality)

Dual

Optimal Revenue = Optimal Virtual Welfare w.r.t. to optimal $\lambda^*$ (Strong Duality)
Strong duality implies Myerson’s result in single-item setting.

- $\Phi^{(\lambda^*)}(v_i) = $ Myerson’s virtual value.

Weak duality:

[Cai-Devanur-Weinberg ’16]: A canonical way for deriving approximately tight upper bounds for the optimal revenue.
For simplicity, assume \( V = [H]^m \subseteq \mathbb{Z}^m \) for some integer \( H \).

Divide the bidder’s type set into \( m \) regions
- \( R_j \) contains all types that have \( j \) as the favorite item.

**Our Flow:**
- No cross-region flow \( (\lambda(v',v) = 0 \text{ if } v, v' \text{ are not in the same region}) \).
- for any \( v', v \in R_j, \lambda(v',v) > 0 \) only if \( v' = v_j \) and \( v' = v_j + 1 \).

Our flow \( \lambda \) has the following two properties: for all \( j \) and \( v \in R_j \)
- \( \Phi_{-j}^{(\lambda)}(v) = v_{-j} \).
- \( \Phi_j^{(\lambda)}(v) = \varphi_j(v_j) \), where \( \varphi_j(\cdot) \) is the Myerson’s Virtual Value function for \( D_j \).
For item $j$:

$$\Phi_j^{(n)}(v) = v_j - \frac{1}{f(v)} \sum_{v_j' > v_j} f(v_j', v_{-j}) = v_j - \frac{1 - F_j(v_j)}{f_j(v_j)}$$

Myerson virtual value function for $D_j$. 

$$\sum_{v_j' > v_j} f(v_j', v_{-j}) = f_{-j}(v_{-j}) \cdot (1 - F_j(v_j))$$
Intuition behind Our Flow

- Virtual Valuation:
  \[ \Phi_j^{(\lambda)}(\nu) = \nu_j - \frac{1}{f(\nu)} \sum_{\nu'} \lambda(\nu', \nu)(\nu'_j - \nu_j) \]

- Intuition:
  - Empty flow \(\Rightarrow\) social welfare.
  - Replace the terms that contribute the most to the social welfare with Myerson’s virtual value.

Our flow \(\lambda\) has the following two properties: for all \(j\) and \(\nu \in R_j\):
- \(\Phi_{-j}^{(\lambda)}(\nu) = \nu_{-j}\).
- \(\Phi_j^{(\lambda)}(\nu) = \varphi_j(\nu_j)\), where \(\varphi_j(\cdot)\) is the Myerson’s Virtual Value function for \(D_j\).
Upper Bound for a Single Bidder

Corollary: $\Phi_j^{(\lambda)}(v) = v_j \cdot \mathbb{I}[v \notin R_j] + \varphi_j(v_j) \cdot \mathbb{I}[v \in R_j].$

Upper Bound for Revenue (single-bidder):

$$\text{REV} \leq \max_{x \in P} L(\lambda, x, p) = \sum_v \sum_j f(v)x_j(v) \cdot (v_j \cdot \mathbb{I}[v \notin R_j] + \varphi_j(v_j) \cdot \mathbb{I}[v \in R_j]).$$

Interpretation: the optimal attainable revenue is no more than the welfare of all non-favorite items plus some term related to the Myerson’s single item virtual values.

Theorem: Selling separately or grand bundling achieves at least $1/6$ of the upper bound above. This recovers the result by Babaioff et. al. [BILW ’14].

Remark: the same upper bound can be easily extended to unit-demand valuations.

Theorem: Posted price mechanism achieves $1/4$ of the upper bound above. This recovers the result by Chawla et. al. [CMS ’10, ’15].
SREV and BREV
[BILW ’14] The optimal revenue of selling \( m \) independent items to an additive bidder, whose valuation \( v \) is drawn from \( D = \times_j D_j \) is no more than \( 6 \max \{ \text{SREV}(D), \text{BREV}(D) \} \).

- SREV(D) is the optimal revenue for selling the items separately.

- Formally, \( \text{SREV}(D) = \sum_j r_j = r \), where \( r_j = \max_x x \cdot \Pr_{v_j}[v_j \geq x] \).

- BREV(D) is the optimal revenue for selling the grand bundle.

- Formally, \( \text{BREV}(D) = \max_x x \cdot \Pr_v[\sum_j v_j \geq x] \).
Single Additive Bidder

Corollary: $\Phi^{(\lambda)}_j(v) \leq v_j \cdot \mathbb{I}[v \notin R_j] + \varphi_j(v_j) \cdot \mathbb{I}[v \in R_j]$.

Goal: upper bound $L(\lambda, x, p)$ for any $x \in P$ using SREV and BREV.

\[
L(\lambda, x, p) = \sum_v \sum_j f(v)x_j(v) \cdot (v_j \cdot \mathbb{I}[v \notin R_j] + \varphi_j(v_j) \cdot \mathbb{I}[v \in R_j])
\]

\[
= \sum_v \sum_j f(v)x_j(v) \cdot v_j \cdot \mathbb{I}[v \notin R_j] + \sum_v \sum_j f(v)x_j(v) \cdot \varphi_j(v_j) \cdot \mathbb{I}[v \in R_j]
\]
Bounding SINGLE

\[ \text{SINGLE} = \sum_v \sum_j f(v) x_j(v) \cdot \varphi_j(v_j) \cdot \mathbb{I}[v \in R_j] \]

\[ = \sum_j \sum_{v_j} f(j) \cdot \varphi_j(v_j) \cdot \left( \sum_{v_j} f_{-j}(v_{-j}) \cdot x_j(v) \cdot \mathbb{I}[v \in R_j] \right) \]

view as the probability of allocating item \( j \) to the bidder when her value for \( j \) is \( v_j \).

- For each item \( j \), this is Myerson’s virtual welfare \( \leq r_j \).

- SINGLE \( \leq r \)
NON-FAVORITE: Core-Tail Decomposition

\[ \text{NON-FAVORITE} = \sum_v \sum_j f(v) x_j(v) \cdot v_j \cdot \mathbb{I}[v \notin R_j] \]

\[ \leq \sum_v \sum_j f(v) \cdot v_j \cdot \mathbb{I}[v \notin R_j] = \sum_j \sum_{v_j} f_j(v_j) \cdot v_j \cdot \Pr_{v_j}[v \notin R_j] \]

\[ \leq \sum_j \sum_{v_j \geq r} f_j(v_j) \cdot v_j \cdot \Pr_{v_j}[^k \neq j, v_k \geq v_j] + \sum_j \sum_{v_j < r} f_j(v_j) \cdot v_j \]

TAIL

CORE
NON-FAVORITE: Bounding the TAIL

- $\text{TAIL} = \sum_j \sum_{v_j \geq r} f_j(v_j) \cdot v_j \cdot \Pr_{v_{-j}}[\exists k \neq j, v_k \geq v_j]$

- Sell each item separately at price $v_j$:

$$v_j \cdot \Pr_{v_{-j}}[\exists k \neq j, v_k \geq v_j] \leq \sum_{k \neq j} v_j \cdot \Pr_{v_k}[v_k \geq v_j] \leq \sum_{k \neq j} r_k \leq r, \forall v_j$$

- Sell each item separately at price $r$:

$$\text{TAIL} \leq \sum_j \sum_{v_j \geq r} f_j(v_j) \cdot r = \sum_j r \cdot \Pr_{v_j}[v_j \geq r] \leq \sum_j r_j \leq r$$
NON-FAVORITE: Bounding the CORE

- $\text{CORE} = \sum_j \sum_{v_j \leq r} f_j(v_j) \cdot v_j = E[v']$

- **Lemma:** $\text{Var}[v'] \leq 2r \cdot r$

- **Corollary:** $\text{Var}[v'] = \sum_j \text{Var}[v'_j] \leq 2r^2$

- **Chebyshev Inequality:** for any random variable $X$, $\text{Pr}[|X - E[X]| \geq a] \leq \frac{\text{Var}[X]}{a^2}$.

- By Chebyshev Inequality,

  $$\text{Pr}[v' < \text{CORE} - 2r] \leq \frac{\text{Var}[v']}{4r^2} \leq \frac{1}{2}$$

- $\text{Pr} [\sum_j v_j \geq \text{CORE} - 2r] \geq 1/2$. If selling the grand bundle at price $\text{CORE} - 2r$, the bidder will buy it with prob. $\geq 1/2$.

- $2\text{BREV} + 2r \geq \text{CORE}$
REV ≤ \( \max \limits_{x \in P} L(\lambda, x, p) \) ≤ SINGLE + TAIL + CORE

- SINGLE ≤ r
- TAIL ≤ r
- CORE ≤ 2BREV + 2r

[BILW ’14] Optimal Revenue ≤ 2BREV + 4SREV.