

**COMP/MATH 553 Algorithmic  
Game Theory  
Lecture 6: Simple Near-Optimal  
Auctions**

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## An overview of today's class



*Discussion of Myerson's Auction*

*Prophet Inequality*

*Prior-Independent Auctions & Bulow-Klemperer Theorem*

# Revenue = Virtual Welfare



[Myerson '81 ] For any single-dimensional environment.

Let  $F = F_1 \times F_2 \times \dots \times F_n$  be the joint value distribution, and  $(x, p)$  be a DSIC mechanism. The expected revenue of this mechanism

$$E_{v \sim F}[\sum_i p_i(v)] = E_{v \sim F}[\sum_i x_i(v) \varphi_i(v_i)],$$

where  $\varphi_i(v_i) := v_i - (1 - F_i(v_i))/f_i(v_i)$  is called bidder  $i$ 's virtual value ( $f_i$  is the density function for  $F_i$ ).

# Myerson's Auction



- ❑ To optimize revenue, we should use the *virtual welfare maximizing allocation rule*

$$- x(v) := \operatorname{argmax}_{x \text{ in } X} \sum_i x_i(v) \varphi_i(v_i)$$

- ❑ If  $F_i$  is regular, then  $\varphi_i(v_i)$  is *monotone* in  $v_i$ .
- ❑ The virtual welfare maximizing allocation rule is *monotone* as well!
- ❑ With the suitable payment rule, this is a DSIC mechanism that maximizes revenue.
- ❑ Same result extends to irregular distributions, but requires extra work (ironing).

# How Simple is Myerson's Auction?



- ❑ Single-item and i.i.d. regular bidders, e.g.  $F_1=F_2=\dots=F_n$
- ❑ All  $\varphi_i(\cdot)$ 's are the same and monotone.
- ❑ The highest bidder has the **highest virtual value!**
- ❑ The optimal auction is the Vickrey auction with reserve price at  $\varphi^{-1}(\theta)$ .
- ❑ Real “killer application” for practice, arguably at eBay.

# How Simple is Myerson's Auction?

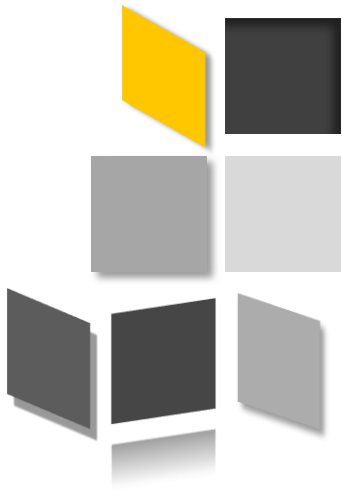


- ❑ Single-item regular bidders but  $F_1 \neq F_2 \neq \dots \neq F_n$
  
- ❑ All  $\varphi_i(\cdot)$ 's are monotone but not the **same**.
  
- ❑ 2 bidders,  $v_1$  uniform in  $[0,1]$ .  $v_2$  uniform in  $[0,100]$ .
  - $\varphi_1(v_1) = 2v_1 - 1$ ,  $\varphi_2(v_2) = 2v_2 - 100$
  
  - Optimal Auction:
    - When  $v_1 > 1/2$ ,  $v_2 < 50$ , sell to 1 at price  $1/2$ .
    - When  $v_1 < 1/2$ ,  $v_2 > 50$ , sell to 2 at price 50.
    - When  $0 < 2v_1 - 1 < 2v_2 - 100$ , sell to 2 at price:  
 $(99 + 2v_1)/2$ , a tiny bit above 50
    - When  $0 < 2v_2 - 100 < 2v_1 - 1$ , sell to 1 at price:  
 $(2v_2 - 99)/2$ , a tiny bit above  $1/2$ .

# How Simple is Myerson's Auction?



- ❑ The payment seems impossible to explain to someone who hasn't studied virtual valuations...
- ❑ In the i.i.d. case, the optimal auction is simply eBay with a smartly chosen opening bid.
- ❑ This weirdness is inevitable if you are 100% confident in your model (i.e., the  $F_i$ 's) and you want every last cent of the maximum-possible expected revenue.
- ❑ Seek out auctions that are simpler, more practical, and more robust than the theoretically optimal auction.
- ❑ Optimality requires complexity, thus we'll only look for approximately optimal solutions.



# Prophet Inequality



# Optimal Stopping Rule for a Game



- ❑ Consider the following game, with  $n$  stages. In stage  $i$ , you are offered a nonnegative prize  $\pi_i$ , drawn from a distribution  $G_i$
- ❑ You are told the distributions  $G_1, \dots, G_n$  in advance, and these distributions are independent.
- ❑ You are told the realization  $\pi_i$  only at stage  $i$ .
- ❑ After seeing  $\pi_i$ , you can either accept the prize and end the game, or discard the prize and proceed to the next stage.
- ❑ The decision's difficulty stems from the trade-off between the risk of accepting a reasonable prize early and then missing out later on a great one, and the risk of having to settle for a lousy prize in one of the final stages.

# Prophet Inequality



## Prophet Inequality [Samuel-Cahn '84]:

There exists a strategy, such that the **expected payoff**  $\geq 1/2 E[\max_i \pi_i]$ . In fact, a threshold strategy suffices.

- Proof: See the board.
- Remark: Our lowerbound only credits  $t$  units of value when more than one prize is above  $t$ . This means that the  $1/2$  applies even if, whenever there are multiple prizes above the threshold, the strategy somehow picks the worst (i.e., smallest) of these.

# Application to Single-item Auctions



- ❑ Single item, regular but non-i.i.d. value distributions
  
- ❑ **Key idea:** think of  $\varphi_i(v_i)^+$  as the  $i$ -th prize. ( $G_i$  is the induced non-negative virtual value distribution from  $F_i$ )
  
- ❑ In a single-item auction, the optimal expected revenue
$$\mathbf{E}_{v \sim F} [\max_i \sum_i x_i(v) \varphi_i(v_i)] = \mathbf{E}_{v \sim F} [\max_i \varphi_i(v_i)^+]$$
(the expected prize of the prophet)
  
- ❑ Consider the following allocation rule
  1. Choose  $t$  such that  $\Pr[\max_i \varphi_i(v_i)^+ \geq t] = 1/2$ .
  
  2. Give the item to a bidder  $i$  with  $\varphi_i(v_i) \geq t$ , if any, breaking ties among multiple candidate winners arbitrarily (subject to monotonicity)

# Application to Single-item Auctions (cont'd)



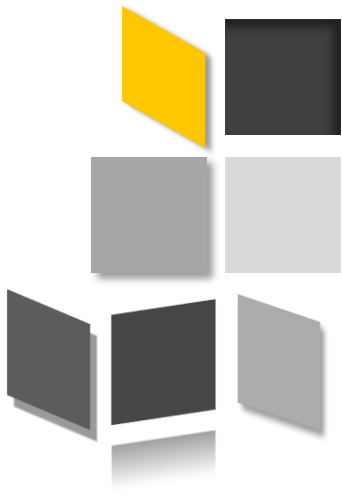
- By Prophet Inequality, any allocation rule that satisfy the above has

$$\mathbf{E}_{\mathbf{v} \sim \mathbf{F}} [\mathbf{max} \sum_i x_i(\mathbf{v}) \varphi_i(\mathbf{v}_i)] \geq \frac{1}{2} \mathbf{E}_{\mathbf{v} \sim \mathbf{F}} [\mathbf{max}_i \varphi_i(\mathbf{v}_i)^+]$$

- Here is a specific monotone allocation rule that satisfies this:

1. Set a reserve price  $r_i = \varphi_i^{-1}(t)$  for each bidder  $i$  with the  $t$  defined above.
2. Give the item to the highest bidder that meets her reserve price (if any).

- The payment is simply the maximum of winner's reserve price and the second highest bid (that meets her own reserve).
- Interesting Open Problem: How about anonymous reserve? We know it's between  $[1/4, 1/2]$ , can you pin down the exact approximation ratio?



# Prior-Independent Auctions

# Another Critique to the Optimal Auction



- ❑ What if your distribution are **unknown**?
- ❑ When there are many bidders and enough past data, it is reasonable to assume you know exactly the value distribution.
- ❑ But if the market is “thin”, you might not be confident or not even know the value distribution.
- ❑ Can you design an auction that does not use any knowledge about the distributions but performs *almost as well as* if you know *everything* about the distributions.
- ❑ Active research agenda, called prior-independent auctions.

# Bulow-Klemperer Theorem

**[Bulow-Klemperer '96]** For any regular distribution  $F$  and integer  $n$ .

$$\mathbb{E}_{v_1, \dots, v_{n+1}} [\text{REV}(\text{Vickrey})] \geq \mathbb{E}_{v_1, \dots, v_n} [\text{REV}(\text{Myerson})]$$

Remark:

- Vickrey's auction is prior-independent!
- This means with the same number of bidders, Vickrey Auction achieves at least  $n-1/n$  fraction of the optimal revenue.
- More competition is better than finding the right auction format.

# Proof of Bulow-Klemperer



- Consider another auction  $M$  with  $n+1$  bidders:
  1. Run Myerson on the first  $n$  bidders.
  2. If the item is unallocated, give it to the last bidder for free.
- This is a *DSIC* mechanism. It has the *same* revenue as Myerson's auction with  $n$  bidders.
- It's allocation rule always give out the item.
- Vickrey Auction also always give out the item, but always to the bidder who has the highest value (also with the highest virtual value).
- Vickrey Auction has the highest virtual welfare among all *DSIC* mechanisms that always give out the item! □