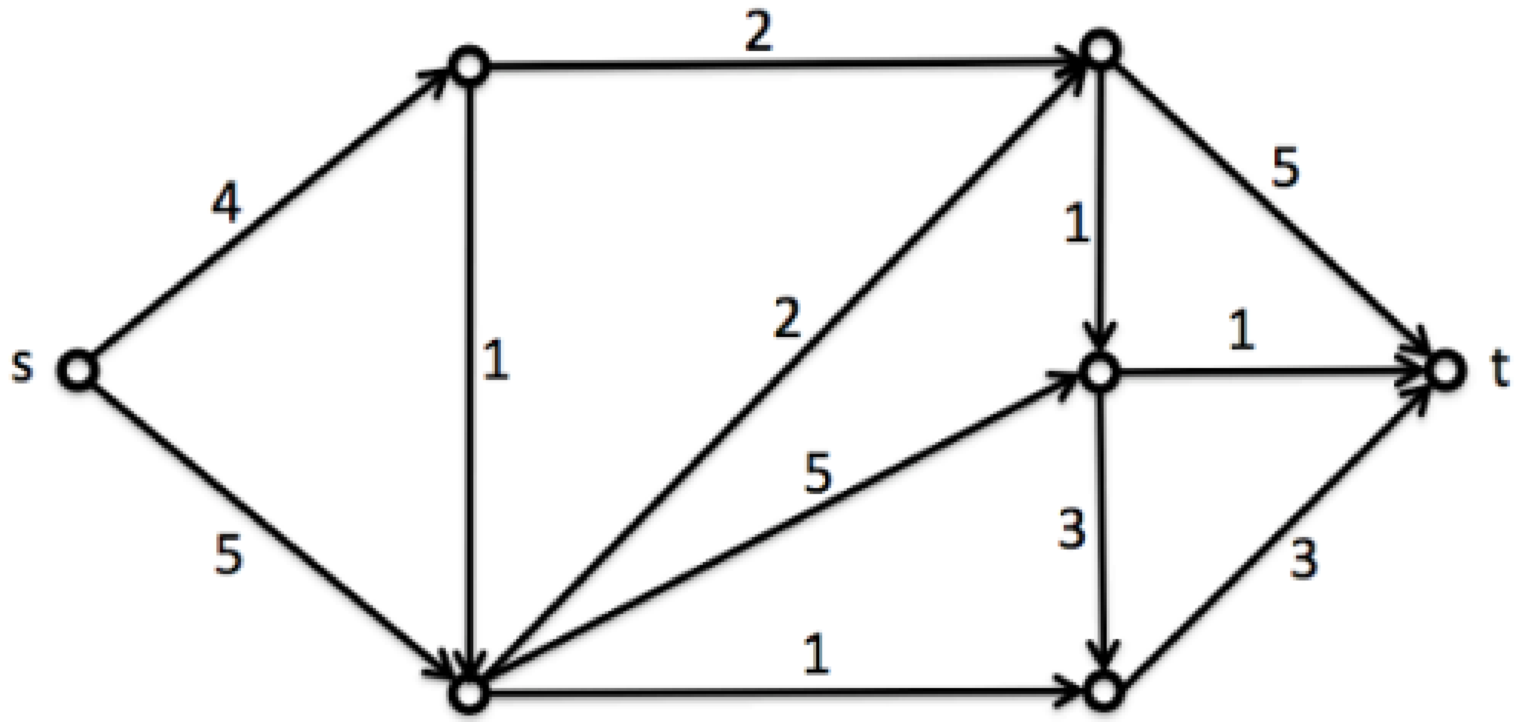
Maximum Flows and Minimum Cuts II

Lecture 6 COMP 610

January 24 2018

Recall: A Capacitated Network

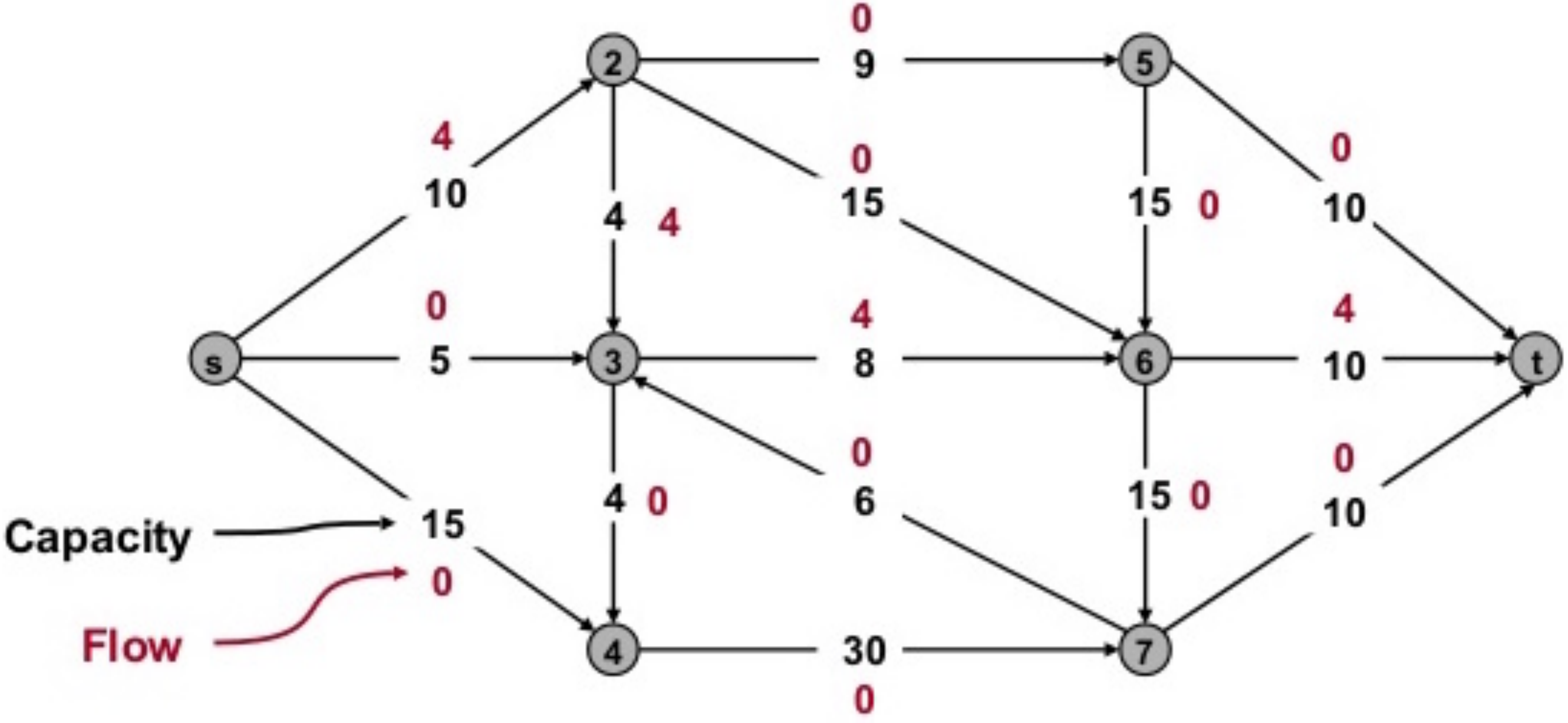
Directed Graph G=(V,E), nonnegative capacity c(e) (sometimes u(e) is used) for each edge e
unique source s and sink t, no edges into s or out of t,



Recall: An s-t Flow

$\forall e \in E$ have $f(e)$, $0 \leq f(e) \leq c(e)$ (capacity constraints)

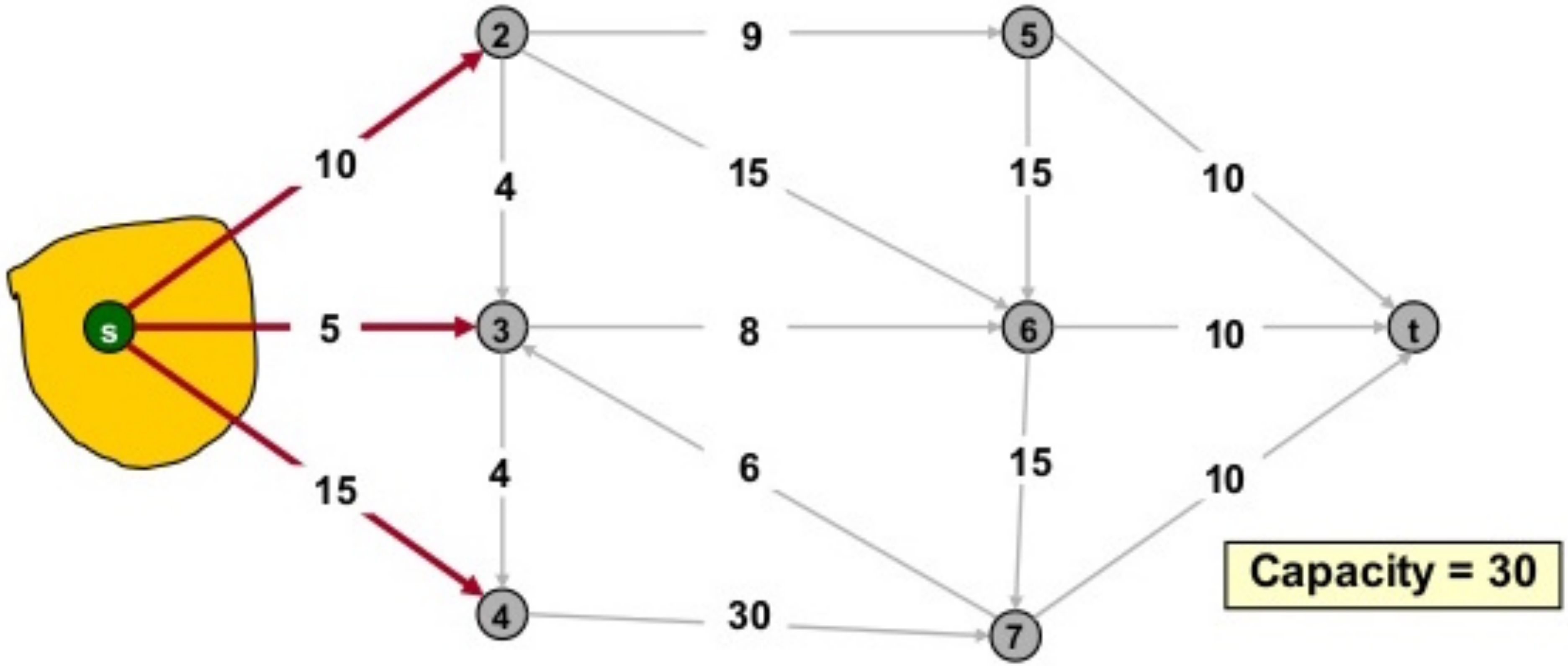
$\forall v \in V$ $\sum_{uv \in E} f(uv) = \sum_{vu \in E} f(uv)$ (flow conservation constraints)



Recall: An s-t Cut

is a partition(A,B) of V with s$\in A, b \in B$.

Its capacity c(A,B)=$\sum_{uv\in E, u\in A, v\in B} c(uv)$



Recall:
The value of a flow and flow across a cut

For all S $\subseteq V$:

$f^{out}(S) = \sum_{uv \in E, u \in S, v \in V-S} f(xy)$

$f^{in}(S) = \sum_{uv \in E, u \in V-S, v \in S} f(xy)$ Type equation here.

$v(f) = f^{out}(\{s\})$

For all s-t cuts (A,B), $v(f) = f^{out}(A) - f^{in}(A) \leq c(A,B)$

Max Flow Min Cut Theorem

For every capacitated network. The maximum volume of an s-t flow is equal to the minimum volume of an s-t cut.

Theorem 7.14 KT

If all the capacities are integer then there is a maximum volume s-t flow f which is integer valued. Can find this flow in $O(v(f)|E|)$ time.

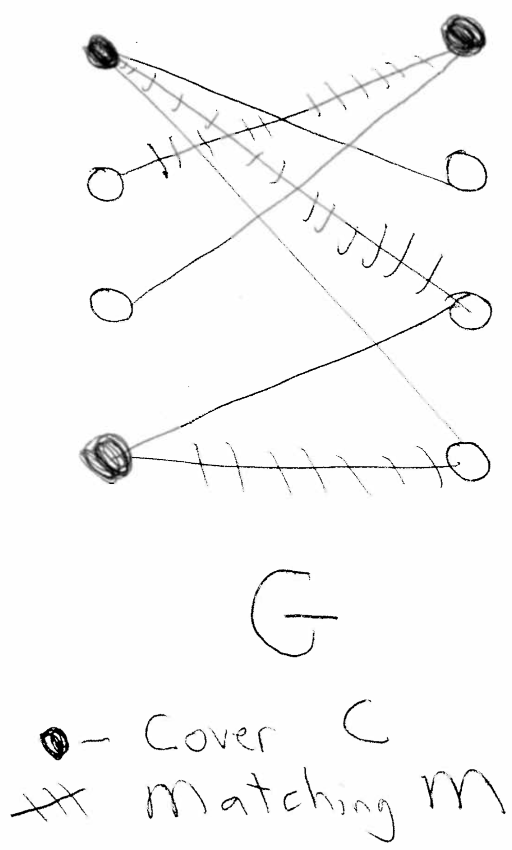
Recall: Matchings and Covers

A matching M is a disjoint set of edges

$C \subseteq V$ is a cover if there is no edge with both endpoints in V-C.

For a matching M and cover C in any G, $|M| \leq |C|$

For Bipartite G:
The Max Size of a Matching=Min Size of a cover



Maximum Number of Edge-Disjoint Paths in a Directed Graph G

V(G')=V(G), E(G')=E(G)-{xs|xs∈ $E(G)\} - \{tx|tx$ ∈E(G)}

Find a integer-valued max value s-t flow f in G'.

Find a set of v(f) edge disjoint paths from s-t in G, only using e if f(e)=1.

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Menger's Theorem(1927): The maximum size of a set of edge disjoint s-t paths in a directed graph equals the minimum size of a set of edges whose removal separates s from t.

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How do we handle this????

For all uv $\in E(G)$, if f(uv)=f(vu)=1, set f(uv)=f(vu)=0. Then find F.

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We want to find out if k planes can handle all the flights.

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There is a source s and an edge from s to every u_i.

There is a sink t and an edge from every v_i to t.

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A path P from s to t points out a set of flights which can be handled by one plane (those for which u_iv_i is an edge of P).

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We can schedule the flights using k planes if there is a set of at most k edge-disjoint paths of G using the edge u_iv_i for every flight i.

Airline Scheduling Via Bipartite Matching

For any schedule using exactly k planes to handle f flights, the corresponding k edge disjoint paths use k edges out of s, k edges into t, the f edges of the form $u_i v_i$, and f-k edges of the form $v_i u_j$ where flight j is reachable from flight i.

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The desired k paths exist precisely if there is a matching of size f-k in the graph G' whose edge set is $\{v_ju_i \mid$ flight i is reachable from flight j$\}$.

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So the number of planes needed is the minimum k for which there is a matching of size n-k in G'. Given the matching we can find the schedule easily, so the time to find an optimal schedule is O(f^3).

Sequential Scheduling Via Bipartite Matching

Image Segmentation Graph.

Graph G=(V,E). V is the set of pixels, E joins neighbours.

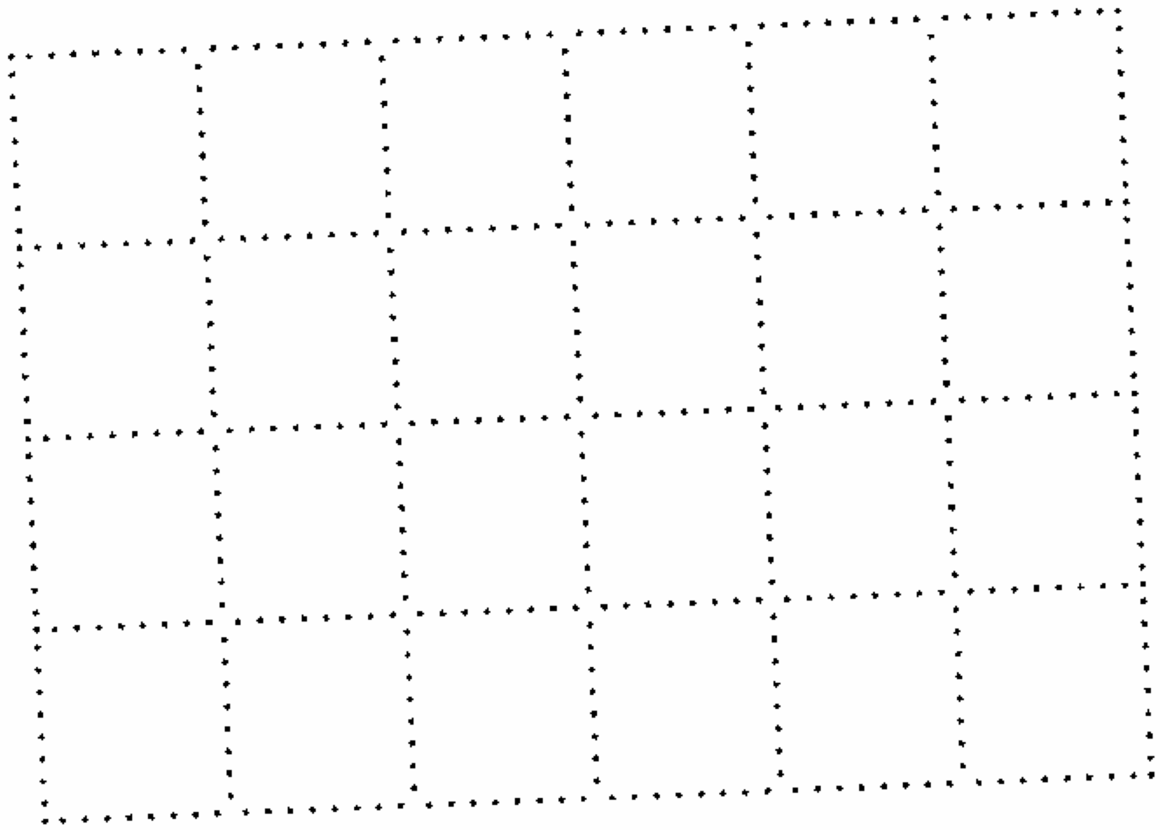

Image Segmentation Problem

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- We want to find a partition of the pixels into a set F of foreground pixels and a set B of background pixels so as to maximize:

$$\sum_{i \in F} a_i + \sum_{i \in B} b_i - \sum_{ij \in E, i \in F, j \in B} p_{ij}$$

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- This is the same as minimizing

$$\sum_{i\in B} a_i + \sum_{i\in F} b_i + \sum_{ij\in E, i\in F, j\in B} p_{ij}$$

Image Segmentation Via Min Cut

- Construct a capacited network whose vertex set is the pixels, s and t.
- Add an edge s_i for every pixel i with capacity a_i
- Add an edge ti for every pixel i with capacity b_i
- For every two neighbouring pixels ij, add edges ij and ji both with capacity p_{ij}

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- We are looking for the minimum capacity cut.