**New Decision Variables**

**Variables:** Interim Allocation rule aka. “REDUCED FORM”:

\[ \{ \pi_i : T_i \rightarrow [0, 1]^n, p_i : T_i \rightarrow \mathbb{R}^+ \}_{i \in [m]} \]

* \[ \pi_{ij}(v_i) : \Pr \left( \vec{t}_{-i} \sim D_{-i}, j \rightarrow i \right) \]

* \[ \hat{p}_i(v_i) : \mathbb{E} \left[ price_i \mid i \text{ valuation } v_i \right] \]
A succinct LP

### Variables:
- \( \pi_{ij}(v_i) \): probability that item \( j \) is allocated to bidder \( i \) if her reported valuation (bid) is \( v_i \) in expectation over every other bidders’ valuations (bids);
- \( p_i(v_i) \): price bidder \( i \) pays if her reported valuation (bid) is \( v_i \) in expectation over every other bidder’s valuations (bids)

### Constraints:
- **BIC:** \[ \sum_j v_{ij} \cdot \pi_{ij}(v_i) - p_i(v_i) \geq \sum_j v_{ij} \cdot \pi_{ij}(v'_i) - p_i(v'_i) \] for all \( v_i \) and \( v'_i \) in \( T_i \)
- **IR:** \[ \sum_j v_{ij} \cdot \pi_{ij}(v_i) - p_i(v_i) \geq 0 \] for all \( v_i \) in \( T_i \)
- Feasibility: exists an auction with this reduced form.

### Objective:
- Expected revenue: \[ \sum_i \sum_{v_i \in T_i} \Pr[t_i = v_i] \cdot p_i(v_i) \]
After solving the succinct LP, we find the optimal reduced form $\pi^*$ and $p^*$.

Can you turn $\pi^*$ and $p^*$ into an auction whose reduced form is exactly $\pi^*$ and $p^*$?

This is crucial, otherwise being able to solve the LP is meaningless.

Will show you a way to implement any feasible reduced form, and it reveals important structure of the revenue-optimal auction!
Implementation of a Feasible Reduced Form
Set of **Feasible** Reduced Forms

- Reduced form is collection \( \{ \pi_i : T_i \rightarrow [0, 1]^n \} \);
- Can view it as a vector \( \vec{\pi} \in \mathbb{R}^n \sum_i |T_i| \);
- Let’s call set of feasible reduced forms \( F(D) \in \mathbb{R}^n \sum_i |T_i| \);

- **Claim 1:** \( F(D) \) is a **convex polytope**.

- **Proof:** *Easy!*
  - A feasible reduced form \( \vec{\pi} \) is implemented by a feasible allocation rule \( M \).
  - \( M \) is a distribution over deterministic feasible allocation rules, of which there is a finite number. So: \( M = \sum_{\ell=1}^k p_{\ell} \cdot M_{\ell} \), where \( M_{\ell} \) is deterministic.
  - Easy to see: \( \vec{\pi} = \sum_{\ell=1}^k p_{\ell} \cdot \vec{\pi}(M_{\ell}) \)

- So, \( F(D) = \) convex hull of reduced forms of feasible deterministic mechanisms
Set of *Feasible* Reduced Forms

Q: Is there a simple allocation rule implementing the corners?
virtual welfare maximizing interim rule when virtual value functions are the $f_i$'s

\[ \pi' \in \arg\max_{\pi' \in F(D)} \{ \pi' \cdot \vec{w} \} \]

expected \textit{virtual} welfare of an allocation rule with interim rule $\pi'$

\[
\begin{align*}
\pi' \cdot \vec{w} &= \sum_i \sum_j \sum_{A \in T_i} \pi'_{ij}(A) w_{ij}(A) \quad \text{---------- (1)} \\
&= \sum_i \sum_j \sum_{A \in T_i} \pi'_{ij}(A) f_{ij}(A) \Pr[t_i = A] \quad \text{--- (2)}
\end{align*}
\]

interpretation: \textit{virtual} value derived by bidder $i$ when given item $j$ when his type is $A$

\[ f_{ij}(A) := \frac{w_{ij}(A)}{\Pr_D[t_i = A]} \]
Is there a simple allocation rule implementing a corner?

virtual welfare maximizing interim rule when virtual value functions are the $f_i$’s

Q: Can you name an algorithm doing this?

A: YES, the VCG allocation rule (w/ virtual value functions $f_i$, $i=1,..,m$)

$$
\pi \in \arg\max_{\vec{\pi}' \in F(D)} \{ \vec{\pi}' \cdot \vec{w} \}
$$

$$
\pi \in \arg\max_{\vec{\pi}' \in F(D)} \{ \vec{\pi}' \cdot \vec{w} \}
$$

virtual welfare derived by bidder $i$ when given item $j$ when his type is $A$

interpretation: virtual value derived by bidder $i$ when given item $j$ when his type is $A$

$$
q_{ij}(A) := \frac{w_{ij}(A)}{Pr_D[t_i = A]}
$$
$F(D)$ is a **Convex Polytope** whose corners are **implementable** by **virtual VCG** allocation rules.

How about implementing any point inside $F(D)$?
Carathéodory’s theorem

If some point $x$ is in the convex hull of $P$ then

$$x = \sum_{p_i \in P} q_i \cdot p_i$$

s.t. $\sum q_i = 1$ and $q_i \geq 0 \ \forall i$

Carathéodory’s Theorem: If a point $x$ of $\mathbb{R}^d$ lies in the convex hull of a set $P$, there is a subset $P'$ of $P$ consisting of $d + 1$ or fewer points such that $x$ lies in the convex hull of $P'$.

For example:

$$x = \frac{1}{4}(0,1) + \frac{1}{4}(1,0) + \frac{1}{2}(0,0)$$
Any point inside $F(D)$ is a convex combination (distribution) over the corners.

The interim allocation rule of any feasible mechanism can be implemented as a distribution over virtual VCG allocation rules.
Structure of the Optimal Auction
Theorem [C.-Daskalaks-Weinberg]: Optimal multi-item auction has the following structure:

1. Bidders submit valuations \((t_1, \ldots, t_m)\) to auctioneer.

2. Auctioneer samples virtual transformations \(f_1, \ldots, f_m\).

3. Auctioneer computes virtual types \(t'_i = f_i(t_i)\).

4. Virtual welfare maximizing allocation is chosen. Namely, each item is given to bidder with highest virtual value for that item (if positive).

5. Prices are charged to ensure truthfulness.
Theorem [C.-Daskalaks-Weinberg]: Optimal multi-item auction has the following structure:

1. Bidders submit valuations \((t_1, \ldots, t_m)\) to auctioneer.
2. Auctioneer samples virtual transformations \(f_1, \ldots, f_m\)
3. Auctioneer computes virtual types \(t'_i = f_i(t_i)\)
4. Virtual welfare maximizing allocation is chosen. Namely, each item is given to bidder with highest virtual value for that item (if positive)
5. Prices are charged to ensure truthfulness.

- Exact same structure as Myerson!
  - in Myerson’s theorem: virtual function = deterministic
  - here, \textit{randomized} (and they must be)
Another difference: in Myerson’s theorem: virtual function is given explicitly, in our result, the transformation is computed by an LP. Is there any structure of our transformation?

In single-dimensional settings, the optimal auction is DSIC. In multi-dimensional settings, this is unlikely to be true. What is the gap between the optimal BIC solution and the optimal DSIC solution?