Markets
“Economics is a science which studies human behavior as a relationship between ends and scarce means which have alternative uses.”

Lionel Robbins (1898 – 1984)
How are scarce resources assigned to alternative uses?
How are scarce resources assigned to alternative uses?

*Prices!*

Parity between demand and supply

*equilibrium prices*
the beginnings of a mathematical theory
Adam Smith (1723-1790)

The Wealth of Nations (1776): the “invisible hand of the economy”

“By pursuing his own interest he frequently promotes that of the society more effectually than when he really intends to promote it.”

“It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest.”
standpoint, a subject of general interest which has so many
different sides.

But the title of this work sets forth not only theoretical
researches; it shows also that I intend to apply to them the
forms and symbols of mathematical analysis. This is a plan
likely, I confess, to draw on me at the outset the condem-
nation of theorists of repute. With one accord they have
set themselves against the use of mathematical forms, and it
will doubtless be difficult to overcome to-day a prejudice
which thinkers, like Smith and other more modern writers,
have contributed to strengthen. The reasons for this preju-
dice seem to be, on the one hand, the false point of view
from which theory has been regarded by the small number
of those who have thought of applying mathematics to it;
and, on the other hand, the false notion which has been
formed of this analysis by men otherwise judicious and well
versed in the subject of Political Economy, but to whom the
mathematical sciences are unfamiliar.
Cournot’s Contributions:

notion of a *demand function*

\[ D = F(p), \text{ where } p \text{ is the price; } F(.) \text{ is assumed continuous, and it is} \]

taken as an empirical proposition that the demand function is decreasing

analysis of a *monopoly*:

- profit-maximizing producer with cost \( f(D) \), for production \( D \);
  discusses decreasing, constant and increasing cost functions

- equations determining *equilibrium price*

*duopoly* model:

two rival producers of a homogeneous product

*unlimited competition, communication of markets on single commodity, ...*
react on each other. An increase in the income of the producers of commodity $A$ will affect the demand for commodities $B$, $C$, etc., and the incomes of their producers, and, by its reaction, will involve a change in the demand for commodity $A$. It seems, therefore, as if, for a complete and rigorous solution of the problems relative to some parts of the economic system, it were indispensable to take the entire system into consideration. But this would surpass the powers of mathematical analysis and of our practical methods of calculation, even if the values of all the constants could be assigned to them numerically. The object
Elements of Pure Economics, or the theory of social wealth, 1874
Leon Walras (1834-1910)

- goal was to solve the problem that was left open by Cournot, i.e. characterize the price equilibrium of a market with many commodities;

- gave system of simultaneous equations for price equilibrium:

  informal argument for the existence of an equilibrium based on the assumption that an equilibrium exists whenever the number of equations equals the number of unknowns

- recognized the need for \textit{dynamics leading to an equilibrium} \textit{tâtonnement}: price adjustment mechanism
Irving Fisher (1867-1947)

hydraulic apparatus for solving markets
with 3 traders with money and a
producer of 3 commodities
Irving Fisher (1867-1947)

*First computational approach!*

1891
Irving Fisher (1867-1947)

Stock Market Crash of 1929:

"Stock prices have reached what looks like a permanently high plateau."

[Shortly before the crisis]

Market is “only shaking out of the lunatic fringe”

[October 21, 1929 (8 days before black Tuesday)]
Arrow-Debreu-McKenzie Theorem

One of the most celebrated theorems in Mathematical Economics.

Established the existence of a market equilibrium under very general conditions using Brouwer/Kakutani’s fixed point theorem. 

it is hence highly non-constructive!
Kenneth Arrow

• Nobel Prize, 1972
Gerard Debreu

- Nobel Prize, 1983
Exchange Market Model (without production)

Consider a marketplace with:

- \( n \) traders (or agents)
- \( k \) goods (or commodities)
  assumed to be \textit{infinitely divisible}

\textit{Utility function} of trader \( i \):

\[ u_i : \mathcal{X}_i \subseteq \mathbb{R}_+^k \rightarrow \mathbb{R}_+ \]

\( \mathcal{X}_i \) non-negative reals
\( \mathbb{R}_+ \) consumption set for trader \( i \)
specifies trader \( i \)'s utility for bundles of goods

\textit{Endowment} of trader \( i \):

\[ e_i \in \mathcal{X}_i \]

amount of goods trader comes to the marketplace with
Suppose the goods in the market are priced according to some price vector \( p \in \mathbb{R}_+^k \).

Under this price vector, each trader would like to sell some of her endowment and purchase an optimal bundle using her income from what s/he sold; thus she solves the following program:

\[
\text{Program}_i(p)
\]

\[
\begin{align*}
\text{max } & \quad u_i(x) \\
\text{s.t. } & \quad p \cdot x \leq p \cdot e_i \\
& \quad x \in \mathcal{X}_i
\end{align*}
\]

**Note:** If \( u_i \) is continuous and \( \mathcal{X}_i \) is compact, then the above program has a well-defined optimum value.
Def: A price vector $p \in \mathbb{R}^k_+$ is called a *competitive market equilibrium* iff there exists a collection of optimal solutions $x_i(p)$ to Program$_i(p)$, for all $i = 1, \ldots, n$, such that the total supply meets the total demand, i.e.

$$\sum_{i=1}^{n} x_i(p) \leq \sum_{i=1}^{n} e_i$$

where $e_i$ represents the total demand and $x_i(p)$ represents the total supply.
Theorem [Arrow-Debreu 1954]: Suppose

(i) $\mathcal{X}_i$ is closed and convex

(ii) $e_i >> 0$, for all $i$ (all coordinates positive)

(iii a) $u_i$ is continuous

(iii b) $u_i$ is quasi-concave

\[ u_i(x) > u_i(y) \implies u_i(\lambda x + (1 - \lambda)y) > u_i(y), \ \forall \lambda \in (0, 1) \]

(iii c) $u_i$ is nonsatiated

\[ \forall y \in \mathcal{X}_i, \ \exists x \in \mathcal{X}_i \text{ s.t. } u_i(x) > u_i(y) \]

Then a competitive market equilibrium exists.
Market Clearing

Nonsatiation + quasi-concavity

⇒ at equilibrium every trader spends all her budget, i.e. if \( x_i(p) \) is an optimal solution to Program \( i(p) \) then

\[
p \cdot x_i(p) = p \cdot e_i
\]

\[
\Rightarrow p \cdot \left( \sum_i x_i(p) - \sum_i e_i \right) = 0
\]

⇒ every good with positive price is fully consumed
A market with no equilibrium

Alice has oranges and apples, but only likes apples.

Bob only has oranges, and likes both oranges and apples.

- if oranges are priced at 0, then Bob’s demand is not well-defined.

- if oranges are priced at > 0, then Alice wants more apples than there are in the market.
Proof of the Arrow-Debreu Theorem

Steps (details on the board)

(i) w.l.o.g. can assume that the $\mathcal{X}_i$ are compact

(ii) by compactness and strict concavity: for all $p$, there exists a unique maximizer $x_i(p)$ of Program$_i(p)$

(iii) by the maximum theorem: $x_i(p)$ is continuous on $p$

(iv) rest of the argument on the board
Utility Functions

Linear utility function (goods are perfect substitutes)

\[ u_i(x) = \sum_j a_{ij} x_j \]

Leontief (or fixed-proportion) utility function

\[ u_i(x) = \min_j \{ a_{ij} x_j \} \]

  e.g. buying ingredients to make a cake
  e.g. rate allocation on a network

Cobb-Douglas utility function

\[ u_i(x) = \prod_j x_j^{a_{ij}}, \quad \text{where} \quad \sum_j a_{ij} = 1 \]

Interpretation: the per-unit fraction of buyer \( i \)'s income used in purchasing good \( j \) is proportional to \( a_{ij} \). i.e. at optimality \( x_{ij} p_j = c a_{ij} \).
Utility Functions

CES utility functions:

\[ u_i(x) = \left( \sum_j u_{ij} \cdot x_j^\rho \right)^{\frac{1}{\rho}} , \quad -\infty < \rho \leq 1 \]

Convention:
- If \( u_{ij} = 0 \), then the corresponding term in the utility function is always 0.
- If \( u_{ij} > 0 \), \( x_j = 0 \), and \( \rho < 0 \), then \( u_i(x) = 0 \) no matter what the other \( x_j \)'s are.

\[ \rho = 1 \quad \Rightarrow \quad \text{linear utility form} \]
\[ \rho \to -\infty \quad \Rightarrow \quad \text{Leontief utility form} \]
\[ \rho \to 0 \quad \Rightarrow \quad \text{Cobb-Douglas form} \]

**elasticity of substitution:** \( \sigma = \frac{1}{1 - \rho} \)
Fisher’s Model

Suppose all endowment vectors are parallel…

\[ e_i = m_i \cdot e, \quad m_i > 0, \quad m_i : \text{scalar}, \ e : \text{vector} \]

⇒ relative incomes of the traders are independent of the prices.

Equivalently, we can imagine the following situation:

- \( n \) traders, with specified money \( m_i \)
- \( k \) divisible goods owned by seller; seller has \( q_j \) units of good \( j \)

Arrow-Debreu Thm  ➔

(under the Arrow-Debreu conditions) there exist prices that the seller can assign on the goods so that the traders spend all their money to buy optimal bundles and supply meets demand
Fisher’s Model with CES utility functions

\[ u_i(x_i) = \left( \sum_j u_{ij} \cdot x_{ij}^\rho \right)^{\frac{1}{\rho}}, \quad -\infty < \rho \leq 1 \]

- Buyers’ optimization program (under price vector $p$):

\[
\max \quad u_i(x_i) \\
\text{s.t. } \sum_j x_{ij} p_j \leq m_i
\]

- Global Constraint:

\[
\sum_i x_{ij} \leq q_j, \quad \forall j \\
x_{ij} \geq 0, \quad \forall j
\]
Eisenberg-Gale’s Convex Program

- The space of feasible allocations is:

\[ \sum_{i} x_{ij} \leq q_j, \quad \forall j \]
\[ x_{ij} \geq 0, \quad \forall j \]

- But how do we aggregate the trader’s optimization problems into one global optimization problem?

  e.g., choosing as a global objective function the sum of the traders’ utility functions won’t work…