6.891	Lecture 11	March 17, 2015

PROPHET INEQUALITY:

Def: A stopping rule τ w.r.t. random variables $X_1, \ldots, X_n$ is a r.v. τ w/ values in $\{1, 2, \ldots, n\}$ and the property that for all $t \in \{1, \ldots, n\}$ the occurrence or non-occurrence of event $\tau = t$ only depends on the values of $X_1, \ldots, X_t$.

[Krengel-Sucheston-Garling '79]: If $X_1, \ldots, X_n$ are independent, non-negative there exists a stopping rule τ w.r.t. $X_1, \ldots, X_n$ such that

$$\mathbb{E}[X_\tau] \geq \frac{1}{2} \mathbb{E}[\max X_i]$$

Proof:
- Let $F_1, \ldots, F_n$ be the distn's of $X_1, \ldots, X_n$
- Let $\tau(\mathfrak{Z}) = \arg\min_t \{X_t \geq \mathfrak{Z}\}$ for some $\mathfrak{Z}$ TBD

  (i.e. stop the first time you see a sample $\geq \mathfrak{Z}$)
- Let $q(\mathfrak{Z}) = \Pr[X_t < \mathfrak{Z}, \forall t] \equiv \prod_t F_t(\mathfrak{Z}^-)$
- $\mathbb{E}[X_{\tau(\mathfrak{Z})}] \geq$

$$\mathfrak{Z} \cdot \boxed{\Pr[\exists t, t' \text{ s.t. } X_t \geq \mathfrak{Z}, X_{t'} \geq \mathfrak{Z}]} + \underbrace{\sum_{t=1}^{n} \mathbb{E}\left[X_t \,\middle|\, \begin{matrix} X_t \geq \mathfrak{Z} \wedge \\ X_{t'} < \mathfrak{Z}, \forall t' \neq t \end{matrix}\right] \cdot \Pr\left[\begin{matrix} X_t \geq \mathfrak{Z} \wedge \\ X_{t'} < \mathfrak{Z}, \forall t' \neq t \end{matrix}\right]}_{(*)}$$

$$(*) = \sum_{t=1}^{n} \mathbb{E}[X_t \mid X_t \geq \tau] \cdot \Pr[X_t \geq \tau] \cdot \prod_{t' \neq t} \Pr[X_{t'} < \tau]$$

$$= \sum_{t=1}^{n} \left(\mathbb{E}[X_t - \tau \mid X_t \geq \tau] + \tau \right) \cdot \Pr[X_t \geq \tau] \cdot \prod_{t' \neq t} F_{t'}(\tau^-)$$

$$= \sum_{t=1}^{n} \left(\mathbb{E}\left[(X_t - \tau)_+ \right] \cdot \prod_{t' \neq t} F_{t'}(\tau^-) + \tau \cdot \left(1 - F_t(\tau^-)\right) \prod_{t' \neq t} F_{t'}(\tau^-) \right)$$

$$\geq q(\tau) \cdot \sum_{t=1}^{n} \mathbb{E}\left[(X_t - \tau)_+ \right] + \tau \cdot \Pr\left[\exists!\ t \text{ s.t. } X_t \geq \tau \right]$$

- Hence: $\mathbb{E}[X_{T(\tau)}] \geq (1 - q(\tau)) \cdot \tau + q(\tau) \cdot \sum_{t=1}^{n} \mathbb{E}\left[(X_t - \tau)_+ \right]$ $(**)$

- On the other hand,

$$\mathbb{E}\left[\max X_t \right] = \mathbb{E}\left[\tau + \max(X_t - \tau) \right]$$

$$\leq \tau + \mathbb{E}\left[\max (X_t - \tau)_+ \right]$$

$$\leq \tau + \sum_{t=1}^{n} \mathbb{E}\left[(X_t - \tau)_+ \right] \qquad (***)$$

Suppose exits τ s.t. $q(\tau) = 1 - q(\tau) = 1/2$. Proof then follows from $(**)$ and $(***)$.

→ If such ξ doesn't exist find ξ s.t.

$$q(\xi) \equiv \mathbb{P}\mathrm{r}\left[X_t < \xi, \forall t\right] \le \frac{1}{2} \le \mathbb{P}\mathrm{r}\left[X_t \le \xi, \forall t\right] \equiv q^+(\xi) \quad (**)$$

Compare the revenue from stopping rules:

$$\tau^+(\xi) = \arg\min_t \{X_t > \xi\}$$

$$\text{and} \quad \tau(\xi) = \arg\min_t \{X_t \geq \xi\}$$

want to show that at least one of these two guarantees $\frac{1}{2}$-approximation

Using the following derivation as for $\mathbb{E}[X_{\tau(\xi)}]$

$$\mathbb{E}\left[X_{\tau^+(\xi)}\right] \geq \xi \cdot \underline{\mathbb{P}\mathrm{r}\left[\exists t, t' \text{ s.t. } X_t > \xi, X_{t'} > \xi\right]} + \underbrace{\sum_{t=1}^{n} \mathbb{E}\left[X_t \,\middle|\, \begin{array}{l} X_t > \xi \wedge \\ X_{t'} \le \xi, \forall t' \neq t \end{array}\right] \cdot \mathbb{P}\mathrm{r}\left[\begin{array}{l} X_t > \xi \wedge \\ X_{t'} \le \xi, \forall t' \neq t \end{array}\right]}_{(*)}$$

$$(*) = \sum_{t=1}^{n} \mathbb{E}\left[X_t \mid X_t > \xi\right] \cdot \mathrm{Pr}\left[X_t > \xi\right] \cdot \prod_{t' \neq t} \mathrm{Pr}\left[X_{t'} \le \xi\right]$$

$$= \sum_{t=1}^{n} \left(\mathbb{E}\left[X_t - \xi \mid X_t > \xi\right] + \xi\right) \cdot \mathrm{Pr}\left[X_t > \xi\right] \cdot \prod_{t' \neq t} F_{t'}(\xi)$$

$$= \sum_{t=1}^{n} \left(\mathbb{E}\left[(X_t - \xi)_+\right] \cdot \prod_{t' \neq t} F_{t'}(\xi) + \xi \cdot \left(1 - F_t(\xi)\right) \prod_{t' \neq t} F_{t'}(\xi)\right)$$

$$\geq \underline{q^+(\xi)} \cdot \sum_{t=1}^{n} \mathbb{E}\left[(X_t - \xi)_+\right] + \xi \cdot \underline{\mathbb{P}\mathrm{r}\left[\exists!\, t \text{ s.t. } X_t > \xi\right]}$$

- <u>Hence</u>: $\mathbb{E}\left[X_{\tau^+(\zeta)}\right] \geq \left(1-q^+(\zeta)\right)\cdot \zeta + q^+(\zeta)\cdot \sum_{t=1}^{n} \mathbb{E}\left[(X_t-\zeta)_+\right]$ $(\ast\ast\ast\ast\ast)$

- Recall from $(\ast\ast)$ that:

$$\mathbb{E}\left[X_{\tau(\zeta)}\right] \geq \left(1-q(\zeta)\right)\cdot \zeta + q(\zeta)\cdot \sum_{t=1}^{n} \mathbb{E}\left[(X_t-\zeta)_+\right] \quad (\ast\ast)$$

and from $(\ast\ast\ast)$ that:

$$\mathbb{E}\left[\max X_t\right] \leq \zeta + \sum_{t=1}^{n} \mathbb{E}\left[(X_t-\zeta)_+\right] \quad (\ast\ast\ast)$$

- Case analysis: - If $\zeta \geq \sum_{t=1}^{n} \mathbb{E}\left[(X_t-\zeta)_+\right]$, then it follows from $(\ast\ast\ast)$, $(\ast\ast)$ & $(\ast\ast\ast\ast)$ that

$$\mathbb{E}\left[X_{\tau(\zeta)}\right] \geq \frac{1}{2}\,\mathbb{E}\left[\max X_t\right]$$

- If $\zeta \leq \sum_{t=1}^{n} \mathbb{E}\left[(X_t-\zeta)_+\right]$, then it follows from $(\ast\ast\ast)$, $(\ast\ast\ast\ast\ast)$ & $(\ast\ast\ast\ast)$ that

$$\mathbb{E}\left[X_{\tau^+(\zeta)}\right] \geq \frac{1}{2}\,\mathbb{E}\left[\max X_t\right].$$

$\boxtimes$

Corollary to Bulow-Klemperer '96:

For any regular distn' F and any n:

$$\mathbb{E}_{v_1,\dots,v_n\sim F}\left[\text{Rev}(\text{Vickrey})\right] \geq \left(1-\frac{1}{n}\right)\mathbb{E}_{v_1,\dots,v_n\sim F}\left[\text{Rev}(\text{Myerson})\right]$$

Proof: • Suffices to show:

$$\mathbb{E}_{v_1,\dots,v_{n-1}\sim F}\left[\begin{matrix}\text{Rev of Myerson}\\ \text{on } v_1,\dots,v_{n-1}\end{matrix}\right] \geq \left(1-\frac{1}{n}\right)\mathbb{E}_{v_1,\dots,v_n\sim F}\left[\begin{matrix}\text{Rev of Myerson}\\ \text{on } v_1,\dots,v_n\end{matrix}\right]$$

• Sample $v_1,\dots,v_{n-1}$ and $v_1', v_2', \dots, v_{n-1}', v_n'$ as follows:

- draw iid samples $u_1,\dots,u_n$ from F
- draw random permutation π
- set $v_i = u_{\pi(i)}, \ \forall\, i=1,\dots,n-1$
  $v_i' = u_{\pi(i)}, \ \forall\, i=1,\dots,n$

• Under above sampling:

① $v_1,\dots,v_{n-1}$ iid from F; hence:

$$\mathbb{E}_{\vec{u},\pi}\left[\begin{matrix}\text{Virtual Welfare of Myerson's}\\ \text{auction on } v_1,\dots,v_{n-1}\end{matrix}\right] = \mathbb{E}_{v_1,\dots,v_{n-1}\sim F}\left[\begin{matrix}\text{Revenue of Myerson's}\\ \text{auction on } v_1,\dots,v_{n-1}\end{matrix}\right]$$

② $v_1',\dots,v_n'$ iid from F; hence:

$$\mathbb{E}_{\vec{u},\pi}\left[\begin{array}{c}\text{VW of Myerson's}\\ \text{auction on } v_1',\dots,v_n'\end{array}\right] = \mathbb{E}_{v_1',\dots,v_n' \sim F}\left[\begin{array}{c}\text{Revenue of Myerson's}\\ \text{auction on } v_1',\dots,v_n'\end{array}\right]$$

③ For each realization of $\vec{u}$:

$$\mathbb{E}_{\pi}\left[\begin{array}{c}\text{Virtual Welfare of Myerson's}\\ \text{auction on } v_1,\dots,v_{n-1}\end{array}\right] \geq$$

$$\left(1-\frac{1}{n}\right)\cdot \mathbb{E}_{\pi}\left[\begin{array}{c}\text{Virtual Welfare of Myerson's}\\ \text{auction on } v_1',\dots,v_n'\end{array}\right]$$