Beluga$^{\mu}$: Programming proofs in context ...

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- Proofs (that a given property is satisfied) are an integral part of the software.

What are good meta-languages to program and reason with formal systems and proofs?

This talk

Design and implementation of Beluga

- Introduction
- Example: Type uniqueness proof
- Writing a proof in Beluga ...
- Wanting more: Programming code transformations
  - Sketching closure conversion
  - Sketching normalization by evaluation
- Conclusion

"The tools we use have a profound (and devious!) influence on our thinking habits, and, therefore, on our thinking abilities."

- Edsger Dijkstra

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Simply typed lambda-calculus

Types and Terms

$$\text{Types } T \quad ::= \quad \mathsf{nat} \mid \mathsf{arr}\, T_1 \; T_2$$

$$\text{Terms M} \quad ::= \quad x \mid \mathsf{lam}\; x{:}T.M \mid \mathsf{app}\; M \; N$$

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Typing Judgment: $\mathsf{oft}\ M\ T$ read as "M has type T"

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$\text{Context } \Gamma \quad ::= \quad \cdot \mid \Gamma, x, \; \text{oft } x \; T$ We are introducing the variable x together with the assumption $\text{oft } x \; T$

Simply typed lambda-calculus

Types and Terms

Types T ::= nat | $T_1 \rightarrow T_2$

Terms M ::= x | lam $x{:}T.M$ | app M N

Typing Judgment: $\Gamma \vdash$ oft M T read as "M has type T in context Γ"

Typing rules

$$\frac{x, u : \text{ oft } x\ T \in \Gamma}{\Gamma \vdash \text{ oft } x\ T}\ u$$

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Talking about derivations

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- What kinds of variables are used? Bound variables, Schematic variables in particular:Meta-variables, Parameter variables, Context variables

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Any mechanization of proofs must deal with these issues; it is just a matter how much support one gets in a given meta-language.

Type uniqueness

Theorem

If $\mathcal{D} : \Gamma \vdash \mathsf{oft}\ M\ T$ and $\mathcal{C} : \Gamma \vdash \mathsf{oft}\ M\ S$ then $\mathcal{E} :\ \mathsf{eq}\ T\ S$.

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Induction on first typing derivation $\mathcal{D}$.

Case 1

$$\mathcal{D} = \cfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, x, u\text{: oft } x\ T \vdash \text{oft } M\ S\end{array}}{\Gamma \vdash \text{oft (lam } x{:}T.M) \text{ (arr } T\ S)}\ \text{t\_lam} \quad \mathcal{C} = \cfrac{\begin{array}{c}\mathcal{C}_1\\ \Gamma, x, u\text{: oft } x\ T \vdash \text{oft } M\ S'\end{array}}{\Gamma \vdash \text{oft (lam } x{:}T.M) \text{ (arr } T\ S')}\ \text{t\_lam}$$

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$\mathcal{E} :\ \mathsf{eq}\ S\ S$ and $S = S'$ by inversion using reflexivity

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Therefore there is a proof for eq $(\mathsf{arr}\ T\ S)\ (\mathsf{arr}\ T\ S')$ by reflexivity.

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Every variable x is associated with a unique typing assumption (property of the context), hence $v = u$ and $S = T$.

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Logical framework LF [HHP'93]

- Compact representation of formal systems and derivations
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On paper proof	Proofs as functions in Beluga
Case analysis	Case analysis and pattern matching
Inversion	Pattern matching using let-expression
Induction Hypothesis	Recursive call

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- Contextual types characterize contextual objects [NPP'08]
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- Context variables parameterize computations
  $\leadsto$ fine grained invariants; distinguish between different contexts
- Recursive types express relationships between contexts and contextual objects
  $\leadsto$ adds expressive power! (See POPL'12)

Step 1: Represent types and lambda-terms in LF

$$\text{Types } T \quad ::= \quad \mathsf{nat} \mid \mathsf{arr}\ T_1\ T_2$$

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Types T ::= nat
| arr T_1 T_2

Terms M ::= x
| lam $x{:}T.M$
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LF representation in Beluga

```
datatype tp:type =
| nat: tp
| arr: tp → tp  → tp;
```

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datatype exp: type =
| lam: tp →  (exp → exp)  → exp
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```
datatype oft: exp → tp → type =
| t_app: oft M (arr T S) → oft N T
    → oft (app M N) S
| t_lam: (Π x:exp.oft x T → oft (M x) S)
    → oft (lam T M) (arr T S);
```

Step 2a: Theorem as type

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Theorem

If $\mathcal{D} : \Gamma \vdash \mathsf{oft}\ M\ T$ and $\mathcal{C} : \Gamma \vdash \mathsf{oft}\ M\ S$ then $\mathcal{E} :\ \mathsf{eq}\ T\ S$.

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```

Read as: "For all contexts `g` of the schema `ctx`, ...

- `[g.oft (M …) T]` and `[ .eq T S]` are contextual types [NPP'08].
- … describes dependency on context.
  `T` is a closed object `(M …)` is an object which may depend on context `g`.

Intrinsic support for contexts

Computation-level Type in Beluga

```
(g:ctx) [g.oft (M ...) T] → [g.oft (M ...) S] →  [.eq T S]
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- Parameterize computation over contexts, Distinguish between contexts.

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  `x:exp, y:exp, u:oft x nat` is ill-formed.

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- Declarations are unique: `b1` is different from `b2`
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- Later declarations overshadow earlier ones
- Weakening, Substitution lemma

Accessing objects in contexts

- How do we access objects from a context?

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- Allow projections on variables and parameter variables only

"Making something variable is easy. Controlling duration of constancy is the trick." *Alan Perlis*

Step 2b: Proofs as Programs

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rec `unique:(g:ctx) [g.oft (M …) T]` $\rightarrow$ `[g.oft (M …) S]` $\rightarrow$ `[.eq T S] =`

Step 2b: Proofs as Programs

```
rec unique:(g:ctx) [g.oft (M …) T] → [g.oft (M …) S] → [.eq T S] =
fn d ⇒ fn c ⇒ case d of
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Step 2b: Proofs as Programs

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rec unique:(g:ctx) [g.oft (M ...) T] → [g.oft (M ...) S] → [.eq T S] =
fn d ⇒ fn c ⇒ case d of
| [g.t_app (D1 ...) (D2 ...)] ⇒                    % Application Case
  let [g.t_app (C1 ...) (C2 ...)] = c in
  let [ .e_ref] = unique [g.D1 ...] [g.C1 ...] in
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Recalll:

```
#q:block x:exp, u:oft x T
#r:block x:exp, u:oft x S
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We also know: #r.1 = #q.1

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```

Recalll:

`#q:block x:exp, u:oft x T`

`#r:block x:exp, u:oft x S`

We also know:	#r.1	=	#q.1
Therefore:	T	=	S

Revisiting the design of Beluga

- Compact adequate representation of derivations and contexts

On paper proof	Implementation in Beluga
Well-formed derivations	Dependent types
Renaming,Substitution	α-renaming, β-reduction in LF

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Comparison

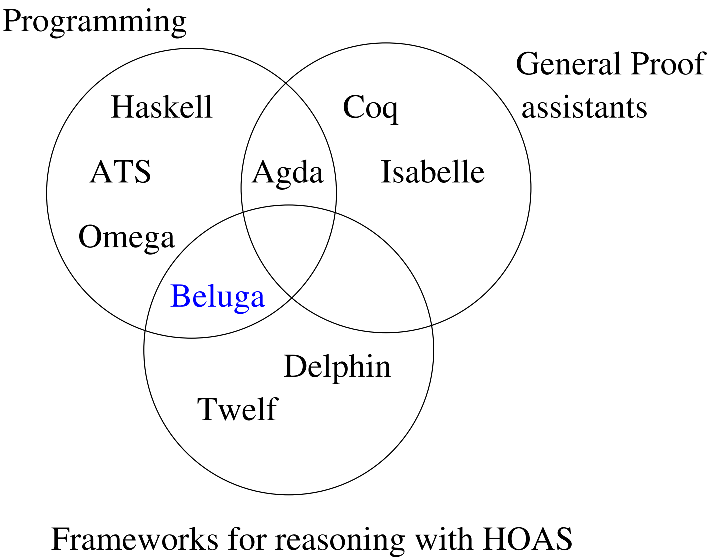
- Twelf [Pf,Sch'99]: Encode proofs as relations
  - Requires lemma to prove injectivity of `arr` constructor.
  - No explicit contexts (cannot express types `T` and `S` and `eq T S` are closed)
  - Parameter case folded into abstraction case
- Delphin [Sch,Pos'08]: Encode proofs as functions
  - Requires lemma to prove injectivity of constructor
  - Cannot express that types `T` and `S` and `eq T S` are closed.
  - Variable carrying continuation as extra argument to handle context lookup
- Abella [Gacek'08], Tac[Baelde'10]: Proof assistants
  - Equality built-into the logic
  - Contexts are represented as lists
  - Requires lemmas about these lists (for example that all assumptions occur uniquely)

This talk

Design and implementation of Beluga

- Introduction
- Example: Type uniqueness
- Writing a proof in Beluga ...
- Wanting more: Programming code transformations
    - Sketching closure conversion
    - Sketching normalization by evaluation
- Conclusion

Three solitudes



Example: Closure conversion

- Translate λ-terms such that bodies only refer to their arguments

Source language		Target language
(lam $y.x + y$) 3	$\Longrightarrow$	(lam $env.env.2 + env.1$) $(3\,,\,x)$

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- Programming in context in Beluga
  - Distinguish between source language `tm` and target language `ctm`
  - Translate `[`ψ`.tm]` where ψ is a source context
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Computation-level Type in Beluga

$\textbf{rec}\ \ \texttt{conv}\ :\texttt{Ctx\_rel}\ [\psi]\ [\phi] \rightarrow [\psi.\ \texttt{tm}] \rightarrow [\phi.\texttt{ctm}]$

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rec `conv :Ctx_rel` $[\psi]$ $[\phi]$ $\to$ `[`ψ`. tm]` $\to$`[`ϕ`.ctm]`

Indexed recursive datatype (POPL'12)

- Example: Relating source and target context

Computation-level data types in Beluga

```
datatype Ctx_rel : {g:ctx}{h:cctx} ctype =
| Rnil  : Ctx_rel [] []
| Rsnoc : Ctx_rel [g] [h]
        → Ctx_rel [g, x:tm] [h,x:ctm] ;
```

Indexed recursive datatype (POPL'12)

- Example: Type preserving context relation

Computation-level data types in Beluga

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datatype  Ctx_trel : {g:tctx}{h:tcctx} ctype =
| Rnil  : Ctx_trel [] []
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```

- Example: Wrapper for contextual objects.

```
datatype  TmVar : {g:tctx} [.tp] → ctype =
| TmVar : {#p:[g.tm T]} TmVar [g] [.T]
;
datatype  CtxObj : {h:cctx} ctype =
| Ctx : {h:cctx} CtxObj [h] ;
```

Indexed recursive datatype (POPL'12)

- Example: Type preserving context relation

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datatype Ctx_trel : {g:tctx}{h:tcctx} ctype =
| Rnil   : Ctx_trel [] []
| Rsnoc : Ctx_trel [g] [h] → Tp_rel [. T] [. S]
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- Example: Wrapper for contextual objects.

```
datatype TmVar : {g:tctx} [.tp] → ctype =
| TmVar : {#p:[g.tm T]} TmVar [g] [.T]
;
datatype CtxObj : {h:cctx} ctype =
| Ctx : {h:cctx} CtxObj [h] ;
```

- Choice how much to push to the computation level

Replacing variables with their projections

- Traverse term in target language by pattern matching on the context

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- Use built-in substitutions to replace `x` with its corresponding projection `proj e N` where `e:envr`.
- Guarantee that **all** variables have been replaced.

Computation in Beluga

```
rec addProjs : (g:cctx) [.nat] → [g, e:envr . ctm] →  [e:envr . ctm] =
fn n ⇒ fn m ⇒ case m of
| [ e:envr . M e ] ⇒ [e:envr . M e]
| [ g, x:ctm , e:envr . M ..  x  e] ⇒
  let  [.N] = n in addProjs [.s N] [g, e:envr . M ..  (proj e N)  e]
;
```

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| [ e:envr . M e ] ⇒ [e:envr . M e]
| [ g, x:ctm , e:envr . M ..  x  e] ⇒
  let [.N] = n in addProjs [.s N] [g, e:envr . M ..  (proj e N)  e]
;
```

- Terminates since context decreases

Converting context to environment

LF representation in Beluga

```
 datatype envr: type =
| nil : envr
| snoc: envr → ctm → envr
and ctm : type =   ...  ;
```

Computation in Beluga

```
rec ctxToEnv : CtxObj [h] → [h . envr] =
fn ctx ⇒ case ctx of
|  Ctx []  ⇒ [ . nil]
|  Ctx [h,x:ctm]  ⇒
  let  [h' . Env .. ] = ctxToEnv (Ctx [h]) in
    [h', x:ctm . snoc (Env ..) x]
;
```

- Convert context to list
- Pattern matching on context

Example: Closure conversion

- Naive Closure conversion [Cave, Pientka'12]
- Type-preserving closure conversion [O. Savary Belanger, M. Boespflug, S. Monnier, B.Pientka]
  - Compact elegant representation
  - Only abstract over the free variables in an expression
  - Enforces also scope preservation
  - Almost proof-less
- Lessons learned:
  - Programming in context requires a new look at existing algorithms
  - Distinguishing between different context natural
  - Indexed data types are key to finding elegant solutions

This talk

Design and implementation of Beluga

- Introduction
- Example: Type uniqueness
- Writing a proof in Beluga ...
- Wanting more: Programming code transformations
  - Sketching closure conversion
  - Sketching normalization by evaluation
- Conclusion

Normalization by evaluation

- Reuse evaluation of computation language to normalize terms in the object language [Berger, Schwichtenberg 91]
- Good benchmark
  - Twelf, Delphin are too weak (to do it directly)
  - Licata and Harper [ICFP'09] cannot express type preservation
  - Coq/Agda lack support for substitutions and binders

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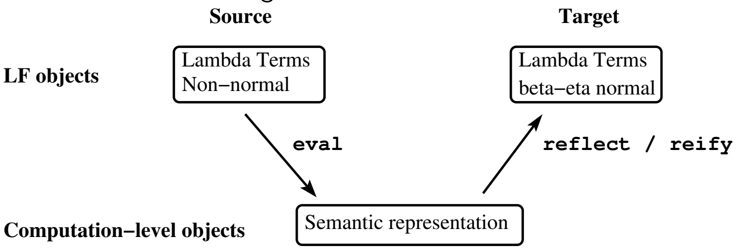
- General idea of NBE in Beluga



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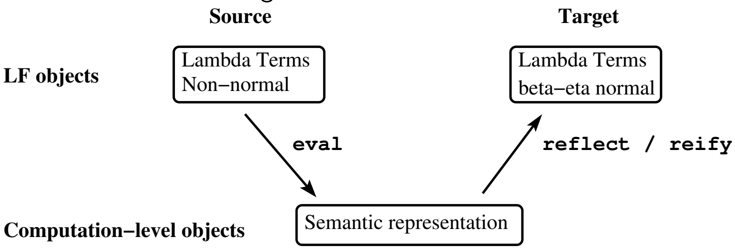

- Evaluation is easy, normalization is hard

NBE in context

Source of type T	Target of type T
$\Gamma \vdash T$	$\Gamma \vdash_n T$ – Normal terms $\Gamma \vdash_r T$ – Neutral terms
Semantic Values of type T $\Gamma \vDash T$	

- Types: $T, S ::= T \Rightarrow S \mid i$
- Definition of semantic values

$$\begin{array}{rcl} \Gamma \vDash i & \equiv_{def} & \Gamma \vdash_n i \\ \Gamma \vDash S \Rightarrow T & \equiv_{def} & \forall \Gamma' \geq \Gamma.\ (\Gamma' \vDash S) \rightarrow (\Gamma' \vDash T) \end{array}$$

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Representation of syntax straightforward

- Source represented in LF using type `tm T`.
- Target represented in LF using type `norm T` and `neut T`.

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How to represent semantic values and context relations?

Defining context extensions using indexed types

- Context `g` is a prefix of context `h`

Computation-level data types in Beluga

```
datatype Extends : {g:ctx} {h:ctx} ctype =
| Zero : Extends [g] [g]
| Succ : Extends [g] [h] → Extends [g] [h,x:neut A]
;
```

- Use indexed types - keyword: **ctype**
- Note: → is overloaded.
  - `tm → tm` is the LF function space : binders in the object language are modelled by LF functions
  - `Extends [g] [h] → Extends [g] [h,x:neut A]` is a computation-level function

Representing target semantic values using indexed types

- Represenation of semantics using computation-level functions

$$
\begin{array}{rcl}
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$$

Computation-level data types in Beluga

```
datatype  Sem : {g:ctx} [. tp] → ctype  =
| Syn :  [g . neut (atomic P)] → Sem [g] [ .atomic P]
| Slam :  ({h:ctx} Extends [g] [h] → Sem [h] [ .S] → Sem [h] [ .T])
           → Sem [g] [ . arr S T]
;
```

- Not a positive definition - we are making no claims regarding strong normalization.

Sketch of normalization by evaluation

- Define mutual recursive functions `reflect` and `reify`

```
rec reflect : [g. neut T] → Sem [g] [ .T] % Recursion on T
and reify   : Sem [g] [ .T] → [g.norm T]  % Recursion on T
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- Map between vars in the source language and their semantic values

```
datatype TmVar : {g:tctx} [.tp] → ctype =
| TmVar : {#p:[g.tm T]} TmVar [g] [.T];
typedef Map : {g:tctx}{h:ctx} ctype = {T:[.tp]} TmVar [g] [.T] → Sem [h] [.T];
```

- Generalized evaluation and normalization followed by reification

```
rec eval     : Map [g] [h] → [g. tm S] → Sem [h] [.S]  = ...
rec evaluate : [. tm S] → Sem [ ] [.S] = fn t ⇒ (eval initialMap t)
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```

- Almost a consistency proof! Currently no termination or positivity checking.

What have we achieved?

- Revised foundation for programming with contexts and contextual LF (joint work with A. Cave [POPL'12])
- Uniform treatment of contextual types, context, ...
- Modular foundation for dependently-typed programming with phase-distinction
  $\Rightarrow$ Generalization of DML and ATS
- Non-termination or effects are allowed
- Effectively write programs to manipulate rich abstract syntax trees and express properties about them
- Release in Sept'12: Support for indexed data types; coverage; type reconstruction; environment-based interpreter; support for holes (partial programs)

Result:

Compact and elegant programming (with) inductive proofs in context

Current work

- Prototype in OCaml (ongoing)
- Extension to coinduction (D. Thibodeau, A. Abel)
- Termination checking (C. Badescu)
- Mixing computations in computation-level types (A. Cave)
- Case study: Certified compiler (O. Savary Belanger)
- Compiling contexts and contextual objects (F. Ferreira)

The end

Thank you!

Download prototype and examples at

`http://complogic.cs.mcgill.ca/beluga/`

Current Belugians: Brigitte Pientka, Mathieu Boespflug, Costin Badescu, Olivier Savary Belanger, Andrew Cave, Francisco Ferreira, Stefan Monnier, David Thibodeau

Interested? - Talk to me! We have funded postdoc and funded PhD positions.