COMP 553: Algorithmic Game Theory

Lecture 20

Fall 2014

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In last lecture, we showed Nash's theorem that a Nash equilibrium exists in every game.

In our proof, we used Brouwer's fixed point theorem as a Black-box.

In today's lecture, we explain Brouwer's theorem, and give an illustration of Nash's proof.

We proceed to prove Brouwer's Theorem using a combinatorial lemma, called Sperner's Lemma, whose proof we also provide.

Brouwer's Fixed Point Theorem

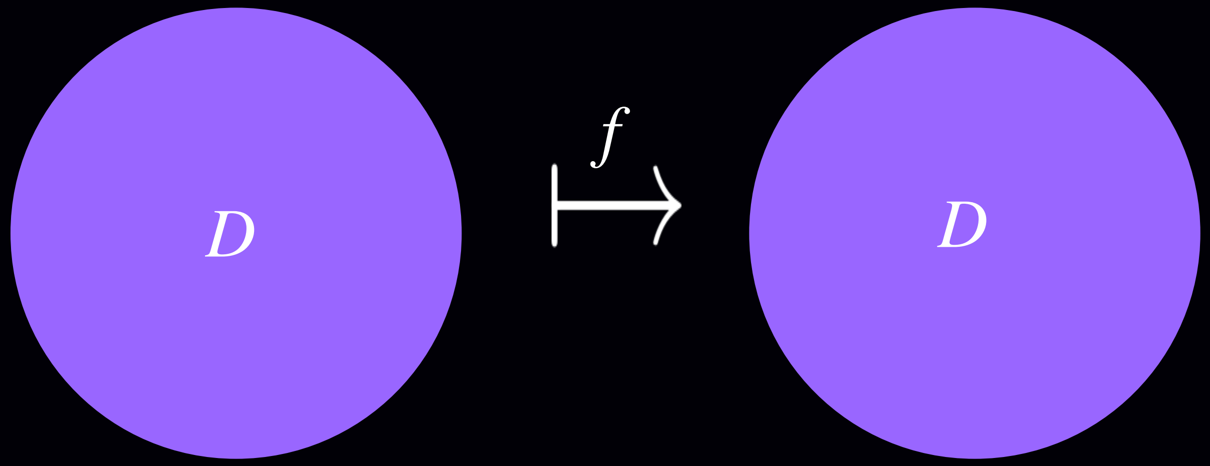
Brouwer's fixed point theorem

Theorem: Let $f : D \longrightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

Then there exists an $x \in D$ s.t. $x = f(x)$.

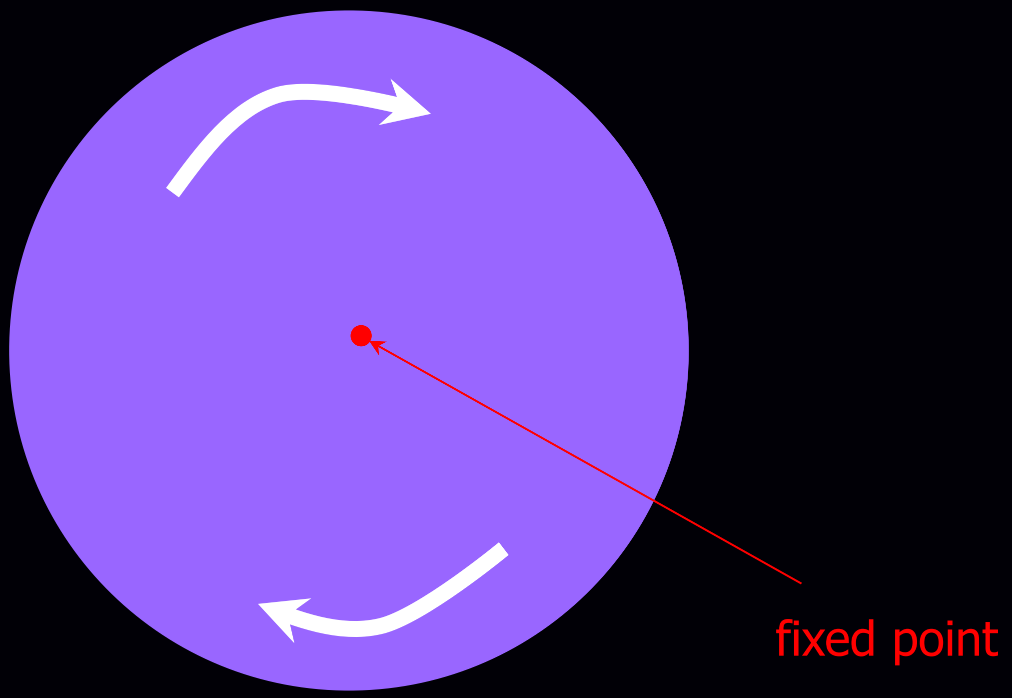
closed and bounded

Below we show a few examples, when D is the 2-dimensional disk.

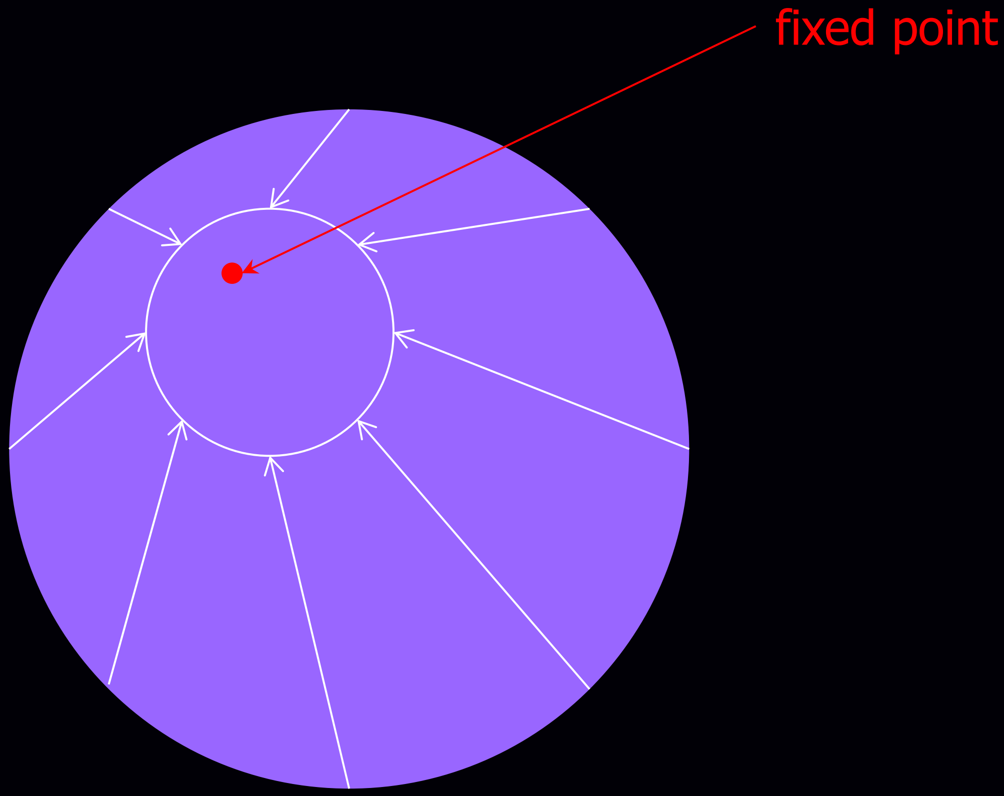


N.B. All conditions in the statement of the theorem are necessary.

Brouwer's fixed point theorem



Brouwer's fixed point theorem



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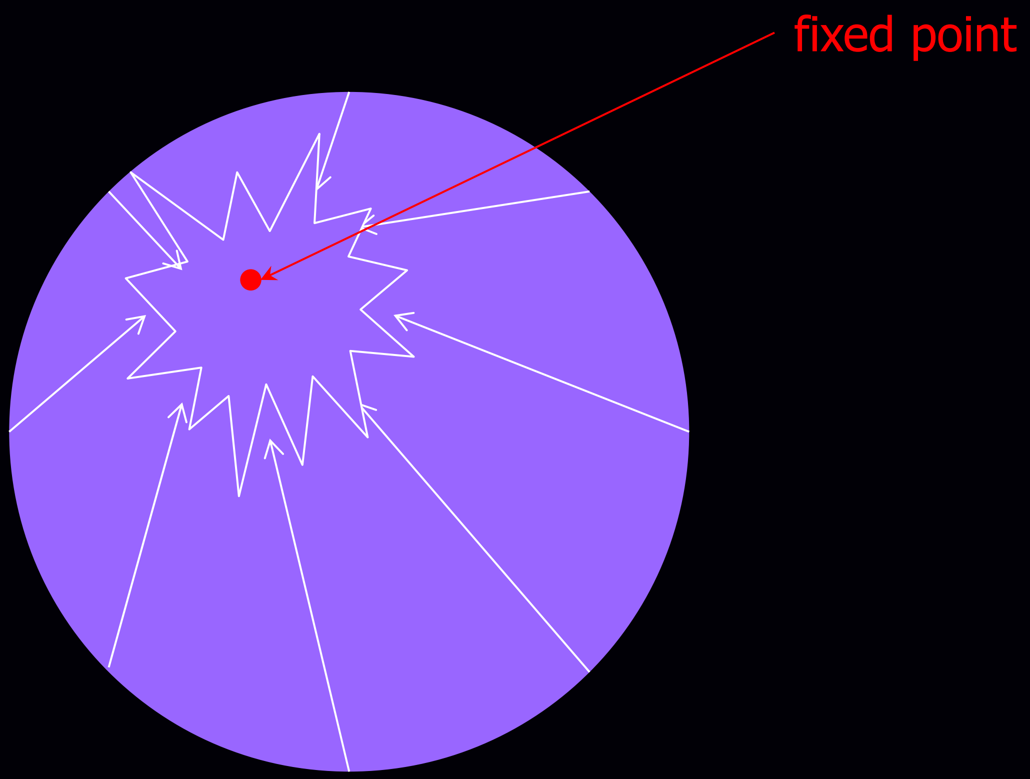

Nash's Proof

Nash's Function

$$\Delta \ni x \overset{f}{\mapsto} y \in \Delta :$$

$$y_p(s_p) := \frac{x_p(s_p) + \text{Gain}_{p;s_p}(x)}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p;s'_p} x)}$$

where: $\text{Gain}_{p;s_p}(x) = \max\{u_p(s_p; x_{-p}) - u_p(x), 0\}$

Visualizing Nash's Construction

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

$\mapsto$

f: $[0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

Visualizing Nash's Construction

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
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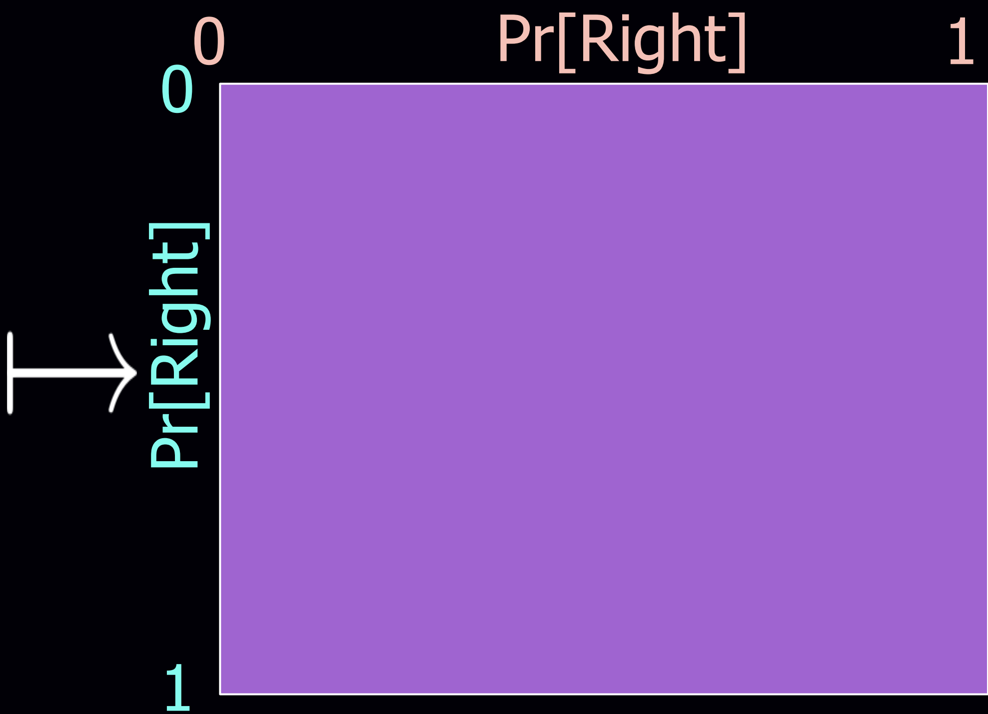
Penalty Shot Game



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Penalty Shot Game

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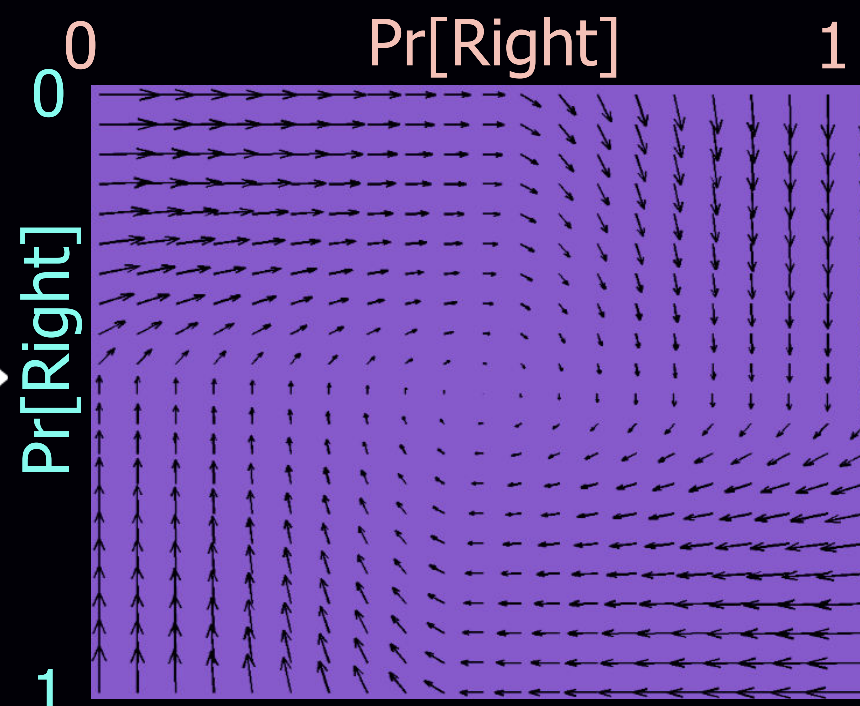

Visualizing Nash's Construction

Kick / Dive	Left	Right
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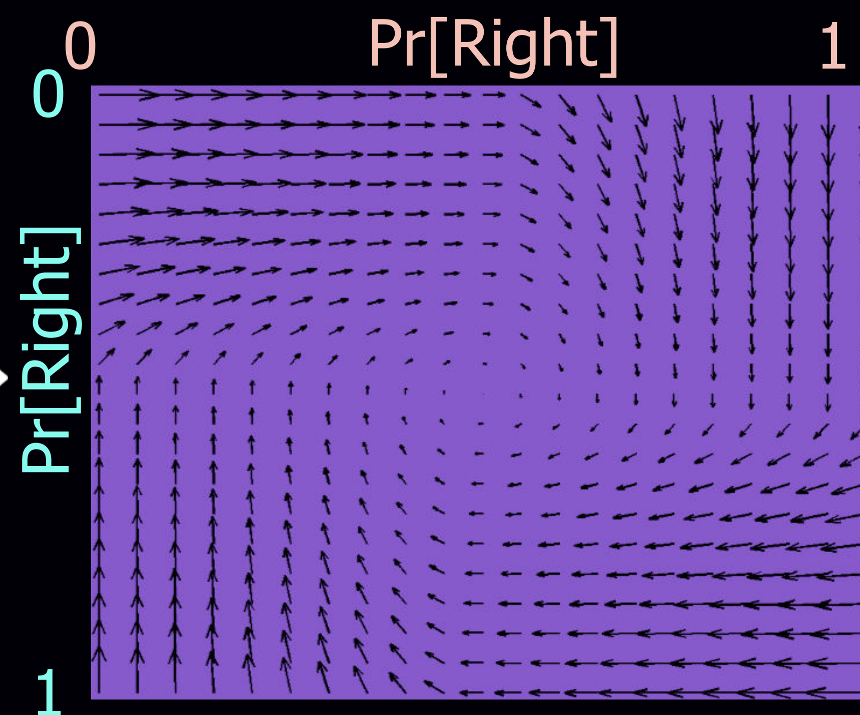
Penalty Shot Game

$\longmapsto$

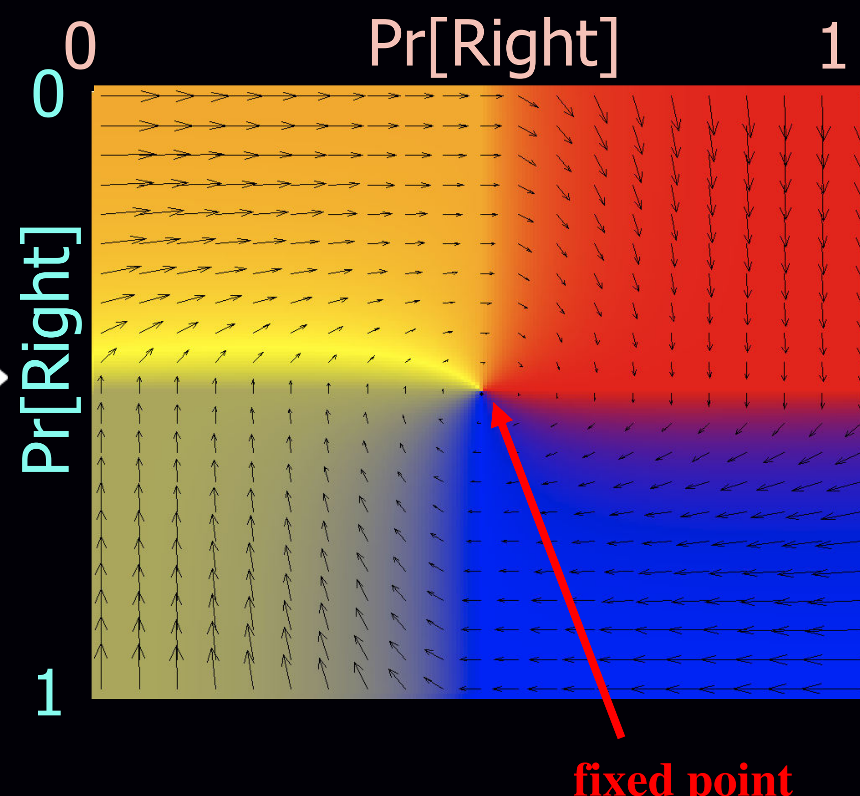


Visualizing Nash's Construction

	Kick / Dive	Left	Right
		½	½
½	Left	1 , -1	-1 , 1
½	Right	-1 , 1	1, -1

Penalty Shot Game

↦



Sperner's Lemma

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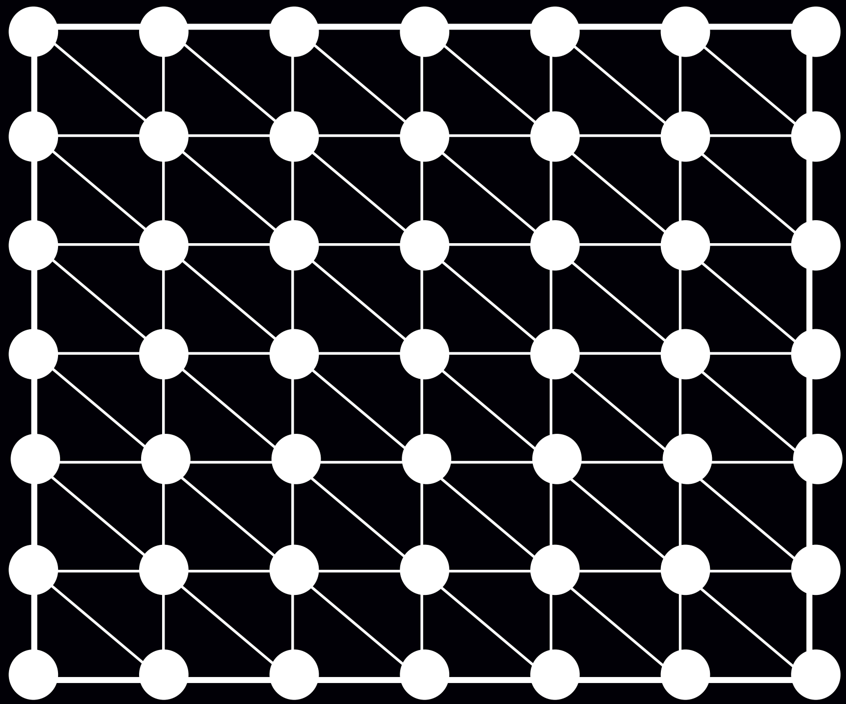
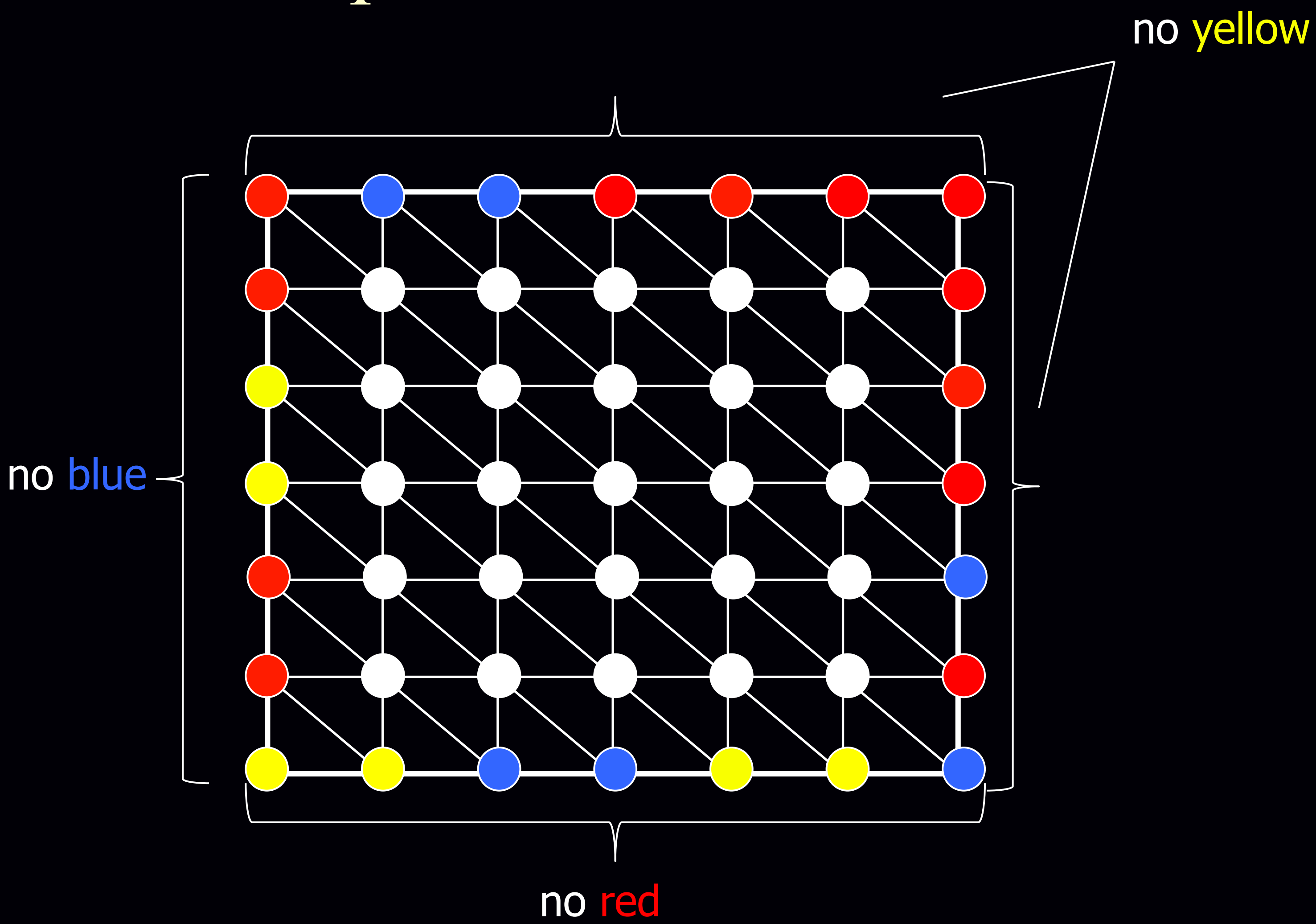
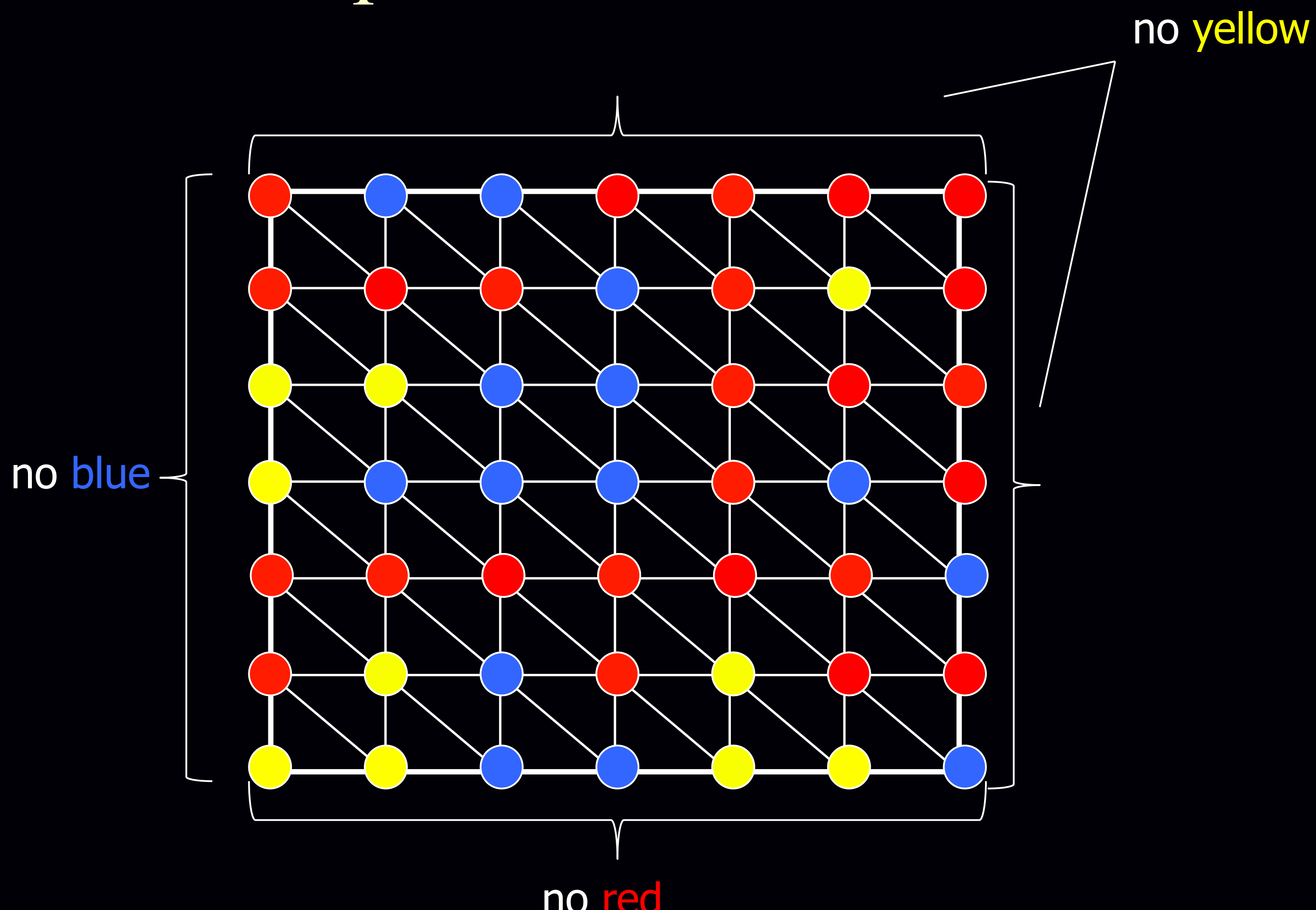


Sperner's Lemma



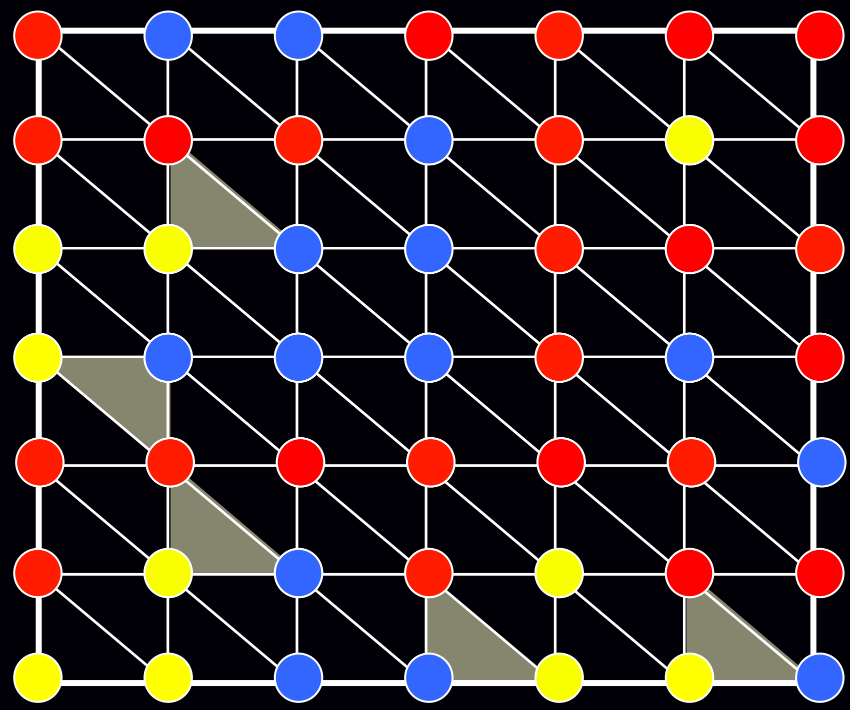
Lemma: Color the boundary using three colors in a legal way.

Sperner's Lemma



Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Sperner's Lemma



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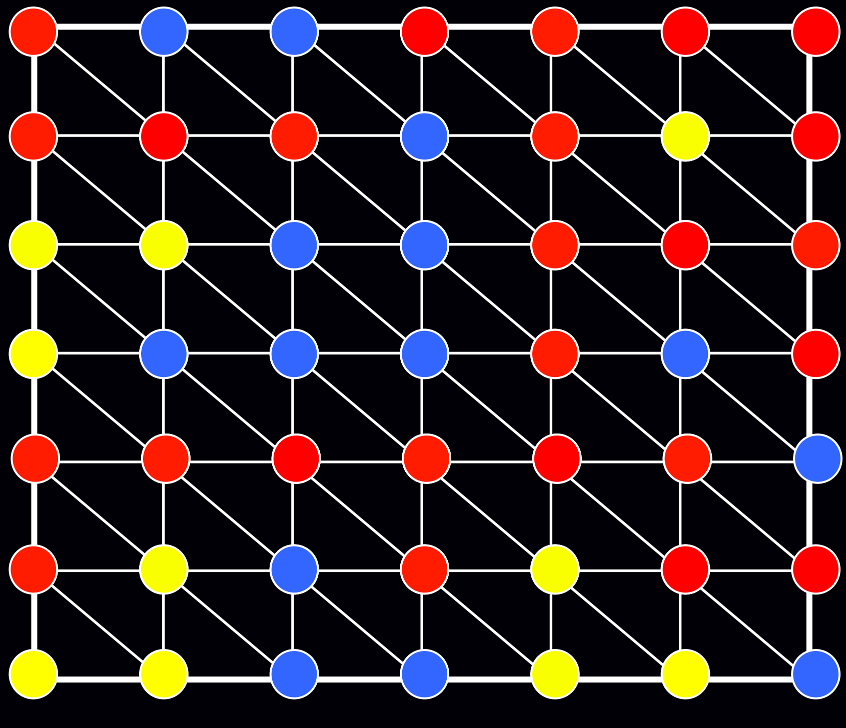
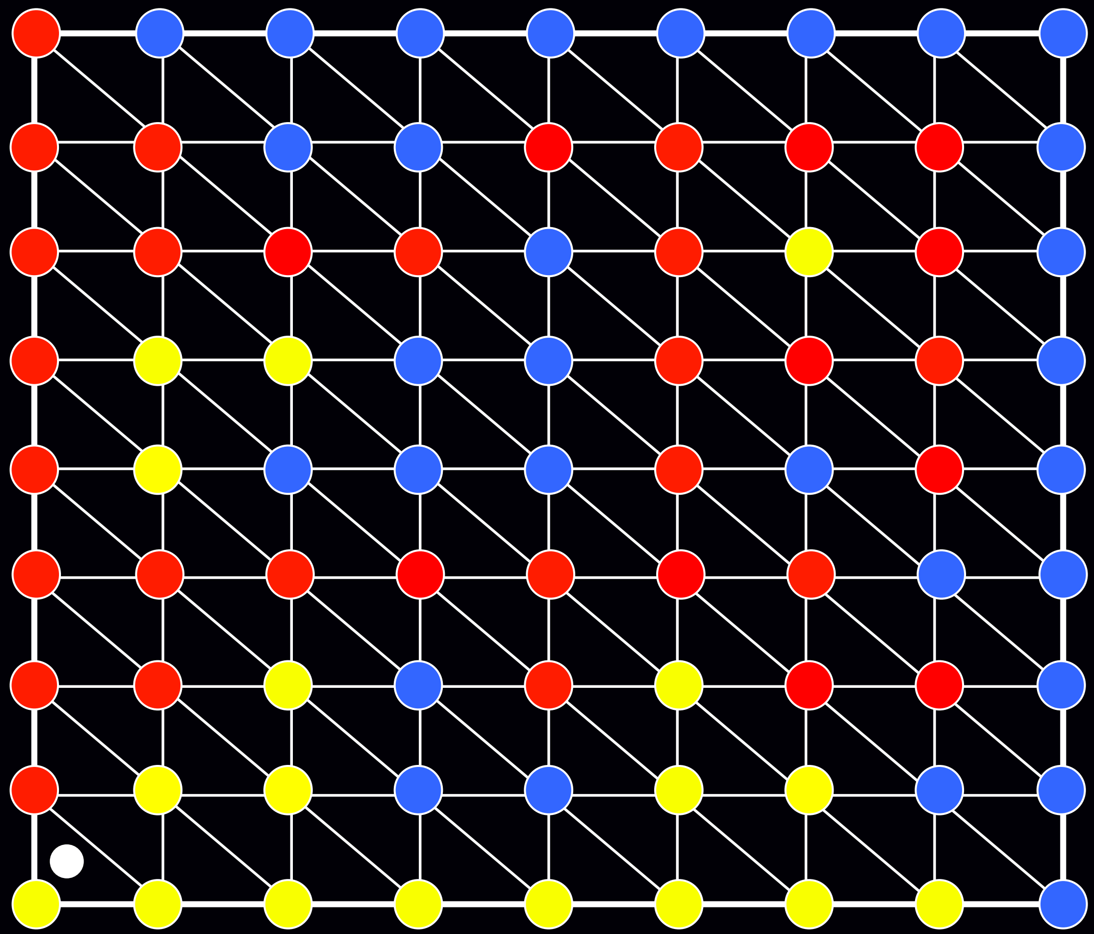
Sperner's Lemma



Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma



For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

Next we define a directed walk starting from the bottom-left triangle.

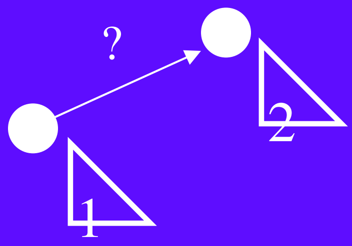
Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Space of Triangles

Transition Rule:

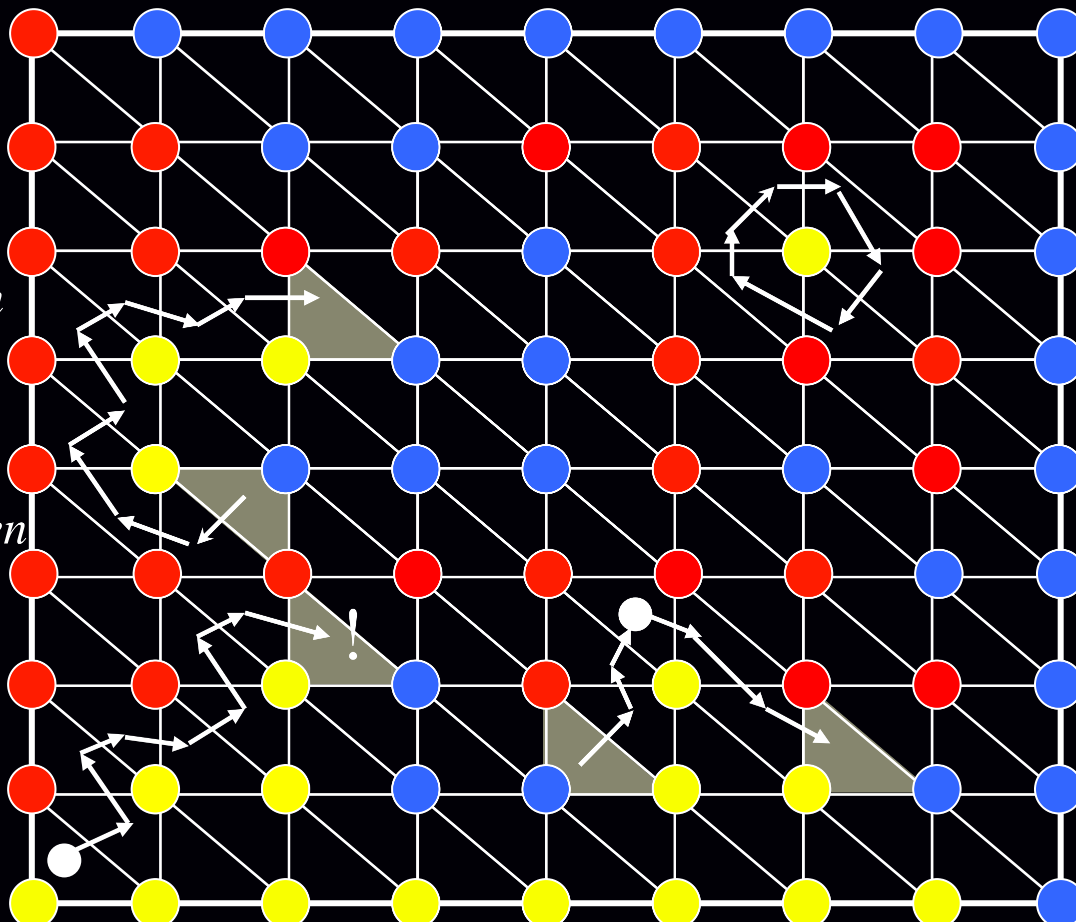
*If ∃ **red** - **yellow** door cross it with **red** on your left hand.*



Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma

Claim**:** *The walk cannot exit the square, nor can it loop around itself in a rho-shape. Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle…*



For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

Next we define a directed walk starting from the bottom-left triangle.

Starting from other triangles we do the same going forward or backward.

Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Brouwer's Fixed Point Theorem

We show that Sperner's Lemma implies Brouwer's Fixed Point Theorem. We start with the 2-dimensional Brouwer problem on the square.

2D-Brouwer on the Square

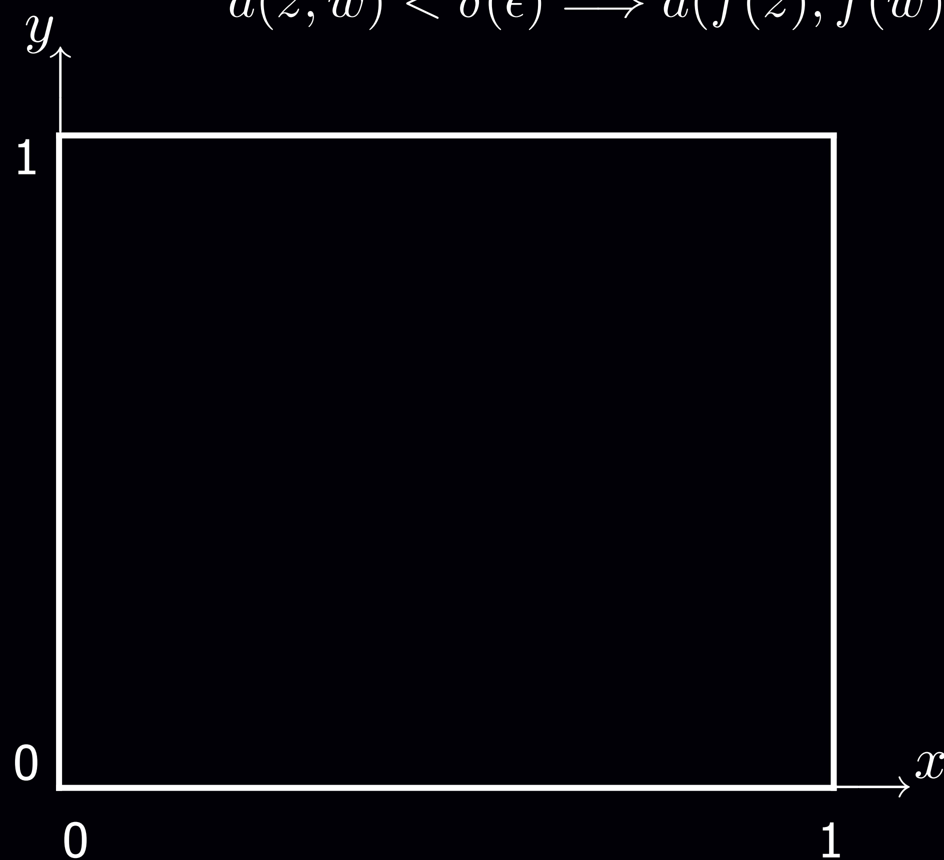
say d is the ℓ_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

↳ must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0, \; \exists \delta(\epsilon) > 0, s.t.$$

$$d(z, w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$



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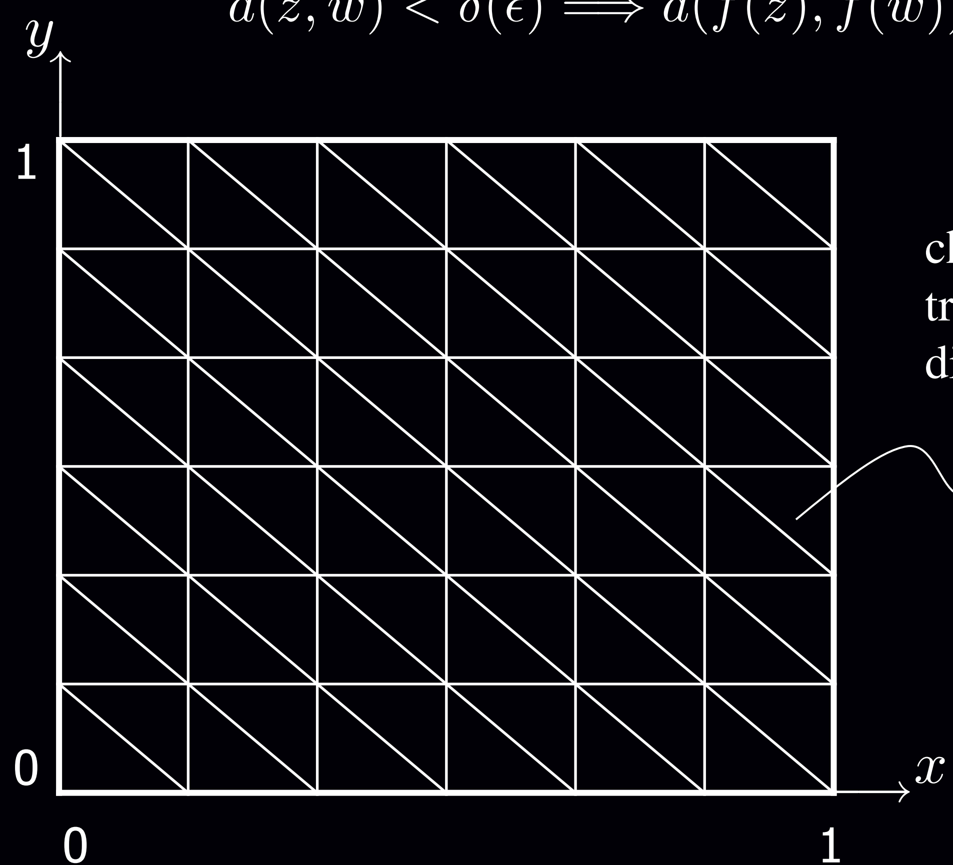

choose some ϵ and triangulate so that the diameter of cells is $\delta < \delta(\epsilon)$

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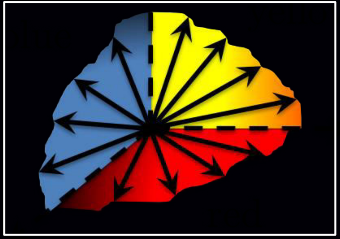
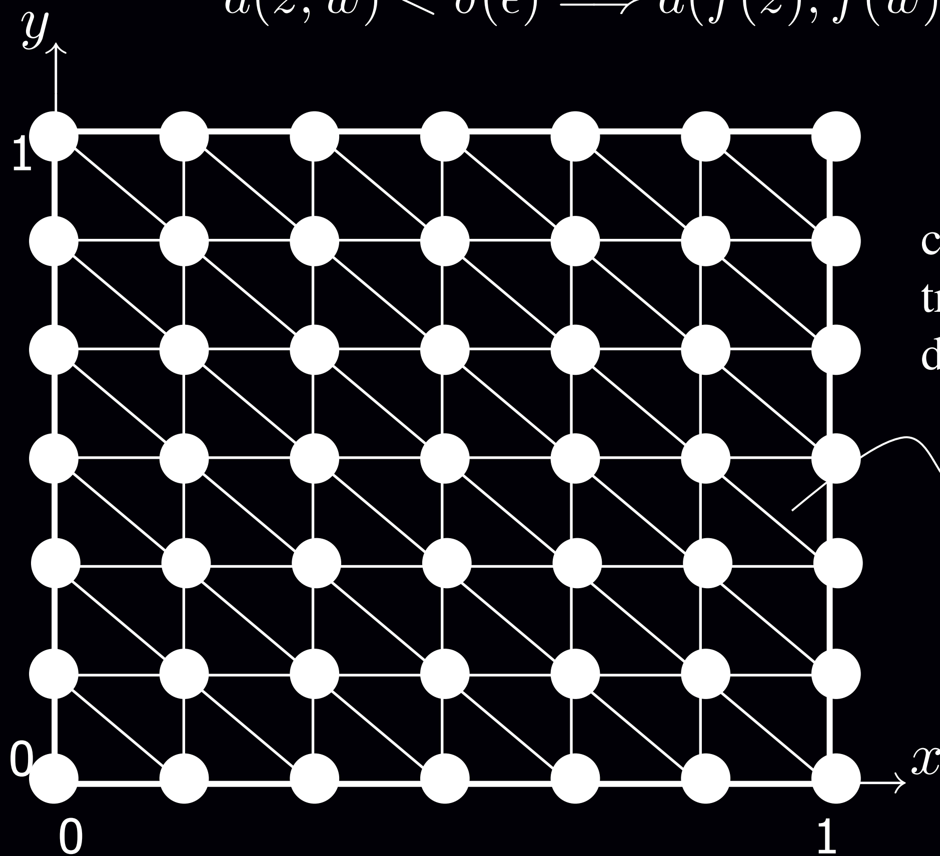
color the nodes of the triangulation according to the direction of

$$f(x) - x$$





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color the nodes of the triangulation according to the direction of

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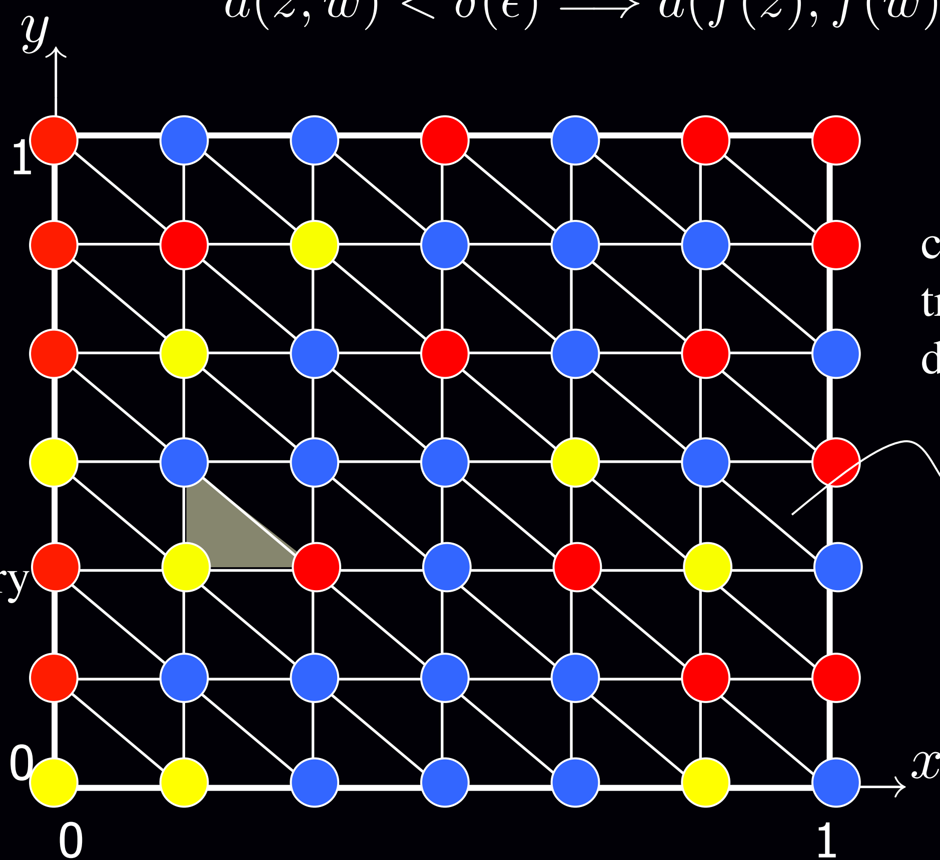
tie-break at the boundary angles, so that the resulting coloring respects the boundary conditions required by Sperner's lemma



choose some ϵ and triangulate so that the diameter of cells is

$$\delta < \delta(\epsilon)$$

find a trichromatic triangle, guaranteed by Sperner

2D-Brouwer on the Square

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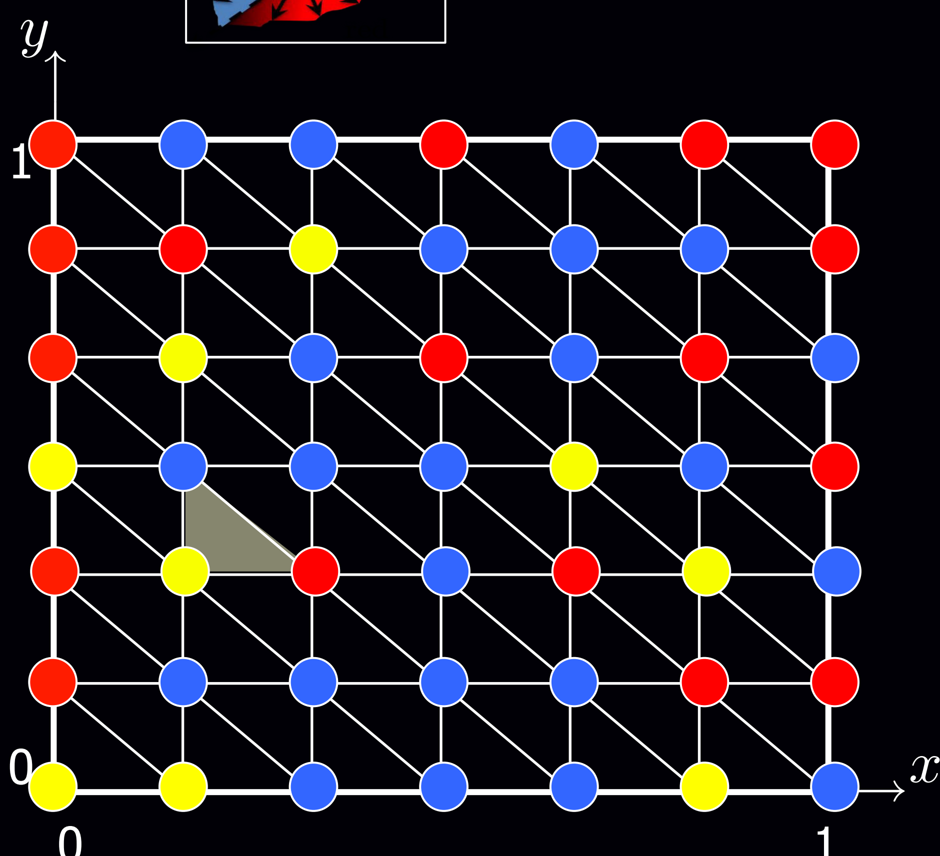
$$d(z,w) < \delta(\epsilon) \Longrightarrow d(f(z), f(w)) < \epsilon$$



Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

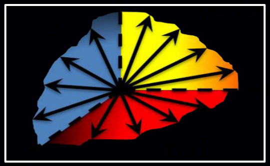
Proof of Claim

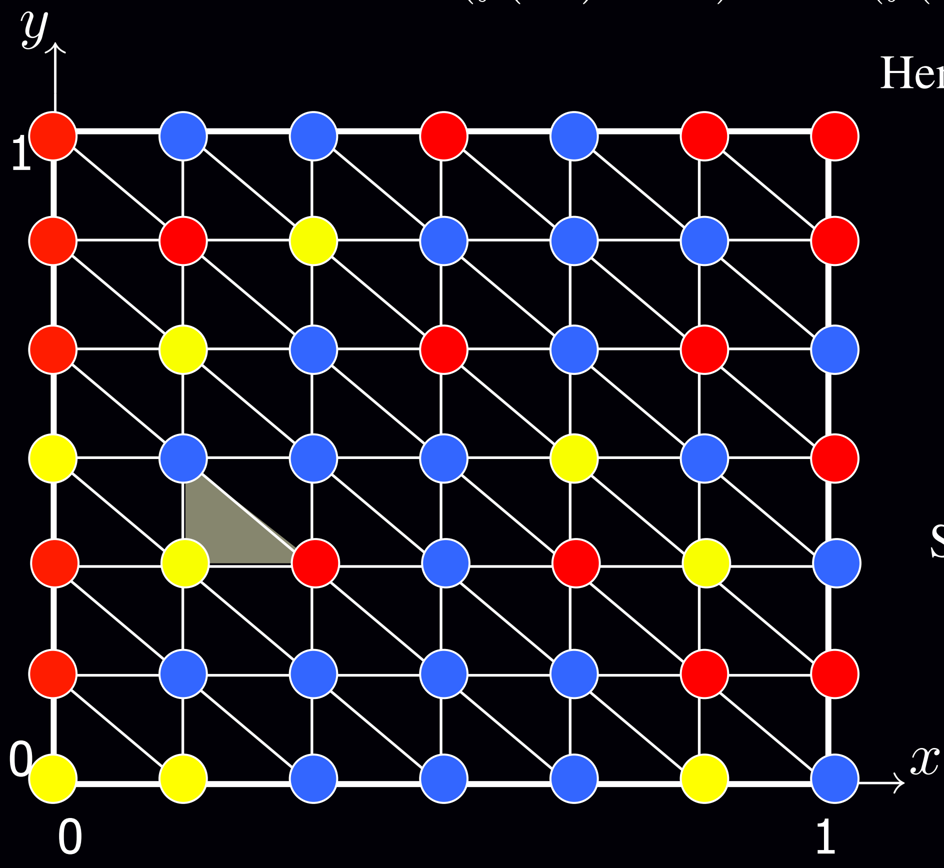
Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then $|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta$.

Proof: Let z^{Y}, z^{R}, z^{B} be the yellow/red/blue corners of a trichromatic triangle.

By the definition of the coloring, observe that the product of

$(f(z^Y) - z^Y)_x$ and $(f(z^B) - z^B)_x$ is ≤ 0.





Hence:

$$|(f(z^Y) - z^Y)_x|$$
$$\leq |(f(z^Y) - z^Y)_x - (f(z^B) - z^B)_x|$$
$$\leq |(f(z^Y) - f(z^B))_x| + |(z^Y - z^B)_x|$$
$$\leq d(f(z^Y), f(z^B)) + d(z^Y, z^B)$$
$$\leq \epsilon + \delta.$$

Similarly, we can show:

$$|(f(z^Y) - z^Y)_y| \leq \epsilon + \delta.$$

∎

2D-Brouwer on the Square

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→ must be uniformly continuous (by the Heine-Cantor theorem)

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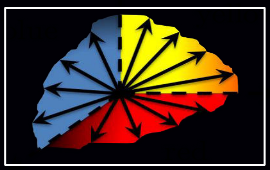

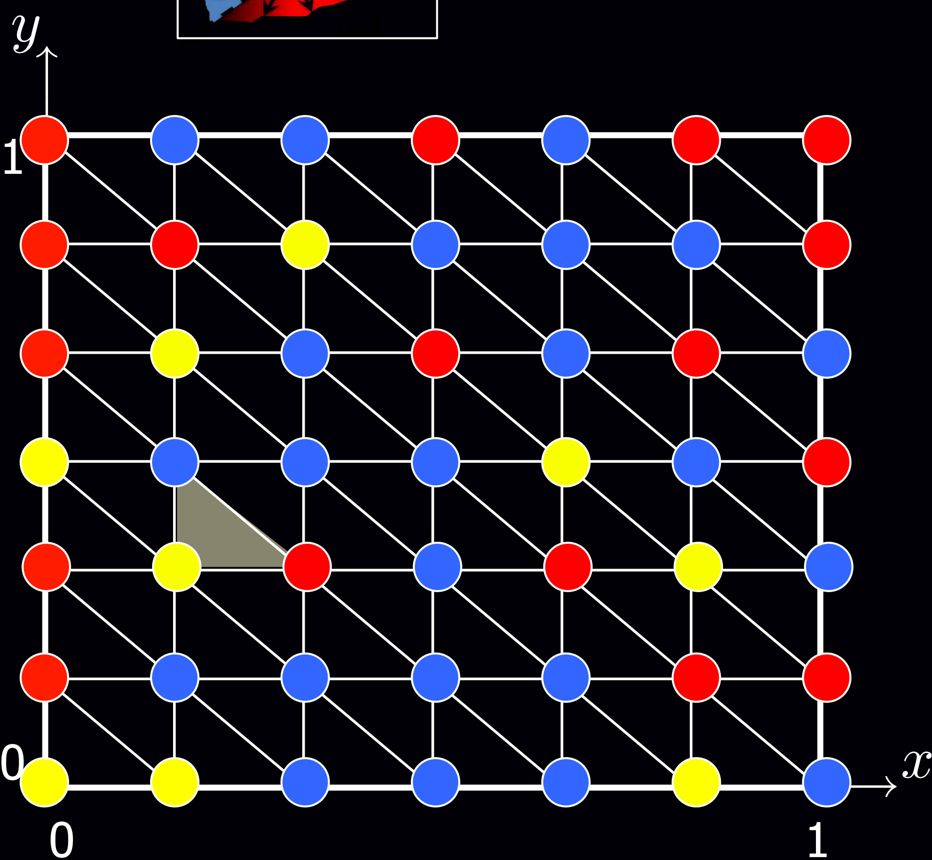

Claim: If z^{Y} is the yellow corner of a trichromatic triangle, then

$$|f(z^{\mathrm{Y}}) - z^{\mathrm{Y}}|_\infty < \epsilon + \delta.$$

Choosing $\delta = \min(\delta(\epsilon), \epsilon)$

$$|f(z^{Y}) - z^{Y}|_\infty < 2\epsilon.$$

2D-Brouwer on the Square

Finishing the proof of Brouwer's Theorem:

- pick a sequence of epsilons: $\epsilon_i = 2^{-i}, i = 1, 2, \ldots$

- define a sequence of triangulations of diameter: $\delta_i = \min(\delta(\epsilon_i), \epsilon_i), i = 1, 2, \ldots$

- pick a trichromatic triangle in each triangulation, and call its yellow corner $z_i^{\mathrm{Y}}, i = 1, 2, \ldots$

- by compactness, this sequence has a converging subsequence $w_i, \quad i = 1, 2, \ldots$ with limit point w^*

Claim: $f(w^*) = w^*.$

Proof: Define the function $g(x) = d(f(x), x)$. Clearly, g is continuous since $d(\cdot, \cdot)$ is continuous and so is f. It follows from continuity that

$$g(w_i) \longrightarrow g(w^*), \text{ as } i \to +\infty.$$

But $0 \leq g(w_i) \leq 2^{-i+1}$. Hence, $g(w_i) \longrightarrow 0$. It follows that $g(w^*) = 0$.

Therefore, $d(f(w^*), w^*) = 0 \implies f(w^*) = w^*.$ ∎

How hard is computing a Nash Equilibrium?

NASH, BROUWER and SPERNER

We informally define three computational problems:

- NASH: find a (appx-) Nash equilibrium in a n player game.
- BROUWER: find a (appx-) fixed point x for a continuous function f().
- SPERNER: find a trichromatic triangle (panchromatic simplex) given a legal coloring.

Function NP (FNP)

A *search problem* L is defined by a relation $R_L(x, y)$ such that

$$R_L(x, y)=1 \quad \text{iff} \quad y \text{ is a solution to } x$$

A search problem is called *total* iff for all x there exists y such that $R_L(x, y) = 1$.

A search problem L belongs to FNP iff there exists an efficient algorithm $A_L(x, y)$ and a polynomial function $p_L(\cdot)$ such that

(i) if $A_L(x, z)=1$ ➔ $R_L(x, z)=1$

(ii) if $\exists\, y$ s.t. $R_L(x, y)=1$ ➔ $\exists\, z$ with $|z| \leq p_L(|x|)$ such that $A_L(x, z)=1$

Clearly, SPERNER $\in$ FNP.

Reductions between Problems

A search problem $L \in$ FNP, associated with $A_L(x, y)$ and p_L, is ***polynomial-time reducible*** to another problem L' $\in$ FNP, associated with $A_{L'}(x, y)$ and $p_{L'}$, iff there exist efficiently computable functions *f, g* such that

(i) x is input to L ➔ $f(x)$ is input to L'

(ii)

$A_{L'}(f(x), y)=1$ ➔ $A_L(x, g(y))=1$

$R_{L'}(f(x), y)=0, \forall y$ ➔ $R_L(x, y)=0, \forall y$

A search problem L is ***FNP-complete*** iff

e.g. SAT

$L \in$ FNP

L' is poly-time reducible to L, for all $L' \in$ FNP

Our Reductions (intuitively)

NASH $\rightsquigarrow$ BROUWER $\rightsquigarrow$ SPERNER $\in$ FNP

both Reductions are polynomial-time

Is then SPERNER FNP-complete?

- With our current notion of reduction the answer is no, because SPERNER always has a solution, while a SAT instance may not have a solution;

- To attempt an answer to this question we need to update our notion of reduction. Suppose we try the following: we require that a solution to SPERNER informs us about whether the SAT instance is satisfiable or not, and provides us with a solution to the SAT instance in the ``yes'' case;

  but if such a reduction existed, it could be turned into a non-deterministic algorithm for checking "no" answers to SAT: guess the solution to SPERNER; this will inform you about whether the answer to the SAT instance is "yes" or "no", leading to $NP = co - NP \ldots$

- Another approach would be to turn SPERNER into a non-total problem, e.g. by removing the boundary conditions; this way, SPERNER can be easily shown FNP-complete, but all the structure of the original problem is lost in the reduction.