A Modal Analysis of Dependently-Typed Meta-Programming

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joint work with
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Special Session in honor of Frank Pfenning at LFMTP’22
Some time in the summer of 1997:

"In the report 'The practice of logical frameworks' by Frank Pfenning (FB Mathematik, February 1996, pre-print 1813) it is stated that the current degree of automation in logical frameworks is not satisfactory. Therefore it is suggested to look for ways to apply techniques from inductive theorem proving in the realm of logical frameworks to automate some of these proofs. "

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This Talk: A Modal Analysis of Dependently-Typed Meta-Programming
What is meta-programming?
What is **meta-programming**?

The **art of writing programs** that **produce** other programs.

• (Quasi)quotation of box $(2 + 2)$ represents an abstract syntax tree (AST) of the expression $2 + 2$.

• Unquote splices in code fragments, for example box $(2 + \text{unbox}(\text{square } 2))$ evaluates to the code box $(2 + 2 \times 2)$ provided \text{square } 2 generates code box $(2 \times 2)$. 
What is **meta-programming**?

The **art of writing programs** that **produce and manipulate** other programs within the same language.
What is meta-programming?

The art of writing programs that produce and manipulate other programs within the same language.

- *(Quasi)quotation* of \texttt{box}(2 + 2) represents an abstract syntax tree (AST) of the expression 2 + 2.

- *Unquote* splices in code fragments, for example \texttt{box}(2 + \texttt{unbox}(\texttt{square} 2)) evaluates to the code \texttt{box}(2 + 2 * 2) provided \texttt{square} 2 generates code \texttt{box}(2 * 2).
Good and Bad of Meta-Programming

The Good:

• Widely used and supported
  (Lisp, Scheme, TemplateHaskell, MetaML, TemplateCoq, …)

• Performance gains due to domain-specific optimizations
  (example: matrix / vector multiplication in machine learning)

The Bad:

• Hard to test the code generation and the generated code
• Hard to reason about code generation and generated code
• Errors are caught during run-time instead of generation time
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The Bad:

- Hard to test the code generation
- Hard to reason about code generation
- Errors are caught during run-time instead of generation time

Types for the rescue!

- Catch errors early at compile-time / generation time
- Reason about code generation and generated code
The Promise:
- Users do not need to learn and use a separate tactic language.
- We can prove properties about tactics themselves.
- Ultimately increases the trust in the overall proof and tactic development.

Current Reality:
- Some external mechanisms requiring extra maintenance of their tactic engines and tactics.
The Promise: Write type-safe tactics or macros within the same language.

- users do not need to learn and use a separate tactic language
- we can prove properties about tactics themselves
- ultimately increases the trust in the overall proof and tactic development

Current Reality: Some external mechanisms requiring extra maintenance of their tactic engines and tactics.
A glimpse of the past

Simply-Typed Meta-Programming
Early 2000: A Modal Analysis of Simply-Typed Metaprogramming

1995
• F. Pfenning and Hao-Chi Wong. On a modal lambda calculus for S4, MFPS’95

1996
• R. Davies and F. Pfenning. A modal analysis of staged computation, POPL’96

2001
• R. Davies and F. Pfenning. A modal analysis of staged computation, JACM’01

F. Pfenning and R. Davies. A judgmental reconstruction of modal logic, MSCS’01

Key Idea:
□τ describes closed code of type τ

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Simply Typed Modal Lambda Calculus with box-modality

- Explicit Modal Lambda Calculus – with box and let-box
  Distinguish between global and local assumptions – two zones

- Implicit **Kripke-style Modal Lambda-Calculus** with box and unbox
  Kripke-style context stack $\Gamma_0; \ldots; \Gamma_n$ where variables at stage $i$ are bound in the context $\Gamma_i$
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### Simply Typed Modal Lambda Calculus with box-modality

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  Kripke-style context stack $\Gamma_0; \ldots; \Gamma_n$ where variables at stage $i$ are bound in
  the context $\Gamma_i$
Modal Lambda-Calculi directly derived from Pfenning and Davies’ work have been used in a wide range of seemingly unconnected applications:

- Contextual Modal Type Theory [Pfenning, Nanevski, Pientka 2008]
- Mechanizing meta-theory [Pientka et al 2008,..., 2019]
- Programming with algebraic effects [Nanevski et al 2021]
- Reasoning about universes in homotopy type [Licata, Shulman, et al 2015, 2018]
- ...
The Unusual Effectiveness of Modal Types

Modal Lambda-Calculi directly derived from Pfenning and Davies’ work have been used in a wide range of seemingly unconnected applications:

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- ...

Yet, a practical and sound modal type theory that supports meta-programming has been elusive.
A glimpse of the future
MINTS— a Modal INtuitionistic Type theory with Stages

Kripke-Style Martin-Löf type theory that supports homogeneous multi-staged programming in the spirit of Scheme or Racket’s quote-unquote style.

supports large elimination and a full cumulative universe hierarchy.
Kripke-Style Martin-Löf type theory that supports homogeneous multi-staged programming in the spirit of Scheme or Racket’s quote-unquote style.

supports large elimination and a full cumulative universe hierarchy.

- Extends Pfenning and Davies’ work to dependent types
- Generate and share code across multiple stages
- Specify strong guarantees of multi-staged program using dependent types (especially large eliminations)
- Reason about multi-staged programs and prove them correct
How do we get there...?
Kripke-style modal lambda-calculus – simply typed!

- Each Kripke world of modal logic corresponds to a stage in the computation
- A term of type $\square \tau$ corresponds to the code of a program of type $\tau$ in a future stage of computation
Local Variable

\[
\frac{x: \tau \in \Gamma_n}{\Gamma_0; \ldots; \Gamma_n \vdash x : \tau}
\]
Kripke-style modal lambda-calculus – simply typed!

Local Variable

\[ \frac{\Gamma \vdash x : \tau}{\Gamma_0; \ldots ; \Gamma_n \vdash x : \tau} \]

Box Introduction (push context onto context stack)

\[ \frac{\Gamma \vdash t : \tau}{\Gamma ; \Gamma_1; \ldots ; \Gamma_n \vdash \text{box } t : \Box \tau} \]

The modal offset \( n \) corresponds to reflexivity and transitivity of the accessibility relation between worlds in the Kripke semantics.
Local Variable

\[\frac{x:\tau \in \Gamma_n}{\Gamma_0; \ldots; \Gamma_n \vdash x : \tau}\]

Box Introduction (push context onto context stack)

\[\frac{\Gamma ; \Gamma ; \vdash t : \tau}{\Gamma ; \Gamma \vdash \text{box } t : \Box \tau}\]

Unbox Elimination (pop context(s) of context stack)

\[\frac{\Gamma ; \Gamma_0 \vdash t : \Box \tau}{\Gamma ; \Gamma_0; \Gamma_1; \ldots; \Gamma_n \vdash \text{unbox}_n t : \tau}\]

The modal offset \( n \) corresponds to reflexivity and transitivity of the accessibility relation between worlds in the Kripke semantics.
Local structural properties and modal transformations

Local structural properties of a context $\Gamma_i$

- Weakening of a context $\Gamma_i$
- Substitution for assumptions in a context $\Gamma_i$

$$
\begin{align*}
\Gamma; \Gamma_0, x: \tau' \vdash t : \tau \\
\Gamma; \Gamma_0 \vdash \lambda x.t : \tau' \rightarrow \tau \\
\Gamma; \Gamma_0 \vdash t' : \tau' \\
\Gamma; \Gamma_0 \vdash (\lambda x.t) t' : \tau \\
\Gamma; \Gamma_0 \vdash [t'/x]t : \tau
\end{align*}
$$
Local structural properties and modal transformations

Local structural properties of a context $\Gamma_i$

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- Substitution for assumptions in a context $\Gamma_i$

Modal transformations (MoTs) between context stacks

- Modal weakening and fusion of context stack $\Gamma; \Gamma_0$

\[
\frac{\Gamma; \Gamma_0; \cdot \vdash t : \tau}{\Gamma; \Gamma_0 \vdash \text{box } t : \Box \tau}
\]

\[
\frac{\Gamma; \Gamma_0; \Gamma_1; \ldots; \Gamma_n \vdash \text{unbox}_n (\text{box } t) : \tau}{\Gamma; \Gamma_0; \Gamma_1; \ldots; \Gamma_n \vdash t' : \tau}
\]

where $t'$ is obtained from $t$ by modal weakening / fusion.
Local structural properties and modal transformations

Local structural properties of a context $\Gamma_i$

- Weakening of a context $\Gamma_i$
- Substitution for assumptions in a context $\Gamma_i$

Modal transformations (MoTs)

- Modal weakening and fusion of context stack

$$
\Gamma ; \Gamma \quad \xrightarrow{\text{modal weakening}} \\
\Gamma ; \Gamma ; \Gamma ; \ldots ; \Gamma_n \quad \xrightarrow{\text{modal fusion}} \\
\Gamma ; \Gamma ; \Gamma ; \ldots ; \Gamma_n \\
$$

where $t'$ is obtained from $t$ by modal weakening / fusion.

Typically, local context and modal context stack transformations are thought of as separate concepts.

This is problematic, since simultaneous substitutions are central in

- Implementations based on explicit substitution
- Normalization proofs using logical relations
- Mechanizations based on de Bruijn

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Towards MINTS by examples
Example: Staged Power Function

\[
\begin{align*}
\text{pow} & : \text{Nat} \rightarrow \square (\text{Nat} \rightarrow \text{Nat}) \\
\text{pow zero} & = \text{box } \lambda \ x \rightarrow 1 \\
\text{pow (succ n)} & = \text{box } \lambda \ x \rightarrow ((\text{unbox}_1 (\text{pow n})) \ x) \ * \ x
\end{align*}
\]
Example: Staged Power Function

\[ \text{pow : Nat} \rightarrow \square (\text{Nat} \rightarrow \text{Nat}) \]
\[ \text{pow zero} = \text{box } \lambda x \rightarrow 1 \]
\[ \text{pow (succ n)} = \text{box } \lambda x \rightarrow ((\text{unbox}_{1} (\text{pow n})) x) * x \]

Typically, code generation does not evaluate code inside a box.

\[ \text{pow 2} = \text{box } \lambda x \rightarrow ((\text{unbox}_{1} (\text{pow 1})) x) * x \]
\[ = \text{box } \lambda x \rightarrow ((\text{unbox}_{1} (\text{box } \lambda y \rightarrow ((\text{unbox}_{1} (\text{pow 0})) y) * y)) x) * x \]
\[ = \text{box } \lambda x \rightarrow ((\lambda y \rightarrow ((\text{unbox}_{1} (\text{pow 0})) y) * y) x) * x \]
\[ = \text{box } \lambda x \rightarrow ((\lambda y \rightarrow ((\text{unbox}_{1} (\text{box } \lambda z \rightarrow 1)) y) * y) x) * x \]
\[ = \text{box } \lambda x \rightarrow ((\lambda y \rightarrow ((\lambda z \rightarrow 1) y) * y) x) * x \]
**Example: Staged Power Function**

\[
\text{pow} : \text{Nat} \rightarrow \Box (\text{Nat} \rightarrow \text{Nat})
\]

\[
\text{pow} \text{ zero} = \text{box} \lambda x \rightarrow 1
\]

\[
\text{pow} (\text{succ n}) = \text{box} \lambda x \rightarrow ((\text{unbox}_1 (\text{pow} \ n)) x) \times x
\]

Typically, code generation does not evaluate code inside a box.

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\text{pow} 2 = \text{box} \lambda x \rightarrow ((\text{unbox}_1 (\text{pow} 1)) x) \times x
\]

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= \text{box} \lambda x \rightarrow ((\text{unbox}_1 (\text{box} \lambda y \rightarrow ((\text{unbox}_1 (\text{pow} 0)) y) \times y)) x) \times x
\]

\[
= \text{box} \lambda x \rightarrow ((\lambda y \rightarrow ((\text{unbox}_1 (\text{pow} 0)) y) \times y) x) \times x
\]

\[
= \text{box} \lambda x \rightarrow ((\lambda y \rightarrow ((\text{unbox}_1 (\text{box} \lambda z \rightarrow 1)) y) \times y) x) \times x
\]

\[
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\]

**What code shall we generate in a type theory?**
Example: Staged Power Function

\[ \text{pow} : \text{Nat} \to \square (\text{Nat} \to \text{Nat}) \]
\[ \text{pow zero} = \text{box} \lambda x \to 1 \]
\[ \text{pow} (\text{succ} \ n) = \text{box} \lambda x \to ((\text{unbox}_1 (\text{pow} \ n)) \ x) \ast x \]

Typically, code generation does not evaluate code inside a box.

\[ \text{pow} 2 = \text{box} \lambda x \to ((\text{unbox}_1 (\text{pow} \ 1)) \ x) \ast x \]
\[ = \text{box} \lambda x \to ((\text{unbox}_1 (\text{box} \lambda y \to ((\text{unbox}_1 (\text{pow} \ 0)) \ y) \ast y)) \ x) \ast x \]
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**What code shall we generate in a type theory?** Evaluation = Normalization!
Example: Staged Power Function

\[
pow : \text{Nat} \rightarrow \Box (\text{Nat} \rightarrow \text{Nat})
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Typically, code generation does not evaluate code inside a box.

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= \text{box} \lambda x \rightarrow ((\lambda y \rightarrow ((\text{unbox}_1 (pow\ 0)) \ y) \ast y) \ x) \ast x
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= \text{box} \lambda x \rightarrow ((\lambda y \rightarrow ((\text{unbox}_1 (\text{box} \lambda z \rightarrow 1)) \ y) \ast y) \ x) \ast x
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= \text{box} \lambda x \rightarrow ((\lambda y \rightarrow ((\lambda z \rightarrow 1) \ y) \ast y) \ x) \ast x
\]

What code shall we generate in a type theory? Evaluation = Normalization!

We typically identify terms up to \(\beta\eta\) (and for example normalize under a \(\lambda\)-abstraction. )

\[\Longrightarrow\ \text{Consequently we normalize also under a box!}\]
Example: Staged Power Function

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Evaluation by Normalization:

\[ \text{pow} 2 = \text{box} \lambda x \to ((\text{unbox}_1 (\text{pow} 1)) x) \times x \]

\[ = \text{box} \lambda x \to ((\text{unbox}_1 (\text{box} \lambda y \to ((\text{unbox}_1 (\text{pow} 0)) y) \times y)) x) \times x \]

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\[ = \text{box} \lambda x \to x \times x \]

Avoids administrative redeces!
Example: Generating N-ary Sum Function

Idea:

- If $n$ is zero, then we return zero;
- If $n$ is one, then we return the identity function;
- If $n$ is two, then we return the function that sums up two arguments, i.e.
  
  \[
  \text{box } \lambda x \ y \to x + y.
  \]
- etc.
Example: Generating N-ary Sum Function

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  \text{box } \lambda x y \rightarrow x + y.
  \]
- etc.

Step 1: Type-level function \texttt{nary} $n$ which computes the type of an $n$-ary function:

\[
\begin{align*}
\text{nary} &: \text{Nat} \rightarrow \text{Se} \\
\text{nary} \text{ zero} &= \text{Nat} \\
\text{nary} \text{ (succ n)} &= \text{Nat} \rightarrow \text{nary} \text{ n}
\end{align*}
\]
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  \text{nary} : \text{Nat} \rightarrow \text{Se}
  \]

  \[
  \text{nary zero} = \text{Nat} \]

  \[
  \text{nary (succ n)} = \text{Nat} \rightarrow \text{nary } n
  \]

**Step 2:** Define \( \text{n-ary-sum } : (n : \text{Nat}) \rightarrow \square (\text{nary } n) \)
Example: Generating N-ary Sum Function

**Idea:**

- If \( n \) is zero, then we return \( \text{zero} \);
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\]
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\text{nary (succ n)} = \text{Nat} \rightarrow \text{nary n}
\]

**Step 2:** Define \( n \)-ary-sum : \((n : \text{Nat}) \rightarrow \square (\text{nary (unbox}_1 \ (\text{lift } n)))\)
Example: Generating N-ary Sum Function

Idea:

- If $n$ is zero, then we return $\text{zero}$;
- If $n$ is one, then we return the identity function;
- If $n$ is two, then we return the function that sums up two arguments, i.e.
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  \text{box } \lambda x y \rightarrow x + y.
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- etc.
Example: Generating N-ary Sum Function

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- If $n$ is zero, then we return $\text{zero}$;
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- If $n$ is two, then we return the function that sums up two arguments, i.e.
  $$\text{box} \lambda x y \rightarrow x + y.$$  
- etc.

Step 2: Implementation $\text{n-ary-sum}$

\[
\begin{align*}
\text{nary-sum} & : (n : \text{Nat}) \rightarrow \square (\text{nary} \ (\text{unbox}_1 \ (\text{lift} \ n))) \\
\text{nary-sum zero} & = \text{box} \ \text{zero} \\
\text{nary-sum} \ (\text{succ} \ \text{zero}) & = \text{box} \ \lambda x \rightarrow x \\
\text{nary-sum} \ (\text{succ} \ (\text{succ} \ n)) & = \text{box} \ \lambda x y \rightarrow (\text{unbox}_1 \ (\text{nary-sum} \ (\text{succ} \ n))) \ (x + y)
\end{align*}
\]
Example: Generating N-ary Sum Function

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\begin{align*}
\text{nary-sum} & : (n : \text{Nat}) \rightarrow \square (\text{nary (unbox}_1 (\text{lift } n))) \\
\text{nary-sum zero} & = \text{box zero} \\
\text{nary-sum (succ zero)} & = \text{box } \lambda \ x \rightarrow \ x \\
\text{nary-sum (succ (succ n))} & = \text{box } \lambda \ x \ y \rightarrow (\text{unbox}_1 (\text{nary-sum (succ n)))) (x \ + \ y)
\end{align*}
\]
Example: Checking that implementation works as intended

**Step 3:** Test that \( n\text{-}ary\text{-}sum \) works as intended.

\[
nary\text{-}sum\text{-}3 : nary\text{-}sum\ 3 \equiv bo\ x\ y\ z \mapsto (x + y) + z
\]

\[
nary\text{-}sum\text{-}3 = \text{refl}
\]

**Step 4:** Prove more general soundness theorems

Summing over a list \( l \) of \( n \) natural numbers returns the same result as generating code using \( nary\text{-}sum \ n \) and then applying it to all the numbers in \( l \).
A glimpse of MINTS:
A Kripke-Style Modal Type Theory
Kripke-style Explicit Formulation: box - unbox (REVISITED)

Kripke Context Stack $\Gamma$

```
\Gamma_0; \ldots; \Gamma_n \vdash t : \tau
```

Term

Type

Local Variable

```
\frac{x: \tau \in \Gamma_n}{\Gamma_0; \ldots; \Gamma_n \vdash x : \tau}
```
Kripke-style Explicit Formulation: box - unbox (REVISITED)

Local Variable

\[ \Gamma_0; \ldots; \Gamma_n \vdash t : \tau \]

\[ x : \tau \in \Gamma_n \]

\[ \Gamma_0; \ldots; \Gamma_n \vdash x : \tau \]

The Problem in the Dependent Typed Setting:

We have dependencies within one context and across the context stack!
Kripke-style Explicit Formulation: box - unbox (REVISITED)

Kripke Context Stack: \[ \Gamma_0; \ldots; \Gamma_n \vdash t : \tau \]

Local Variable:

\[ \Gamma_n = \Gamma, x : \tau, \Gamma' \]

\[ \Gamma_0; \ldots; \Gamma_n \vdash x : [w] \tau \]

The Problem in the Dependent Typing Setting:

We have dependencies within one context and across the context stack!

Solution: Ordinary weakening wrt to a context
**Unbox Elimination** (pop context(s) of context stack)

\[
\frac{\Gamma; \Gamma_0 \vdash t : \Box \tau}{\Gamma \vdash \text{unbox}_n \ t : \tau}
\]
Unbox Elimination (pop context(s) of context stack)

\[ \Gamma; \Gamma_0 \vdash t : \square \tau \]
\[ \Gamma; \Gamma_0; \Gamma_1 \ldots ; \Gamma_n \vdash \text{unbox}_n \ t : \tau \]

The Problem in the Dependentely Typed Setting:

We have dependencies within one context and across the context stack!
Unbox Elimination \((\text{pop context(s) of context stack})\)

\[
\Gamma; \Gamma_0 \vdash t : \Box \tau \\
\Gamma; \Gamma_0; \Gamma_1 \ldots; \Gamma_n \vdash \text{unbox}_n t : \tau
\]

The Problem in the Dependently Typed Setting:

For the premise we have:

- \(\Gamma; \Gamma_0; \cdot \vdash \tau : \text{Se}\)

In the conclusion we need:

- \(\Gamma; \Gamma_0; \Gamma_1 \ldots; \Gamma_n \vdash \tau : \text{Se}\)

**Required:** Weaken with \(\Gamma_1\) and Modal Weakening wrt context stack \(\Gamma_2; \ldots \Gamma_n\)
A general transformation between context stacks

Unified (Simultaneous) Substitutions

Local Context $\Gamma, \Delta := \cdot | \Gamma, x:\tau$

Context Stack $\vec{\Gamma}, \vec{\Delta} := \epsilon; \vec{\Gamma}; \Gamma$

Local Substitution $\sigma := \cdot | \sigma, t/x$

Substitution Stack $\vec{\sigma} := \epsilon; \sigma | \vec{\sigma}; \uparrow^n \sigma$

$$
\frac{\vec{\Gamma} \vdash \vec{\sigma} : \vec{\Delta}}{
\vec{\Gamma}; \Gamma_1; \ldots; \Gamma_n \vdash \vec{\sigma} : \vec{\Delta} 
}
$$

$$
\frac{\vec{\Gamma}; \Gamma_1; \ldots; \Gamma_n \vdash \vec{\sigma}; \uparrow^n \sigma : \vec{\Delta} ; \Delta}
$$
Unified Substitution Operation

\[
[\overrightarrow{\sigma}; \uparrow^k\sigma]x := \sigma(x) \quad \text{lookup } x \text{ in } \sigma
\]

\[
[\overrightarrow{\sigma}; \uparrow^k\sigma](\lambda x. t) := \lambda x.[\overrightarrow{\sigma}; \uparrow^k(\sigma, x/x)]t
\]

\[
[\overrightarrow{\sigma}; \uparrow^k\sigma](s \ t) := [\overrightarrow{\sigma}; \uparrow^k\sigma]s \ [\overrightarrow{\sigma}; \uparrow^k\sigma]t
\]

\[
[\overrightarrow{\sigma}; \uparrow^k\sigma](\text{box } t) := \text{box } [\overrightarrow{\sigma}; \uparrow^k\sigma; \uparrow^1.]t
\]

\[
[\overrightarrow{\sigma}; \uparrow^k\sigma](\text{unbox}_n \ t) := ?
\]
Unified Substitution Operation

\[ [\overrightarrow{\sigma}; \uparrow^k \sigma] x \ := \ \sigma(x) \] lookup \( x \) in \( \sigma \)

\[ [\overrightarrow{\sigma}; \uparrow^k \sigma](\lambda x. t) \ := \ \lambda x.[\overrightarrow{\sigma}; \uparrow^k(\sigma, x/x)] t \]

\[ [\overrightarrow{\sigma}; \uparrow^k \sigma](s \ t) \ := \ [\overrightarrow{\sigma}; \uparrow^k] s \ [\overrightarrow{\sigma}; \uparrow^k] t \]

\[ [\overrightarrow{\sigma}; \uparrow^k \sigma](\text{box } t) \ := \ \text{box } [\overrightarrow{\sigma}; \uparrow^k; \uparrow^1] t \]

\[ [\overrightarrow{\sigma}; \uparrow^k \sigma](\text{unbox}_n \ t) \ := ? \]

Recall – **Unbox Elimination** (pop context(s) of context stack)

\[
\begin{align*}
\Gamma; \Gamma_0 &\vdash t : \Box \tau \\
\Gamma; \Gamma_0; \ldots; \Gamma_n &\vdash \text{unbox}_n \ t : \tau
\end{align*}
\]

\[ \implies \text{truncate the unified substitution } [\overrightarrow{\sigma}; \uparrow^k \sigma] \text{ to apply it to } t \text{ on the rhs} \]
Truncation and Truncation Offset

Typing of unified substitution: \( \Gamma \vdash \sigma : \Delta \).

**Truncation** : \( \sigma \mid n \)

- Returns a prefix of \( \sigma \) with domain context stack \( \Delta \mid n \)

**Truncation Offset** \( O(\sigma, n) = k \)

- Sums over all the modal offsets in the truncated part of \( \sigma \)
- Used to adjust the range \( \Gamma \) s.t. \( \sigma \mid n \) remains well-typed.
Truncation and Truncation Offset

**Typing of unified substitution:** \( \Gamma \vdash \sigma : \Delta \).

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- Sums over all the modal offsets in the the truncated part of \( \sigma \)
- Used to adjust the range \( \Gamma \) s.t. \( \sigma \mid n \) remains well-typed.

**Typing of truncated substitution:** \( (\Gamma \mid k) \vdash (\sigma \mid n) : (\Delta \mid n) \)
Revisiting Unbox

**Typing**

\[
\Gamma_0 ; \Gamma \vdash t : \square \tau \quad \Gamma_0 ; \Gamma \vdash \tau : \text{Se} \quad \Gamma_0 ; \Gamma_1 ; \ldots ; \Gamma_n \vdash \text{unbox}_n \ t : \left[ I ; \uparrow^n \cdot \right] \tau
\]

**Applying unified substitution**

\[
[\sigma ; \uparrow^k \sigma](\text{unbox}_n \ t) := \text{unbox}_{n'} \left[ \sigma' \right] t
\]

where

\[
\sigma' = (\sigma ; \uparrow^k \sigma) | n
\]

Truncate \( \sigma ; \uparrow^k \sigma \)

and

\[
n' = \mathcal{O}(\sigma ; \uparrow^k \sigma , n)
\]

Compute the truncation offset
Unified Substitutions are key.
Point 1: Unified Substitutions enable normalization for MINTS

Normalization by Evaluation algorithm for MINTS following Abel’13

- Algebraic uniform characterization of the Kripke-structure based on unified substitutions (syntax) using untyped modal transformation (semantics)
- A core modal dependent type theory as an explicit substitution calculus → our NbE algorithm applies to all 4 subsystems of S4
- Completeness proof for the NbE algorithm uses a partial equivalence relation (PER) model for the untyped domain terms together with untyped modal transformations.
- Soundness proof for the NbE algorithm based on a Kripke glueing model
  Exploits a special class of unified substitutions, restricted to weakenings.
- NbE soundness and completeness mechanized in Agda (11K) only exploits function extensionality and induction-recursion
- Mechanization exposes common oversimplification in how cumulativity of universes
Point 2: Contextual Box and Unbox to Handle Open Code

\[ \tau ::= \ldots \]

Terms \( t ::= \ldots \), box \( (\rightarrow \Delta \vdash \tau) \)

Introduction

\[ \Gamma ; \rightarrow \Delta \vdash \vdash t : \tau \]

\[ \rightarrow \Gamma \vdash \vdash \Box (\rightarrow \Delta \vdash \tau) : \Box (\rightarrow \Delta \vdash \tau) \]

Elimination

\[ \Gamma | n \vdash \vdash t : \Box (\rightarrow \Delta \vdash \tau) \]

\[ \rightarrow \Gamma \vdash \vdash \vdash \sigma :: \rightarrow \Delta \vdash \Box (t, \vdash \sigma) : \Box \]

Previously:
The modal offset at the unbox only allowed for modal weakening.

Now:
Unified substitution allows for modal weakening and instantiation of variables.
Point 2: Contextual Box and Unbox to Handle Open Code

Types $\tau := \ldots \mid \text{box } (\Delta \vdash \tau)$

Terms $t := \ldots \mid \text{box } (\Delta \vdash t) \mid \text{unbox } (t, \vec{\sigma})$ where $\vec{\sigma}$ is a partial unified substitution.
Point 2: Contextual Box and Unbox to Handle Open Code

Types  \( \tau := \ldots \mid \text{box} (\overrightarrow{\Delta} \vdash \tau) \)

Terms  \( t := \ldots \mid \text{box} (\overrightarrow{\Delta} \vdash t) \mid \text{unbox} (t, \overrightarrow{\sigma}) \) where \( \overrightarrow{\sigma} \) is a partial unified substitution.

**Introduction**

\[
\frac{\overrightarrow{\Gamma} ; \overrightarrow{\Delta} \vdash t : \tau}{\overrightarrow{\Gamma} \vdash \text{box} (\overrightarrow{\Delta} \vdash t) : \text{box} (\overrightarrow{\Delta} \vdash \tau)}
\]
Point 2: Contextual Box and Unbox to Handle Open Code

Types  \( \tau := \ldots \mid \text{box} (\vec{\Delta} \vdash \tau) \)

Terms  \( t := \ldots \mid \text{box} (\vec{\Delta} \vdash t) \mid \text{unbox} (t, \vec{\sigma}) \) where \( \vec{\sigma} \) is a partial unified substitution.

### Introduction

\[
\frac{\vec{\Gamma}; \vec{\Delta} \vdash t : \tau}{\vec{\Gamma} \vdash \text{box} (\vec{\Delta} \vdash t) : \text{box} (\vec{\Delta} \vdash \tau)}
\]

### Elimination

\[
\frac{\vec{\Gamma} \mid n \vdash t : \text{box} (\vec{\Delta} \vdash \tau) \quad \vec{\Gamma} \vdash \vec{\sigma} :: \vec{\Delta}}{\vec{\Gamma} \vdash \text{unbox} (t, \vec{\sigma}) : [\vec{\Gamma}; \vec{\sigma}] \tau}
\]

where \( \mathcal{O}(\vec{\sigma}) = n \)

- **Previously:** The modal offset at the unbox only allowed for modal weakening
- **Now:** Unified substitution allow for modal weakening and instantiation of the variables \( \vec{\Delta} \)
Current Status

- **MINTS**: Kripke-style modal type theory (joint work with J. Z. Hu and J. Jang)
- Categorical view of modal lambda-calculi (joint work with J. Z. Hu) [MFPS’22]
- System F-style meta-programming with pattern matching (joint work with J. Jang, S. Gélineau, S. Monnier) [POPL’22]
Lesson 1: Logic through the Curry-Howard isomorphism allows us to gain a deeper understanding of computational phenomena.

Lesson 2: While other approaches exist to support type-safe generation of typed code, they are not logically motivated.

Lesson 3: Unified substitutions provide a general concept to capture transformation between context stacks both syntactically and semantically.

Lesson 4: Good first step towards a dependently typed foundation for meta-programming!

What’s next? – How to support pattern matching in MINTS.