COMP/MATH 533: Algorithmic Game Theory September 8, 2014

Lecture 2: Mechanism Design Basics

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NOTE: The content of these notes has not been formally reviewed by the lecturer. It is recommended that they are read critically.

1 Preliminaries

Mechanism design is the science of rule making. How is one to make good rules so as to ensure that given goals are achieved?

Where usually systems are given and predictions must be made, mechanism design investigates the converse: given some end goals, a system must be constructed to satisfy these goals.

In this course, one may separate systems into two categories: those that are influenced by money and those which are not. An example of the former case are auctions whereas an example of the latter is the construction of a system aimed at sending children into designated kindergartens. Indeed, understandably, money should not influence acceptance of a child. Other examples include online auctions websites such as eBay, online advertisement spots sales and spectrum auctions.

Example 1. *An* online marketplace is a web platform that aggregates products from multiple third-parties under a single system and offers its users a mechanism to purchase these products. Products are often purchased through an auction. A few examples of online marketplaces include ebay.com, priceline.com, hotwire.com and skyauction.com. It is worth noting that different online marketplaces use different auction formats (first-price sealed bid, Vickrey auction, etc.) depending on their various goals.

Example 2. *A* sponsored search refers to advertisers' websites being explicitly listed alongside search engines' results. The search queries trigger relevant ads to be displayed. Search engines organize auctions where advertisers bid on the placement of their ads on the search engines' pages.

Example 3. *A* spectrum auction refers to the process whereby a government auctions the licenses to transmit signals of a certain range to telecommunication companies. Countries are often divided into various geographical regions, each with their own license that companies bid for. As with online marketplaces, different spectrum auctions use different auction formats.

2 Single Item Auction

A simple first step into the study of game theory in auctions carries one to the single item auction event.

Definition 1. *A* single item auction *is an auction where the auctioneer has a single item to sell and in which n bidders participate. Bidder i considers the item to be of a certain value v_i to him/her. The values of $v_1, \cdots, v_n$ are private and known only to the corresponding bidder. The goal is for the auctioneer to give the item to the bidder who values it the most, i.e. with highest v_i. A natural thing to do is to give the item to the highest bidder, but the choice of p the selling price of the item is to be chosen with care. Note that the auctioneer informs the bidders of how p will be chosen before they bid. Then, one may define a quasilinear utility function for each bidder i as follows:*

$$u_i = \begin{cases} v_i - p & \text{if bidder } i \text{ wins} \\ 0 & \text{if bidder } i \text{ loses} \end{cases}$$

Definition 2. *A* sealed-bid auction *is an auction in which all bidders submit their bids b_i at the same time so that no one knows the amounts that were bid. The auctioneer collects the bids and chooses who wins and sells it at the price p as defined previously.*

Remark 1. *The choice of p matters. Suppose that the auctioneer sets $p = 0$ regardless of the bids b_i. Then the strategy for the bidders is to bid for the highest possible number among all other bidders which comes back to the problem of who can find the biggest real number.*

Theorem 1. *If two bidders are playing against each other in a sealed-bid single item auction, the selling price p of the item is set to be $p = \max(b_1, b_2)$ and v_i is drawn from a uniform distribution on the interval $[0, 1]$ then bidding $b_i = \frac{1}{2}v_i$ for $i \in \{1, 2\}$ is a Nash equilibrium.*

Proof: Without loss of generality, assume you are bidder 1. v_1 is known but v_2 is only known to be drawn from a uniform distribution on the interval $[0, 1]$. Suppose that bidder 2 bids $\frac{v_2}{2}$ and you are to bid b_1. Let u_1 be the quasilinear utility function as previously seen.

Then

$$
\begin{aligned}
u_1 &= \mathrm{P}\left(b_1 > \frac{v_2}{2}\right) \cdot (v_1 - b_1) \\
&= \mathrm{P}\left(2b_1 > v_2\right) \cdot (v_1 - b_1) \\
&= 2b\left(v_1 - b_1\right)
\end{aligned}
$$

By completion of squares, one gets

$$
u_1 = \frac{v_1^2}{2} - 2\left(b_1 - \frac{v_1}{2}\right)^2
$$

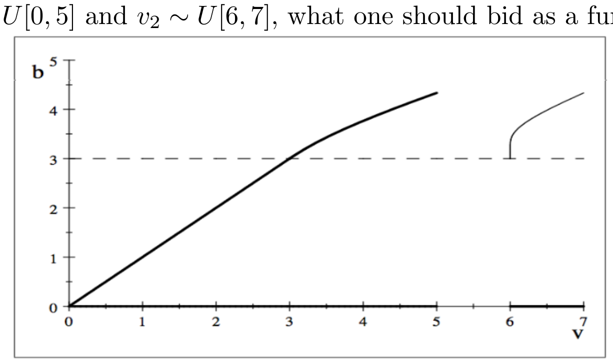
which is maximized when $b_1 = \frac{v_1}{2}$. Alternatively, one could compute $\frac{\partial u_1}{\partial b_1}$ and compare with 0 to find extrema of u_1 which should yield the same answer. □

Remark 2. *The theorem above asserted a Nash equilibrium in the case of 2 bidders but for n bidders, $b_i = \frac{v_i}{n}$ is the Nash equilibrium (the proof is similar).*

This simplistic model allows for a concise answer but when more difficult mechanisms appear, the Nash equilibria are harder to compute. For example, if the v_i were not drawn from a uniform distribution from 0 to 1, but rather from an unknown distribution F then $b_i(v_i) = \mathbb{E}\left(\max_{j:j\neq i} v_j | v_j \leq v_i\right)$.

Example 4. *Suppose for example that v_1 is drawn from a uniform distribution on $[0, 5]$ and v_2 is drawn from a uniform distribution on $[6, 7]$ then figure 1 shows the bid that bidder 1 should submit as a function of his/her value.*

Figure 1: When $v_1 \sim U[0, 5]$ and $v_2 \sim U[6, 7]$, what one should bid as a function of his/her value



In the case where each v_i are drawn from different unknown distributions, then the Nash equilibria become more complicated and can be seen in figure 2.

Figure 2: An example of a more complex Nash equilibrium

$$v_1(b) = \frac{36}{(2b-6)\left(\frac{1}{5}\right)e^{\frac{9}{4}+\frac{6}{6-2b}} + 24 - 4b},$$

$$v_2(b) = 6 + \frac{36}{(2b-6)(20)e^{-\frac{9}{4}-\frac{6}{6-2b}} - 4b}.$$

3 Second Price / Vickrey Auction

Definition 3. *The* second price *or* Vickrey auction *is similar to the single item auction described in definition 1 except that p is set to be the second highest bid.*

Lemma 1. *In the second price or Vickrey auction, every bidder has a dominant strategy which is to set $b_i = v_i$. This maximizes bidder i's utility u_i. Furthermore, $u_i \geq 0$ in all cases.*

Proof: Suppose you are bidder i. Let $B = max_{j:j\neq i} b_j$ that is the maximum bid excluding yours. Then consider the following cases - the columns indicate how much you bid, the rows indicate how big is your value compared to B and the cells indicate your utility u_i :

	$b_i = v_i$	$b_i > v_i$	$b_i < v_i$
Case 1: $v_i > B$	$v_i - B > 0$	$v_i - B > 0$	$\begin{cases} v_i - B > 0 & \text{if } b_i > B \\ 0 & \text{if } b_i \leq B \end{cases}$
Case 2: $v_i \leq B$	0	$\begin{cases} v_i - B \leq 0 & \text{if } b_i > B \\ 0 & \text{if } b_i \leq B \end{cases}$	0

In all cases, bidding v_i yields the best results with regards to the alternatives. □

4 Some Desirable Properties

The second price or Vickrey auction has three desirable properties of mechanisms.

4.1 Dominant-Strategy Incentive Compatible

Abbreviated to DSIC, the best strategy to use in mechanisms that are DSIC is to act truthfully. Then one is guaranteed to have a utility greater than or equal to 0. In the second price or Vickrey auction, bidding true to your value v_i yielded the best results.

4.2 Performance Guarantee

The design of any system has a certain goal in mind and one hopes to design in such a way that the goal is maximized. An example of goal is to maximize the system designer's revenue. Alternatively, one may have a goal to maximize the social welfare for reasons such as eliminating competition by offering the most to clients.

Social Welfare: This property says that a model should maximize the population's happiness. In the auctions' case, the population is satisfied at best when the item is given to the person that values it the most. Such is the case in the second price or Vickrey auction as the winner is the one who values the most the item, with the highest v_i.

4.3 Polynomial Time

The computation of the p value is done in a linear time as it needs to scan for all the bids and get the second highest.

These properties are desirable in all mechanisms and as such will be a set of goals one must keep in mind in model design.

5 A Case Study: Sponsored Search Auction

An application of the auction concept can be seen in the domain of web advertising. On a given webpage, there are k slots where advertisements can be put, which can be compared to k items at an auction. However, each slot j has a different click-through rate (CTR) α_j which corresponds to the probability that a user will click on that ad if the ad is displayed and varies depending on the slot. Furthermore, each bidder i has some value v_i which corresponds to the value he/she gets from having a user click on his/her ad. Then for bidder i, the total value of having slot j becomes $\alpha_j v_i$.

Complications one might see are the multiple items, their different CTR's. The goals remain similar to those described in section 4.