COMP/MATH 553 Algorithmic Game Theory
Lecture 6: Simple Near-Optimal Auctions

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Yang Cai

An overview of today's class

- *Discussion of Myerson's Auction*
- *Prophet Inequality*
- *Prior-Independent Auctions & Bulow-Klemperer Theorem*

Revenue = Virtual Welfare

[Myerson '81  **]** For any single-dimensional environment.

Let $F = F_1 \times F_2 \times \ldots \times F_n$ be the joint value distribution, and (x,p) be a DSIC mechanism. The expected revenue of this mechanism

$$E_{v\sim F}[\Sigma_i\ p_i(v)] = E_{v\sim F}[\Sigma_i\ x_i(v)\ \varphi_i\ (v_i)],$$

where $\varphi_i\ (v_i) := v_i - (1-F_i(v_i))/f_i(v_i)$ is called bidder i's virtual value (f_i is the density function for F_i).

Myerson's Auction

- To optimize revenue, we should use the ***virtual welfare maximizing allocation rule***

  - $x(v) := \text{argmax}_{x \text{ in } X} \Sigma_i x_i(v)\, \varphi_i(v_i)$

- If F_i is regular, then $\varphi_i(v_i)$ is ***monotone*** in v_i.

- The virtual welfare maximizing allocation rule is ***monotone*** as well!

- With the suitable payment rule, this is a DSIC mechanism that maximizes revenue.

- Same result extends to irregular distributions, but requires extra work (ironging).

How Simple is Myerson's Auction?

- Single-item and i.i.d. regular bidders, e.g. $\boldsymbol{F_1=F_2=\ldots=F_n}$
- All $\boldsymbol{\varphi_i\,()}$'s are the same and monotone.
- The highest bidder has the **highest virtual value**!
- The optimal auction is the Vickrey auction with reserve price at $\boldsymbol{\varphi^{-1}(0)}$.
- Real "killer application" for practice, arguably at eBay.

How Simple is Myerson's Auction?

- Single-item regular bidders but $F_1 \neq F_2 \neq \ldots \neq F_n$
- All $\varphi_i\,()$'s are monotone but not the **same**.
- 2 bidders, v_1 uniform in [0,1]. v_2 uniform in [0,100].
  - $\varphi_1(v_1) = 2v_1 - 1,\ \varphi_2(v_2) = 2v_2 - 100$
  - Optimal Auction:
    - When $v_1 > ½$, $v_2 < 50$, sell to 1 at price ½.
    - When $v_1 < ½$, $v_2 > 50$, sell to 2 at price 50.
    - When $0 < 2v_1 - 1 < 2v_2 – 100$, sell to 2 at price: $(99+2v_1)/2$, a tiny bit above 50
    - When $0 < 2v_2 - 100 < 2v_1 - 1$, sell to 1 at price: $(2v_2 - 99)/2$, a tiny bit above ½.

How Simple is Myerson's Auction?

- The payment seems impossible to explain to someone who hasn't studied virtual valuations...
- In the i.i.d. case, the optimal auction is simply eBay with a smartly chosen opening bid.

- This weirdness is inevitable if you are 100% confident in your model (i.e., the F_i's) and you want every last cent of the maximum-possible expected revenue.

- Seek out auctions that are simpler, more practical, and more robust than the theoretically optimal auction.

- Optimality requires complexity, thus we'll only look for approximately optimal solutions.

Prophet Inequality

Optimal Stopping Rule for a Game

- Consider the following game, with n stages. In stage i, you are offered a nonnegative prize π_i, drawn from a distribution G_i
- You are told the distributions $G_1, \ldots, G_n$ in advance, and these distributions are independent.
- You are told the realization π_i only at stage i.
- After seeing π_i, you can either accept the prize and end the game, or discard the prize and proceed to the next stage.
- The decision's difficulty stems from the trade-off between the risk of accepting a reasonable prize early and then missing out later on a great one, and the risk of having to settle for a lousy prize in one of the final stages.

Prophet Inequality

Prophet Inequality [Samuel-Cahn '84]:
There exists a strategy, such that the **expected payoff ≥ 1/2 $E[\max_i \pi_i]$**. In fact, a threshold strategy suffices.

- Proof: See the board.

- Remark: Our lowerbound only credits ***t*** units of value when more than one prize is above ***t***. This means that the ½ applies even if, whenever there are multiple prizes above the threshold, the strategy somehow picks the worst (i.e., smallest) of these.

Application to Single-item Auctions

- Single item, regular but non-i.i.d. value distributions

- Key idea: think of $\varphi_i(v_i)^+$ as the i-th prize. (G_i is the induced non-negative virtual value distribution from F_i)

- In a single-item auction, the optimal expected revenue

  $\mathbf{E}_{v\sim F}\,[\max \Sigma_i\, x_i(v)\, \varphi_i\,(v_i)] = \mathbf{E}_{v\sim F}\,[\max_i \varphi_i(v_i)^+]$ (the expected prize of the prophet)

- Consider the following allocation rule
  1. Choose t such that $\Pr[\max_i \varphi_i\,(v_i)^+ \geq t] = 1/2$.
  2. Give the item to a bidder i with $\varphi_i\,(v_i) \geq t$, if any, breaking ties among multiple candidate winners arbitrarily (subject to monotonicity)

Application to Single-item Auctions (cont'd)

- By Prophet Inequality, any allocation rule that satisfy the above has

$$\mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\,[\mathbf{max}\ \Sigma_i\, x_i(v)\ \varphi_i\ (v_i)] \geq ½\ \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\,[max_i\,\varphi_i(v_i)^+]$$

- Here is a specific monotone allocation rule that satisfies this:
  1. Set a reserve price $r_i = \varphi_i^{-1}(t)$ for each bidder i with the t defined above.
  2. Give the item to the highest bidder that meets her reserve price (if any).

- The payment is simply the maximum of winner's reserve price and the second highest bid (that meets her own reserve).

- Interesting Open Problem: How about anonymous reserve? We know it's between [1/4, 1/2], can you pin down the exact approximation ratio?

Prior-Independent Auctions

Another Critique to the Optimal Auction

- What if your distribution are unknown?
- When there are many bidders and enough past data, it is reasonable to assume you know exactly the value distribution.
- But if the market is "thin", you might not be confident or not even know the value distribution.
- Can you design an auction that does not use any knowledge about the distributions but performs ***almost as well as*** if you know ***everything*** about the distributions.
- Active research agenda, called prior-independent auctions.

Bulow-Klemperer Theorem

[Bulow-Klemperer '96] For any regular distribution F and integer n.

$$\mathbb{E}_{v_1,\ldots,v_{n+1}}[\text{REV}(\text{Vickrey})] \geq \mathbb{E}_{v_1,\ldots,v_n}[\text{REV}(\text{Myerson})]$$

Remark:

- Vickrey's auction is prior-independent!
- This means with the same number of bidders, Vickrey Auction achieves at least n-1/n fraction of the optimal revenue.
- More competition is better than finding the right auction format.

Proof of Bulow-Klemperer

- Consider another auction M with n+1 bidders:
  1. Run Myerson on the first n bidders.
  2. If the item is unallocated, give it to the last bidder for free.

- This is a ***DSIC*** mechanism. It has the ***same*** revenue as Myreson's auction with n bidders.

- It's allocation rule always give out the item.

- Vickrey Auction also always give out the item, but always to the bidder who has the highest value (also with the highest virtual value).

- Vickrey Auction has the highest virtual welfare among all DSIC mechanisms that always give out the item! □