In last lecture, we showed Nash’s theorem that a Nash equilibrium exists in every game.

In our proof, we used Brouwer’s fixed point theorem as a Black-box.

In today’s lecture, we explain Brouwer’s theorem, and give an illustration of Nash’s proof.

We proceed to prove Brouwer’s Theorem using a combinatorial lemma, called Sperner’s Lemma, whose proof we also provide.
Brouwer’s Fixed Point Theorem
Brouwer’s fixed point theorem

**Theorem:** Let \( f : D \rightarrow D \) be a continuous function from a convex and compact subset \( D \) of the Euclidean space to itself. Then there exists an \( x \in D \) s.t. \( x = f(x) \).

N.B. All conditions in the statement of the theorem are necessary.

Below we show a few examples, when \( D \) is the 2-dimensional disk.
Brouwer’s fixed point theorem

fixed point
Brouwer’s fixed point theorem
Brouwer’s fixed point theorem
Nash’s Proof
Nash’s Function

\[ \Delta \ni x \mapsto y \in \Delta : \]

\[ y_p(s_p) := \frac{x_p(s_p) + \text{Gain}_{p; s_p}(x)}{1 + \sum_{s'_p \in S_p} \text{Gain}_{p; s'_p} x} \]

where:

\[ \text{Gain}_{p; s_p}(x) = \max\{u_p(s_p; x_{-p}) - u_p(x), 0\} \]
Visualizing Nash’s Construction

Penalty Shot Game

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$f: [0,1]^2 \rightarrow [0,1]^2$, continuous such that fixed points $\equiv$ Nash eq.
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Fixed point $\leftrightarrow$ Nash eq.
Sperner’s Lemma
Sperner’s Lemma
Sperner’s Lemma

Lemma: Color the boundary using three colors in a legal way.
Sperner’s Lemma

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For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

Next we define a directed walk starting from the bottom-left triangle.
Proof of Sperner’s Lemma

Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.
Proof of Sperner’s Lemma

Claim: The walk cannot exit the square, nor can it loop around itself in a rho-shape. Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle...

For convenience we introduce an outer boundary, that does not create new tri-chromatic triangles.

Next we define a directed walk starting from the bottom-left triangle.

Starting from other triangles we do the same going forward or backward.

Lemma: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.
Proof of Brouwer’s Fixed Point Theorem

We show that Sperner’s Lemma implies Brouwer’s Fixed Point Theorem. We start with the 2-dimensional Brouwer problem on the square.
2D-Brouwer on the Square

Suppose $f: [0,1]^2 \rightarrow [0,1]^2$, continuous

must be uniformly continuous (by the Heine-Cantor theorem)

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0, \text{ s.t.}$$

$$d(z, w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon$$

say $d$ is the $\ell_\infty$ norm
2D-Brouwer on the Square

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choose some \( \epsilon \) and triangulate so that the diameter of cells is \( \delta < \delta(\epsilon) \)
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color the nodes of the triangulation according to the direction of $f(x) - x$

tie-break at the boundary angles, so that the resulting coloring respects the boundary conditions required by Sperner’s lemma

find a trichromatic triangle, guaranteed by Sperner

choose some $\epsilon$ and triangulate so that the diameter of cells is $\delta < \delta(\epsilon)$

say $d$ is the $\ell_\infty$ norm
Suppose $f : [0,1]^2 \rightarrow [0,1]^2$, continuous must be uniformly continuous (by the Heine-Cantor theorem)

Claim: If $z_Y$ is the yellow corner of a trichromatic triangle, then

$$|f(z_Y) - z_Y|_{\infty} < \epsilon + \delta.$$
**Proof of Claim**

**Claim:** If $z^Y$ is the yellow corner of a trichromatic triangle, then $|f(z^Y) - z^Y|_\infty < \epsilon + \delta$.

**Proof:** Let $z^Y, z^R, z^B$ be the yellow/red/blue corners of a trichromatic triangle.

By the definition of the coloring, observe that the product of 

$$(f(z^Y) - z^Y)_x \text{ and } (f(z^B) - z^B)_x$$

is $\leq 0$.

Hence:

$$|(f(z^Y) - z^Y)_x| \leq |(f(z^Y) - z^Y)_x - (f(z^B) - z^B)_x|$$

$$\leq |(f(z^Y) - f(z^B))_x| + |(z^Y - z^B)_x|$$

$$\leq d(f(z^Y), f(z^B)) + d(z^Y, z^B)$$

$$\leq \epsilon + \delta.$$

Similarly, we can show:

$$|(f(z^Y) - z^Y)_y| \leq \epsilon + \delta.$$
2D-Brouwer on the Square

Suppose \( f : [0,1]^2 \to [0,1]^2 \), continuous

must be uniformly continuous (by the Heine-Cantor theorem)

\[ \forall \epsilon > 0, \exists \delta(\epsilon) > 0, \text{ s.t.} \]
\[ d(z, w) < \delta(\epsilon) \implies d(f(z), f(w)) < \epsilon \]

Claim: If \( z^Y \) is the yellow corner of a trichromatic triangle, then

\[ |f(z^Y) - z^Y|_\infty < \epsilon + \delta. \]

Choosing \( \delta = \min(\delta(\epsilon), \epsilon) \)

\[ |f(z^Y) - z^Y|_\infty < 2\epsilon. \]
2D-Brouwer on the Square

Finishing the proof of Brouwer’s Theorem:

- pick a sequence of epsilons: $\epsilon_i = 2^{-i}, i = 1, 2, \ldots$

- define a sequence of triangulations of diameter: $\delta_i = \min(\delta(\epsilon_i), \epsilon_i), i = 1, 2, \ldots$

- pick a trichromatic triangle in each triangulation, and call its yellow corner $z^Y_i, i = 1, 2, \ldots$

- by compactness, this sequence has a converging subsequence $w_i, i = 1, 2, \ldots$

Claim: $f(w^*) = w^*$.

Proof: Define the function $g(x) = d(f(x), x)$. Clearly, $g$ is continuous since $d(\cdot, \cdot)$ is continuous and so is $f$. It follows from continuity that

$$g(w_i) \longrightarrow g(w^*), \text{ as } i \rightarrow +\infty.$$

But $0 \leq g(w_i) \leq 2^{-i+1}$. Hence, $g(w_i) \longrightarrow 0$. It follows that $g(w^*) = 0$.

Therefore, $d(f(w^*), w^*) = 0 \implies f(w^*) = w^*$. 
How hard is computing a Nash Equilibrium?
We informally define three computational problems:

- **NASH**: find a (appx-) Nash equilibrium in a n player game.
- **BROUWER**: find a (appx-) fixed point $x$ for a continuous function $f()$.
- **SPERNER**: find a trichromatic triangle (panchromatic simplex) given a legal coloring.
Function NP (FNP)

A search problem \( L \) is defined by a relation \( R_L(x, y) \) such that

\[
R_L(x, y) = 1 \quad \text{iff} \quad y \text{ is a solution to } x
\]

A search problem is called \textit{total} iff for all \( x \) there exists \( y \) such that \( R_L(x, y) = 1 \).

A search problem \( L \) belongs to FNP iff there exists an efficient algorithm \( A_L(x, y) \) and a polynomial function \( p_L(\cdot) \) such that

(i) if \( A_L(x, z) = 1 \) \( \Rightarrow \) \( R_L(x, z) = 1 \)

(ii) if \( \exists \ y \text{ s.t. } R_L(x, y) = 1 \) \( \Rightarrow \) \( \exists \ z \text{ with } |z| \leq p_L(|x|) \text{ such that } A_L(x, z) = 1 \)

Clearly, SPERNER \( \in \) FNP.
Reductions between Problems

A search problem \( L \in \text{FNP}, \) associated with \( A_L(x, y) \) and \( p_L, \) is *polynomial-time reducible* to another problem \( L' \in \text{FNP}, \) associated with \( A_{L'}(x, y) \) and \( p_{L'}, \) iff there exist efficiently computable functions \( f, g \) such that

(i) \( x \) is input to \( L \) \( \Rightarrow \) \( f(x) \) is input to \( L' \)

(ii) \[
A_{L'}(f(x), y) = 1 \quad \Rightarrow \quad A_L(x, g(y)) = 1 \\
R_{L'}(f(x), y) = 0, \ \forall \ y \quad \Rightarrow \quad R_L(x, y) = 0, \ \forall \ y
\]

A search problem \( L \) is *FNP-complete* iff

- \( L \in \text{FNP} \)
- \( L' \) is poly-time reducible to \( L, \) for all \( L' \in \text{FNP} \)

e.g. SAT
Our Reductions (intuitively)

NASH $\leadsto$ BROUWER $\leadsto$ SPERNER $\in$ FNP

both Reductions are polynomial-time

Is then SPERNER FNP-complete?

- With our current notion of reduction the answer is no, because SPERNER always has a solution, while a SAT instance may not have a solution;

- To attempt an answer to this question we need to update our notion of reduction. Suppose we try the following: we require that a solution to SPERNER informs us about whether the SAT instance is satisfiable or not, and provides us with a solution to the SAT instance in the "yes" case;

  but if such a reduction existed, it could be turned into a non-deterministic algorithm for checking "no" answers to SAT: guess the solution to SPERNER; this will inform you about whether the answer to the SAT instance is "yes" or "no", leading to $NP = co - NP$ ...

- Another approach would be to turn SPERNER into a non-total problem, e.g. by removing the boundary conditions; this way, SPERNER can be easily shown FNP-complete, but all the structure of the original problem is lost in the reduction.