COMP/MATH 533: Algorithmic Game Theory October 20, 2014

Lecture 13

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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Arrow's theorem

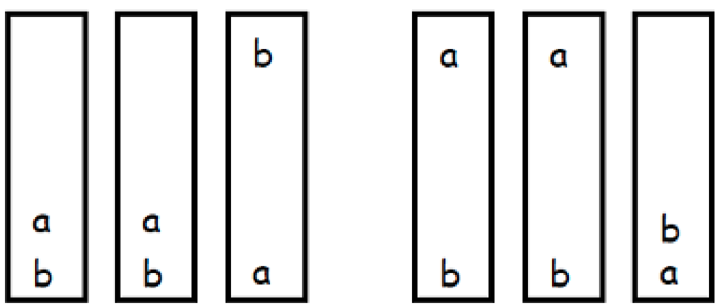
Theorem 1. Arrows theorem *states that there exists no social welfare function (which aggregates everyones preference order into one common order) that can satisfy all three desirable properties of a voting scheme.*

- *Unanimity* - if everyone has the same preference or order the social welfare function should give the same order in its output.

  i.e. $W(<, <, \ldots, <) =<, \forall < in\, L$ Where W is the social welfare function that aggregates voters preferences into a common order and L is the set of linear orders in the alternatives (permutation).

- *Independence of irrelevant alternatives (IIA)* - The order betweens any two alternatives a and b should be only dependant on what everyone votes for a and b and should be independent of the presence and order of any other alternatives. i,e if the social welfare function ranks a above b for the current votes, and if we change the votes, but do not change the order between a and b in each of the vote, the function should still rank a above b in its output.

  In the pictorial example below the set of three votes on both left and right should produce the same outcome by the social welfare function for the ordering/preference between a and b irrespective of the presence/absence of other alternatives between them among the two scenarios.

- *Not dictatorial* - There should not be a person i in the system who can dictate the outcome of the social welfare function according to his own preference or order independent of the choices of others.

  i.e there should not be a voter i such that

  $W(<_1, <_2, \ldots, <_n) = <_i, \forall <_1, <_2, \ldots, <_n \, in\, L$

Arrows impossibility theorem states that if there are at least three candidates, the three properties listed above cannot be simultaneously satisfied.

1.1 Proof of Arrow's theorem

The proof will be obtained through sequence of Lemmas, where we will see that if we insist on unanimity and independence of irrelevant alternatives, then we will get a dictatorship in the social welfare function.

Lemma 1 (Pairwise Unanimity)**.** *if everyone thinks a is better than b, the social welfare function should think that a is better than b*

Proof: Let's construct a voting profile $<^{'} = (a, b, c, d, e, \ldots)$

If everyone submits $<^{'}$, by unanimity, we know that $W(<^{'}, <^{'}, <^{'}, ... <^{'}, <^{'}) = <^{'}$.

On the other hand, $\forall\, i\; a <_i b \Leftrightarrow a <^{'} b$

By IIA, we know $a < b \Leftrightarrow a <^{'} b$. Thus, $a < b$. □

Lemma 2 (Neutrality)**.** *Let $<_1, <_2, \ldots, <_n$ and $<_1^{'}, <_2^{'}, \ldots, <_n^{'}$ be two profiles, $<= W(<_1, <_2, \ldots, <_n)$ and $<^{'} = W(<_1^{'}, <_2^{'}, \ldots, <_n^{'})$. If $\forall i\ a <_i b \Leftrightarrow c <_i^{'} d$, then $a < b \Leftrightarrow c <^{'} d$.*

Proof: We prove this Lemma by case analysis.

Case 1: $c \neq b$. We only show the case where $a < b$, the case where $a > b$ can be proved similarly.

Create a single preference π_i from $<_i$ and $<_i^{'}$: where c is just below a and d just above b.

i.e.

- if $a <_i b$ then $d >_{\pi_i} b >_{\pi_i} a >_{\pi_i} c$

- if $b <_i a$ then $a >_{\pi_i} c >_{\pi_i} d >_{\pi_i} b$

Input all π_i's to the social welfare function W and we get $<_\pi$.

We must have:

- Since $a <_i b \Leftrightarrow a <_{\pi_i} b$, by IIA, we have $a <_\pi b \Leftrightarrow a < b \because a <_\pi b$. Thus, $a <_\pi b$.
- $c <_\pi a$ from construction of statement of proof as c is always lower than a.
- $b <_\pi d$ from construction of statement of proof as b is always lower than d.

$\therefore c <_\pi d$ using above constructs.

Since $c <_{\pi_i} d \Leftrightarrow c <_i^{'} d (a <_i b \Rightarrow c <_i^{'} d$, similarly for $b <_i a$), by IIA, $c <_\pi d \Leftrightarrow c <^{'} d$. Thus, $c <^{'} d$.

Case 2: $c = b$.

Again we will show the case where $a < b$, you are asked to prove the $a > b$ case in Problem set 2.

Create a single preference $<_i^*$ for all i based on $<_i$,

- if $a <_i b$, create $<_i^*$ such that $d >_i^* a >_i^* b = c$.

- if $a >_i b$ create $<_i^*$ such that $a >_i^* b = c >_i^* d$.

Notice that $a <_i b \Leftrightarrow a <_i^* d$ for all i. Since $a \neq b$, we can apply Case 1 on this case[1] and show $a < b \Leftrightarrow a <^* d$. Since $a < b$, $a <^* d$. Now we look at a and c, as $a >_i^* c$ for all i, Lemma 1 tells us $a >^* c$. Therefore, $d >^* c$.

On the other hand, because $c <_i^* d \Leftrightarrow c <_i^{'} d$, by IIA, $c >^{'} d \Leftrightarrow c >^* d$. Thus, $c <^{'} d$.

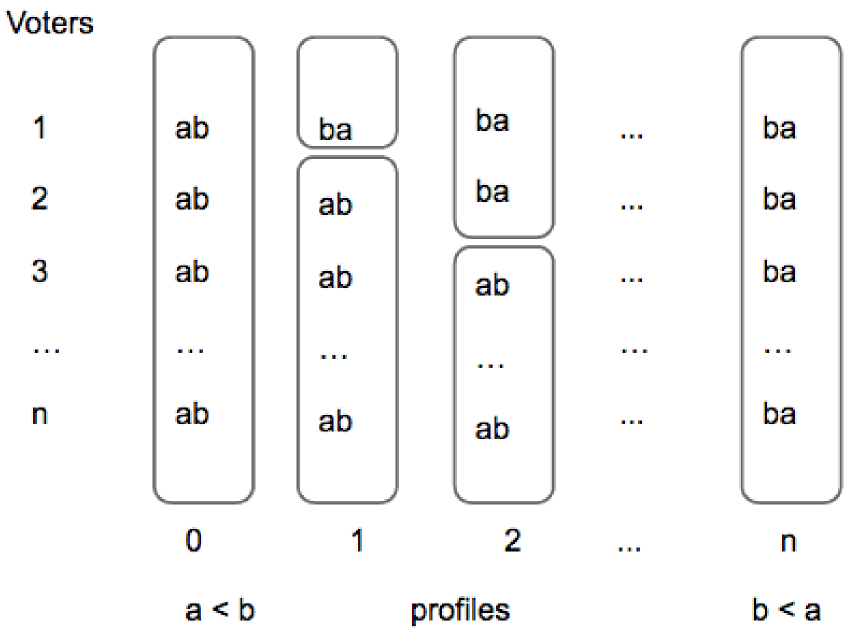
□

1.2 Proof of Arrow's theorem: $i*$ is the dictator

Consider a sequence of votes, we care only about a and b in this case. In profile 0, everyone favors a over b. In profile 1, we switch the order for voter 1, and in profile 2, we do it for voter 2 as well and so forth till n^{th} round.

Now we have $n + 1$ profiles.

[1]Think of $<_i^*$ is the order $<^{'}$, and a is the candidate "c" in Case 1.



We know that in profile $0, a$ is above b and in n^{th} profile, b is above a. This implies that at some point the order must have swapped.

This means that the change must have happened at least at some profile i^* (the first profile in which $b < a$)

Claim 1. *Claim : i^* is the dictator. we can show that social welfare function will always use i^*'s profile. for any $<_1, <_2, \ldots, <_n$ and $<= W(<_1, <_2, \ldots, <_n)$ and c, d in A. If $c <_{i^*} d$ then $c < d$.*

Proof: Take $e \neq c$, d and create a new preference list $<_i^{'}$ for all i by modifying $<_i$ in the following way:

- for $i < i^*$, move e to the bottom of the preference list $<_i^{'}$
- for $i > i^*$, move e to the top of the preference list $<_i^{'}$
- for $i = i^*$, move e to the middle of c and d, such that $c <_i^{'} e <_i^{'} d$.

Now compare the votes in $<_{i'}$ for c, e with the votes for a, b in the $(i^* - 1)$-th preference profile

Voters	$<_i^{'}$	$<_{\pi_i^{(i^*-1)}}$
1	ec	ba
2	ec	ba
$\vdots$	$\ldots$	$\ldots$
$i^* - 1$	ec	ba
i^*	ce	ab
$i^* + 1$	ce	ab
$\vdots$	$\ldots$	$\ldots$
n	ce	ab

Let $<^{'} = W(<_1^{'}, \ldots, <_n^{'})$ and $<_{\pi^{(i^*-1)}} = W(<_{\pi_1^{(i^*-1)}}, \ldots, <_{\pi_n^{(i^*-1)}})$. Since $\forall i, e <_i^{'} c \Leftrightarrow b <_{\pi_i^{(i^*-1)}} a$, by Lemma 2, $e <^{'} c \Leftrightarrow b <_{\pi^{(i^*-1)}} a$. Thus, $c <^{'} e$ (1).

Similarly, we compare the votes for e and d in $<_{i'}$ with the votes for a, b in the i^*-th preference profile.

Voters	$<_i^{'}$	$<_{\pi_i^{i^*}}$
1	ed	ba
2	ed	ba
$\vdots$	...	...
$i^* - 1$	ed	ba
i^*	ed	ba
$i^* + 1$	de	ab
$\vdots$	...	...
n	de	ab

Since $e <_i^{'} d \Leftrightarrow b <_{\pi_i^{i^*}} a$
$e <^{'} d \Leftrightarrow b < a$
$\Rightarrow e <^{'} d$ (2)
Combining (1) and (2), we know $c <^{'} d$.
And hence i^* is the dictator

$\square$