THIS DOCUMENT IS THE ONLINE-ONLY APPENDIX TO:

Higher-order term indexing using substitution trees

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A. DETAILLED PROOFS OF PREVIOUS THEOREMS

THEOREM A.1 SOUNDNESS OF MSLG FOR OBJECTS.
(PREVIOUS THM. 5.2 ON PAGE 21)

(1) *If* $(\Delta, \Omega); \Gamma \vdash M_1 \sqcup M_2 : A \Longrightarrow M/(\Omega', \theta_1, \theta_2)$ *and* $(\Delta, \Omega); \Gamma \vdash M_1 \Leftarrow A$ *and* $(\Delta, \Omega); \Gamma \vdash M_2 \Leftarrow A$ *then* $(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega'$ *and* $(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega'$ *and* $M_1 = [\![\theta_1]\!]M$ *and* $M_2 = [\![\theta_2]\!]M$ *and* $(\Delta, \Omega'); \Gamma \vdash M \Leftarrow A$.

(2) *If* $(\Delta, \Omega); \Gamma \vdash R_1 \sqcup R_2 : P \Longrightarrow R/(\Omega', \theta_1, \theta_2)$ *and* $(\Delta, \Omega); \Gamma \vdash R_1 \Rightarrow P$ *and* $(\Delta, \Omega); \Gamma \vdash R_2 \Rightarrow P$ *then* $(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega'$ *and* $(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega'$ *and* $R_1 = [\![\theta_1]\!]R$ *and* $R_2 = [\![\theta_2]\!]R$ *and* $(\Delta, \Omega'); \Gamma \vdash R \Rightarrow P$.

(3) *If* $(\Delta, \Omega); \Gamma \vdash S_1 \sqcup S_2 : A > P \Longrightarrow S/(\Omega', \theta_1, \theta_2)$ *and* $(\Delta, \Omega); \Gamma \vdash S_1 > A \Rightarrow P$ *and* $(\Delta, \Omega); \Gamma \vdash S_2 > A \Rightarrow P$ *then* $(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega'$ *and* $(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega'$ *and* $(\Delta, \Omega'); \Gamma \vdash S > A \Rightarrow P$ *and* $S_1 = [\![\theta_1]\!]S$ *and* $S_2 = [\![\theta_2]\!]S$.

PROOF. Simultaneous induction on the structure of the first derivation. We give here a few cases.

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash \lambda x.M_1 \sqcup \lambda x.M_2 : A_1 \to A_2 \Longrightarrow \lambda x.M/(\Omega', \theta_1, \theta_2)$

$(\Delta, \Omega); \Gamma, x{:}A_1 \vdash M_1 \sqcup M_2 : A_2 \Longrightarrow M/(\Omega', \theta_1, \theta_2)$	by premise
$(\Delta, \Omega); \Gamma \vdash \lambda x.M_1 \Leftarrow A_1 \to A_2$	by assumption
$(\Delta, \Omega); \Gamma, x{:}A_1 \vdash M_1 \Leftarrow A_2$	by inversion
$(\Delta, \Omega); \Gamma \vdash \lambda x.M_2 \Leftarrow A_1 \to A_2$	by assumption
$(\Delta, \Omega); \Gamma, x{:}A_1 \vdash M_2 \Leftarrow A_2$	by inversion
$(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega'$	by i.h.
$(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega'$	by i.h.
$M_1 = [\![\theta_1]\!]M$	by i.h.
$\lambda x.M_1 = \lambda x.[\![\theta_1]\!]M$	by rule
$\lambda x.M_1 = [\![\theta_1]\!](\lambda x.M)$	by contextual substitution definition

$M_2 = [\![\theta_2]\!]M$ by i.h.
$\lambda x.M_2 = \lambda x.[\![\theta_2]\!]M$ by rule
$\lambda x.M_2 = [\![\theta_2]\!](\lambda x.M)$ by contextual substitution definition
$(\Delta, \Omega'); \Gamma, x{:}A_1 \vdash M \Leftarrow A_2$ by i.h.
$(\Delta, \Omega'); \Gamma \vdash \lambda x.M \Leftarrow A_1 \rightarrow A_2$ by rule

Case. $\mathcal{D} = (\Delta; \Omega); \Gamma \vdash R_1 \sqcup R_2 : P \Longrightarrow R/(\Omega', \theta_1, \theta_2)$

$(\Delta, \Omega); \Gamma \vdash R_1 \sqcup R_2 : P \Longrightarrow R/(\Omega', \theta_1, \theta_2)$ by premise
$(\Delta, \Omega); \Gamma \vdash R_1 \Leftarrow P$ by ass
$(\Delta, \Omega); \Gamma \vdash R_1 \Rightarrow P$ by rule
$(\Delta, \Omega); \Gamma \vdash R_2 \Leftarrow P$ by ass
$(\Delta, \Omega); \Gamma \vdash R_2 \Rightarrow P$ by rule
$(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega'$ by i.h.
$(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega'$ and
$R_1 = [\![\theta_1]\!]R$ and $R_2 = [\![\theta_2]\!]R$ and $(\Omega', \Delta); \Gamma \vdash R \Rightarrow P$
$(\Delta, \Omega'); \Gamma \vdash R \Leftarrow P$ by rule

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \sqcup u[\pi_\Gamma] : P \Longrightarrow u[\pi_\Gamma]/(\cdot, \cdot, \cdot)$

$u{::}P[\Psi] \in \Delta$ and $\Delta; \Gamma \vdash \pi \Leftarrow \Psi$ by premise
$(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \Rightarrow P$ by assumption
$u[\pi_\Gamma] = u[\pi_\Gamma]$ by reflexivity
$(\Delta, \Omega) \vdash \cdot \Leftarrow \cdot$ by rule
$\Delta; \Gamma \vdash u[\pi_\Gamma] \Rightarrow P$ by rule

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \sqcup R : P \Longrightarrow i[\mathsf{id}_\Gamma]/(i{::}P[\Gamma],\ \hat{\Gamma}.u[\pi_\Gamma]/i,\ \hat{\Gamma}.R/i)$

$u{::}P[\Psi] \in \Delta$ and $\Delta; \Gamma \vdash \pi \Leftarrow \Psi$ by premise
$(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash R \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash R \Leftarrow P$ by rule
$(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \Leftarrow P$ by rule
$u[\pi_\Gamma] = [\![\hat{\Gamma}.u[\pi_\Gamma]/i]\!]i[\mathsf{id}_\Gamma]$
$u[\pi_\Gamma] = u[\pi_\Gamma]$ by reflexivity
$R = [\![\hat{\Gamma}.R/i]\!]i[\mathsf{id}_\Gamma]$
$R = R$ by reflexivity
$(\Delta, \Omega) \vdash \hat{\Gamma}.R/i \Leftarrow i{::}P[\Gamma]$ by rule using assumption
$(\Delta, \Omega) \vdash u[\pi_\Gamma]/i \Leftarrow i{::}P[\Gamma]$ by rule using assumption
$(\Delta, i{::}P[\Gamma]); \Gamma \vdash \mathsf{id}_\Gamma \Leftarrow \Gamma$ by definition
$(\Delta, i{::}P[\Gamma]); \Gamma \vdash i[\mathsf{id}_\Gamma] \Rightarrow P$ by rule

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash c \cdot S_1 \sqcup c \cdot S_2 : P \Longrightarrow c \cdot S/(\Omega',\ \theta_1,\ \theta_2)$

$(\Delta, \Omega); \Gamma \vdash S_1 \sqcup S_2 : A > P \Longrightarrow S/(\Omega', \theta_1, \theta_2)$ by premise
$(\Delta, \Omega); \Gamma \vdash c \cdot S_1 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash S_1 > A \Rightarrow P$ by inversion
$(\Delta, \Omega); \Gamma \vdash c \cdot S_2 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash S_2 > A \Rightarrow P$ by inversion

$S_1 = [\![\theta_1]\!]S$ by i.h.
$S_2 = [\![\theta_2]\!]S$ by i.h.
$(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega'$ by i.h.
$(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega'$ by i.h.
$c \cdot S_1 = c \cdot\ [\![\theta_1]\!]S$ by rule
$c \cdot S_1 = [\![\theta_1]\!](c \cdot S)$ by contextual substitution definition
$c \cdot S_2 = c \cdot\ [\![\theta_2]\!]S$ by rule
$c \cdot S_2 = [\![\theta_2]\!](c \cdot\ S)$ by contextual substitution definition
$(\Delta, \Omega'); \Gamma \vdash S > A \Rightarrow P$ by i.h.
$(\Delta, \Omega'); \Gamma \vdash c \cdot S \Rightarrow P$ by rule

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash R_1 \sqcup R_2 : P \Longrightarrow i[\mathsf{id}_\Gamma]/(i{::}P[\Gamma],\ \hat{\Gamma}.R_1/i,\ \hat{\Gamma}.R_2/i)$

$R_1 = H_1 \cdot S_1$ and $R_2 = H_2 \cdot S_2$ and $H_1 \neq H_2$ by premise
$(\Delta, \Omega); \Gamma \vdash H_1 \cdot S_1 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash H_1 \cdot S_1 \Leftarrow P$ by rule
$(\Delta, \Omega); \Gamma \vdash H_2 \cdot S_2 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash H_2 \cdot S_2 \Leftarrow P$ by rule
$H_1 \cdot S_1 = [\![\hat{\Gamma}.(H_1 \cdot S_1)/i]\!](i[\mathsf{id}_\Gamma])$ by contextual substitution definition
$H_1 \cdot S_1 = H_1 \cdot S_1$ by reflexivity
$H_2 \cdot S_2 = [\![\hat{\Gamma}.(H_2 \cdot S_2)/i]\!](i[\mathsf{id}_\Gamma])$ by contextual substitution definition
$H_2 \cdot S_2 = H_2 \cdot S_2$ by reflexivity
$(\Delta, i{::}P[\Gamma]); \Gamma \vdash \mathsf{id}_\Gamma \Leftarrow \Gamma$ by definition
$(\Delta, i{::}P[\Gamma]); \Gamma \vdash i[\mathsf{id}_\Gamma] \Rightarrow P$ by rule

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash (M_1; S_1) \sqcup (M_2; S_2) : (A_1 \to A_2) > P$
$\Longrightarrow (M; S)/(\Omega', \theta, \theta')$

$(\Delta, \Omega); \Gamma \vdash M_1 \sqcup M_2 : A_1 \Longrightarrow M/(\Omega_1,\ \theta_1,\ \theta_2)$ by premise
$(\Delta, \Omega); \Gamma \vdash S_1 \sqcup S_2 : A_2 > P \Longrightarrow S/(\Omega_2,\ \theta_1',\ \theta_2')$
$\Omega' = (\Omega_1, \Omega_2)$, $\theta = (\theta_1, \theta_1')$, $\theta' = (\theta_2, \theta_2')$
$(\Delta, \Omega); \Gamma \vdash (M_1; S_1) > A_1 \to A_2 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash M_1 \Leftarrow A_1$ by inversion
$(\Delta, \Omega); \Gamma \vdash S_1 > A_2 \Rightarrow P$
$(\Delta, \Omega); \Gamma \vdash (M_2; S_2) > A_1 \to A_2 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash M_2 \Leftarrow A_1$ by inversion
$(\Delta, \Omega); \Gamma \vdash S_2 > A_2 \Rightarrow P$
$M_1 = [\![\theta_1]\!]M$ by i.h.
$M_2 = [\![\theta_2]\!]M$ by i.h.
$(\Delta, \Omega_1); \Gamma \vdash M \Leftarrow A_1$ by i.h.
$(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega_1$ by i.h.
$(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega_1$ by i.h.
$(\Delta, \Omega'); \Gamma \vdash M \Leftarrow A_1$ by weakening
$S_1 = [\![\theta_1']\!]S$ by i.h.
$S_2 = [\![\theta_2']\!]S$ by i.h.
$(\Delta, \Omega_2); \Gamma \vdash S > A_2 \Rightarrow P$ by i.h.
$(\Delta, \Omega) \vdash \theta_1' \Leftarrow \Omega_2$ by i.h.
$(\Delta, \Omega) \vdash \theta_2' \Leftarrow \Omega_2$ by i.h.
$(\Delta, \Omega',); \Gamma \vdash S > A_2 \Rightarrow P$ by weakening

$(\Delta, \Omega) \vdash (\theta_1, \theta_1') \Leftarrow \Omega'$ — θ_1 and θ_1' refer to distinct meta-variables by typing rules for contextual substitutions
$(\Delta, \Omega) \vdash (\theta_2, \theta_2') \Leftarrow \Omega'$ — θ_2 and θ_2' refer to distinct meta-variables by typing rules for contextual substitutions
$M_1 = [\![\theta_1, \theta_1']\!]M$ — by lemma weakening
$M_2 = [\![\theta_2, \theta_2']\!]M$ — by lemma weakening
$S_1 = [\![\theta_1, \theta_1']\!]S$ — by lemma weakening
$S_2 = [\![\theta_2, \theta_2']\!]S$ — by lemma weakening
$(M_1; S_1) = ([\![\theta_1, \theta_1']\!]M; [\![\theta_1, \theta_1']\!]S)$ — by rule
$(M_1; S_1) = [\![\theta_1, \theta_1']\!](M; S)$ — by contextual substitution definition
$(M_2; S_2) = ([\![\theta_2, \theta_2']\!]M; [\![\theta_2, \theta_2']\!]S)$ — by rule
$(M_2; S_2) = [\![\theta_2, \theta_2']\!](M; S)$ — by contextual substitution definition
$(\Delta, \Omega'); \Gamma \vdash (M\ ;\ S) > A_1 \rightarrow A_2 \Rightarrow P$ — by rule

□

Theorem A.2 Completeness of mslg of terms.
(previous Thm. 5.3 on page 21)

(1) *If $\Delta, \Omega \vdash \theta_1 \Leftarrow \Omega'$ and $\Delta, \Omega \vdash \theta_2 \Leftarrow \Omega'$ and θ_1 and θ_2 are incompatible and $\Delta, \Omega; \Gamma \vdash M_1 \Leftarrow A$, $\Delta; \Gamma \vdash M_2 \Leftarrow A$, and $\Delta, \Omega'; \Gamma \vdash M \Leftarrow A$ and $M_1 = [\![\theta_1]\!]M$ and $M_2 = [\![\theta_2]\!]M$ then there exists a contextual substitution θ_1^*, θ_2^*, and a modal context Ω^*, such that $(\Delta, \Omega); \Gamma \vdash M_1 \sqcup M_2 : A \Longrightarrow M/(\Omega^*, \theta_1^*, \theta_2^*)$ and $\theta_1^* \subseteq \theta_1$, $\theta_2^* \subseteq \theta_2$ and $\Omega^* \subseteq \Omega'$*

(2) *If $\Delta, \Omega \vdash \theta_1 \Leftarrow \Omega'$ and $\Delta, \Omega \vdash \theta_2 \Leftarrow \Omega'$ and θ_1 and θ_2 are incompatible and $\Delta, \Omega; \Gamma \vdash R_1 \Rightarrow P$, $\Delta; \Gamma \vdash R_2 \Rightarrow P$, and $\Omega', \Delta; \Gamma \vdash R \Rightarrow P$ and $R_1 = [\![\theta_1]\!]R$ and $R_2 = [\![\theta_2]\!]R$ then there exists a contextual substitution θ_1^*, θ_2^*, and a modal context Ω^*, such that $(\Delta, \Omega); \Gamma \vdash R_1 \sqcup\!\!\!\!\_\, R_2 : P \Longrightarrow R/(\Omega^*, \theta_1^*, \theta_2^*)$ and $\theta_1^* \subseteq \theta_1$, $\theta_2^* \subseteq \theta_2$ and $\Omega^* \subseteq \Omega'$*

(3) *If $\Delta, \Omega \vdash \theta_1 \Leftarrow \Omega'$ and $\Delta, \Omega \vdash \theta_2 \Leftarrow \Omega'$ and θ_1 and θ_2 are incompatible and $(\Delta, \Omega); \Gamma \vdash S_1 > A \Rightarrow P$, $(\Delta, \Omega); \Gamma \vdash S_2 > A \Rightarrow P$, and $(\Delta, \Omega'); \Gamma \vdash S > A \Rightarrow P$ and $S_1 = [\![\theta_1]\!]S$ and $S_2 = [\![\theta_2]\!]S$ then there exists a contextual substitution θ_1^*, θ_2^*, and a modal context Ω^*, such that $(\Delta, \Omega); \Gamma \vdash S_1 \sqcup\!\!\!\!\_\, S_2 : A \Longrightarrow S/(\Omega^*, \theta_1^*, \theta_2^*)$ and $\theta_1^* \subseteq \theta_1$, $\theta_2^* \subseteq \theta_2$ and $\Omega^* \subseteq \Omega'$.*

Proof. Simultaneous induction on the structure of M, R, and S. We give a few cases.

Case. $R = u[\pi_\Gamma]$ and $u{::}P[\Gamma] \in \Delta$
$(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \Rightarrow P$ — by assumption
$R_1 = [\![\theta_1]\!](u[\pi_\Gamma])$ — by assumption
$R_1 = u[\pi_\Gamma]$ — by contextual substitution definition
$R_2 = [\![\theta_2]\!](u[\pi_\Gamma])$ — by assumption
$R_2 = u[\pi_\Gamma]$ — by contextual substitution definition
$(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \sqcup\!\!\!\!\_\, u[\pi_\Gamma] : P \Longrightarrow u[\pi_\Gamma]/(\cdot, \cdot, \cdot)$ — by rule
$\cdot \subseteq \Omega'$, $\cdot \subseteq \theta_1$, $\cdot \subseteq \theta_2$

Case. $M = \lambda x.M'$.

$M_1 = [\![\theta_1]\!](\lambda x.M')$ by assumption
$M_1 = \lambda x.[\![\theta_1]\!]M'$ by contextual substitution definition
$M_1' = [\![\theta_1]\!]M'$ and $M_1 = \lambda x.M_1'$ by inversion
$M_2 = [\![\theta_2]\!](\lambda x.M')$ by assumption
$M_2 = \lambda x.[\![\theta_2]\!]M'$ by contextual substitution definition
$M_2' = [\![\theta_2]\!]M'$ and $M_2 = \lambda x.M_2'$ by inversion
$(\Delta, \Omega'); \Gamma \vdash \lambda x.M' \Leftarrow A_1 \to A_2$ by assumption
$(\Delta, \Omega'); \Gamma, x{:}A_1 \vdash M' \Leftarrow A_2$ by inversion
$(\Delta, \Omega); \Gamma \vdash \lambda x.M_1' \Leftarrow A_1 \to A_2$ by assumption
$(\Delta, \Omega); \Gamma, x{:}A_1 \vdash M_1' \Leftarrow A_2$ by inversion
$(\Delta, \Omega); \Gamma \vdash \lambda x.M_2' \Leftarrow A_1 \to A_2$ by assumption
$(\Delta, \Omega); \Gamma, x{:}A_1 \vdash M_2' \Leftarrow A_2$ by inversion
$(\Delta, \Omega); \Gamma, x{:}A_1 \vdash M_1' \sqcup M_2' : A_2 \Longrightarrow M'/(\Omega^*, \theta_1^*, \theta_2^*)$ by i.h.
$\Omega^* \subseteq \Omega'$, $\theta_1^* \subseteq \theta_1$, $\theta_2^* \subseteq \theta_2$
$(\Delta, \Omega); \Gamma \vdash \lambda x.M_1' \sqcup \lambda x.M_2' : A_1 \to A_2 \Longrightarrow \lambda x.M'/(\Omega^*, \theta_1^*, \theta_2^*)$ by rule

Case. $R = i[\mathsf{id}_\Gamma]$

$(\Delta; \Omega); \Gamma \vdash i[\mathsf{id}_\Gamma] \Rightarrow P$ by assumption
$i{::}P[\Gamma] \in \Omega$ by inversion
$R_1 = [\![\theta_1]\!](i[\mathsf{id}_\Gamma])$ by assumption
$R_2 = [\![\theta_2]\!](i[\mathsf{id}_\Gamma])$ by assumption
$\hat{\Gamma}.R'/i \in \theta_1$ and $\hat{\Gamma}.R''/i \in \theta_2$ by assumption
R' and R'' are incompatible by assumption
$R_1 = R'$ by contextual substitution definition
$R_2 = R''$ by contextual substitution definition

Sub-Case 1. : $R_1 = u[\pi_\Gamma]$ and $R_2 = R''$

$(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \sqcup R'' : P \Longrightarrow i[\mathsf{id}_\Gamma]/(i{::}P[\Gamma], \hat{\Gamma}.u[\pi_\Gamma]/i, \hat{\Gamma}.R''/i)$ by rule
$i{::}P[\Gamma] \subseteq \Omega'$, $(\hat{\Gamma}.u[\pi_\Gamma]/i) \subseteq \theta_1$, $(\hat{\Gamma}.R''/i) \subseteq \theta_2$

Sub-Case 2. : $R_1 = R'$ and $R_2 = u[\pi_\Gamma]$

$(\Delta, \Omega); \Gamma \vdash R' \sqcup u[\pi_\Gamma] : P \Longrightarrow i[\mathsf{id}_\Gamma]/(i{::}P[\Gamma], \hat{\Gamma}.R'/i, \hat{\Gamma}.u[\pi_\Gamma]/i)$ by rule
$(i{::}P[\Gamma] \subseteq \Omega'$, $(\hat{\Gamma}.u[\pi_\Gamma]/i) \subseteq \theta_2$, $(\hat{\Gamma}.R'/i) \subseteq \theta_1$

Sub-Case 3. : $R_1 = H_1 \cdot S_1$ and $R_2 = H_2 \cdot S_2$

$H_1 \cdot S_1$ is incompatible with $H_2 \cdot S_2$ and $H_1 \neq H_2$ by assumption
$(\Delta, \Omega); \Gamma \vdash H_1 \cdot S_1 \sqcup H_2 \cdot S_2 : P \Longrightarrow i[\mathsf{id}_\Gamma]/(i{::}P[\Gamma], \hat{\Gamma}.(H_1 \cdot S_1)/i, \hat{\Gamma}.(H_2 \cdot S_2)/i)$
by rule
$(i{::}P[\Gamma]) \subseteq \Omega'$, $(\hat{\Gamma}.H_1 \cdot S_1/i) \subseteq \theta_1$, $(\hat{\Gamma}.H_2 \cdot S_2/i) \subseteq \theta_2$

□

THEOREM A.3 SOUNDNESS FOR MSLG OF SUBSTITUTIONS.
(PREVIOUS THM. 5.5 ON PAGE 22)
If $(\Delta, \Omega_1) \vdash \rho_1 \sqcup \rho_2 : \Omega_2 \Longrightarrow \rho/(\Omega, \theta_1, \theta_2)$ *and*
$(\Delta, \Omega_1) \vdash \rho_1 \Leftarrow \Omega_2$ *and* $(\Delta, \Omega_1) \vdash \rho_2 \Leftarrow \Omega_2$
then $(\Delta, \Omega) \vdash \rho \Leftarrow \Omega_2$, $(\Delta, \Omega_1) \vdash \theta_1 \Leftarrow \Omega$, $(\Delta, \Omega_1) \vdash \theta_2 \Leftarrow \Omega$, *and*
$[\![\theta_1]\!]\rho = \rho_1$ *and* $[\![\theta_2]\!]\rho = \rho_2$

PROOF. Induction on the first derivation.

Case. $\mathcal{D} = (\Delta, \Omega_1) \vdash \cdot : \cdot \Longrightarrow \cdot/(\cdot, \cdot, \cdot)$

$\cdot = \cdot$ by syntactic equality
$\cdot = [\![\cdot]\!](\cdot)$ contextual substitution definition

Case. $\mathcal{D} = (\Delta, \Omega_1) \vdash (\rho_1, \hat{\Psi}.R_1/i) \sqcup (\rho_2, \hat{\Psi}.R_2/i) : (\Omega_2, i{::}P[\Psi])$
$\Longrightarrow (\rho, \hat{\Psi}.R/i)/((\Omega, \Omega'), (\theta_1, \theta_1'), (\theta_2, \theta_2'))$

$(\Delta, \Omega_1) \vdash \rho_1 \sqcup \rho_2 : \Omega_2 \Longrightarrow \rho/(\Omega, \theta_1, \theta_2)$ by premise
$(\Delta, \Omega_1); \Psi \vdash R_1 \sqcup R_2 : P \Longrightarrow R/(\Omega', \theta_1', \theta_2')$ by premise
$(\Delta, \Omega_1) \vdash (\rho_1, \hat{\Psi}.R_1/i) \Leftarrow (\Omega_2, i{::}P[\Psi])$ by assumption
$(\Delta, \Omega_1) \vdash \rho_1 \Leftarrow \Omega_2$ by inversion
$(\Delta, \Omega_1); \Psi \vdash R_1 \Leftarrow P$
$(\Delta, \Omega_1); \Psi \vdash R_1 \Rightarrow P$ by inversion
$(\Delta, \Omega_1) \vdash (\rho_2, \hat{\Psi}.R_2/i) \Leftarrow (\Omega_2, i{::}P[\Psi])$ by assumption
$(\Delta, \Omega_1) \vdash \rho_2 \Leftarrow \Omega_2$ by inversion
$(\Delta, \Omega_1); \Psi \vdash R_2 \Rightarrow P$
$(\Delta, \Omega'); \Psi \vdash R \Rightarrow P$ by soundness theorem 5.2
$(\Delta, \Omega'); \Psi \vdash R \Leftarrow P$ by rule
$R_1 = [\![\theta_1']\!]R,\ \Delta, \Omega_1 \vdash \theta_1' \Leftarrow \Omega'$ by soundness theorem 5.2
$R_2 = [\![\theta_2']\!]R,\ \Delta, \Omega_1 \vdash \theta_2' \Leftarrow \Omega'$ by soundness theorem 5.2
$R_1 = [\![\theta_1, \theta_1']\!]R$ by weakening
$R_2 = [\![\theta_2, \theta_2']\!]R$ by weakening
$\rho_1 = [\![\theta_1]\!]\rho$ by i.h.
$\rho_2 = [\![\theta_2]\!]\rho$ by i.h.
$\rho_1 = [\![\theta_1, \theta_1']\!]\rho$ by weakening lemma
$\rho_2 = [\![\theta_2, \theta_2']\!]\rho$ by weakening lemma
$(\rho_1, \hat{\Psi}.R_1/i) = ([\![\theta_1, \theta_1']\!]\rho,\ [\![\theta_1, \theta_1']\!]\hat{\Psi}.R/i)$ by rule
$(\rho_2, \hat{\Psi}.R_2/i) = ([\![\theta_2, \theta_2']\!]\rho,\ [\![\theta_2, \theta_2']\!]\hat{\Psi}.R/i)$ by rule
$(\rho_1, \hat{\Psi}.R_1/i) = [\![\theta_1, \theta_1']\!](\rho, \hat{\Psi}.R/i)$ by contextual substitution definition
$(\rho_2, \hat{\Psi}.R_2/i) = [\![\theta_2, \theta_2']\!](\rho, \hat{\Psi}.R/i)$ by contextual substitution definition
$(\Delta, \Omega) \vdash \rho \Leftarrow \Omega_2$ by i.h.
$(\Delta, \Omega, \Omega') \vdash \rho \Leftarrow \Omega_2$ by weakening
$(\Delta, \Omega, \Omega'); \Psi \vdash R \Leftarrow P$ by weakening
$(\Delta, \Omega, \Omega') \vdash (\rho, \hat{\Psi}.R/i) \Leftarrow (\Omega_2, i{::}P[\Psi])$ by rule
$\Delta, \Omega_1) \vdash (\theta_1, \theta_1') \Leftarrow (\Omega, \Omega')$ by typing rules
$\Delta, \Omega_1) \vdash (\theta_2, \theta_2') \Leftarrow (\Omega, \Omega')$ by typing rules

□

THEOREM A.4 COMPLETENESS FOR MSLG OF CONTEXTUAL SUBSTITUTIONS. (PREVIOUS THM. 5.6 ON PAGE 23)
If $(\Delta, \Omega) \vdash \theta_1 \Leftarrow \Omega'$ *and* $(\Delta, \Omega) \vdash \theta_2 \Leftarrow \Omega'$ *and* θ_1 *and* θ_2 *are incompatible and* $\rho_1 = [\![\theta_1]\!]\rho$ *and* $\rho_2 = [\![\theta_2]\!]\rho$ *then* $(\Delta, \Omega) \vdash \rho_1 \sqcup \rho_2 : \Omega_1 \Longrightarrow \rho/(\Omega^*, \theta_1^*, \theta_2^*)$ *such that* $\Omega^* \subseteq \Omega'$, $\theta_1^* \subseteq \theta_1$, $\theta_2^* \subseteq \theta_2$.

PROOF. Induction on the structure of ρ.

Case. $\rho = \cdot$

$\rho_1 = [\![\theta_1]\!](\cdot)$ by assumption
$\rho_1 = \cdot$ and $\Omega_1 = \cdot$ by inversion
$\rho_2 = [\![\theta_2]\!](\cdot)$ by assumption
$\rho_2 = \cdot$ and $\Omega_1 = \cdot$ by inversion
$(\Delta, \Omega) \vdash \cdot \sqcup \cdot : \cdot \Longrightarrow \cdot/(\cdot, \cdot, \cdot)$ by rule
$\cdot \subseteq \Omega_1,\ \cdot \subseteq \theta_1,\ \cdot \subseteq \theta_2$

Case. $\rho = (\rho', \hat{\Psi}.R/i)$

$\rho_1' = [\![\theta_1]\!](\rho', \hat{\Psi}.R/i)$ by assumption
$\rho_1' = ([\![\theta_1]\!](\rho'),\ \hat{\Psi}.[\![\theta_1]\!]R/i)$ by contextual substitution definition
$\rho_1' = (\rho_1, \hat{\Psi}.R_1/i)$ by equality
$\rho_1 = [\![\theta_1]\!]\rho'$
$R_1 = [\![\theta_1]\!]R$
$\rho_2' = [\![\theta_2]\!](\rho', \hat{\Psi}.R/i)$ by assumption
$\rho_2' = ([\![\theta_2]\!](\rho'),\ \hat{\Psi}.[\![\theta_2]\!]R/i)$ by contextual substitution definition
$\rho_2' = (\rho_2, \hat{\Psi}.R_2/i)$ by equality
$\rho_2 = [\![\theta_2]\!]\rho'$
$R_2 = [\![\theta_2]\!]R$
$(\Delta, \Omega); \Psi \vdash R_1 \sqcup R_2 : P \Longrightarrow R/(\Omega^*, \theta_1^*, \theta_2^*)$ by completeness lemma 5.3
$\Omega^* \subseteq \Omega',\ \theta_1^* \subseteq \theta_1,\ \theta_2^* \subseteq \theta_2$
$(\Delta, \Omega) \vdash \rho_1 \sqcup \rho_2 : \Omega_1 \Longrightarrow \rho'/(\Omega^{**}, \theta_1^{**}, \theta_2^{**})$ by i.h.
$\Omega^{**} \subseteq \Omega',\ \theta_1^{**} \subseteq \theta_1,\ \theta_2^{**} \subseteq \theta_2$
$(\Delta, \Omega) \vdash (\rho_1, \hat{\Psi}.R_1/i) \sqcup (\rho_2, \hat{\Psi}.R_2/i) : (\Omega_1, i{::}P[\Psi])$
$\Longrightarrow (\rho', \hat{\Psi}.R/i)/((\Omega^{**}, \Omega^*),\ (\theta_1^{**}, \theta_1^*),\ (\theta_2^{**}, \theta_2^*))$ by rule
$(\Omega^{**}, \Omega^*) \subseteq \Omega',\ (\theta_1^{**}, \theta_1^*) \subseteq \theta_1,\ (\theta_2^{**}, \theta_2^*) \subseteq \theta_2$

□

Lemma A.5 Insertion of substitution into tree.
(previous Lemma 5.7 on page 25)
If $\Delta \vdash C \sqcup \delta : \Omega \Longrightarrow (V, S)$ *and* $\Delta \vdash \delta \Leftarrow \Omega$ *and for any* $(\Omega_i{\vdash}\rho_i \twoheadrightarrow C') \in C$ *with* $\Delta, \Omega_i \vdash \rho_i \Leftarrow \Omega$ *then*

(1) *for any* $(N_i, \theta_2) \in V$ *where* $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, *we have* $[\![\theta_2]\!]\rho_i = \delta$.
(2) *for any* $(N_i, \Omega' \vdash \rho', \theta_1, \theta_2) \in S$ *where* $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, *we have* $[\![\theta_2]\!]\rho' = \delta$ *and* $[\![\theta_1]\!]\rho' = \rho_i$.

Proof. By structural induction on the first derivation and by using the previous soundness lemma for mslg of substitutions (lemma 5.5).

Case. $\mathcal{D} = \dfrac{}{\Delta \vdash \mathsf{nil} \sqcup \delta : \Omega \Longrightarrow (\cdot, \cdot)}$.
Trivially true.

$$\mathcal{D} = \dfrac{\begin{array}{ll}\Delta & \vdash C \sqcup \delta : \Omega \Longrightarrow (V, S)\\ \Delta, \Omega_1 & \vdash \rho_1 \sqcup \delta : \Omega \Longrightarrow \mathsf{id}_\Omega/(\Omega, \rho_1, \delta)\end{array}}{\Delta \vdash [(\Omega_1{\vdash}\rho_1 \twoheadrightarrow C_1), C] \sqcup \delta : \Omega \Longrightarrow (V, S)}\ NC$$

(*Case.*)

By i.h., for any $(N_i, \theta_2) \in V$, $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, we have $[\![\theta_2]\!]\rho_i = \delta$ and for any $(N_i, \Omega' \vdash \rho', \theta'_1, \theta'_2) \in S$ where $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, we have $[\![\theta'_2]\!]\rho' = \delta$ and $[\![\theta'_1]\!]\rho' = \rho_i$.

$$\textit{Case.}\ \mathcal{D} = \frac{\begin{array}{ll}\Delta & \vdash C \sqcup \delta : \Omega \Longrightarrow (V, S) \\ \Delta, \Omega_1 & \vdash \rho_1 \sqcup \delta : \Omega \Longrightarrow \rho_1/(\Omega_1, \mathsf{id}_{\Omega_1}, \theta_2)\end{array}}{\Delta \vdash [(\Omega_1 \vdash \rho_1 \twoheadrightarrow C_1), C] \sqcup \delta : \Omega \Longrightarrow ((V\ ,\ (\Omega_1 \vdash \rho_1 \twoheadrightarrow C_1)), S)}\ FC$$

By i.h., for any $(N_i, \theta_2) \in V$, $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, we have $[\![\theta_2]\!]\rho_i = \delta$ and for any $(N_i, (\Omega' \vdash \rho', \theta'_1, \theta'_2)) \in S$ where $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, we have $[\![\theta'_2]\!]\rho' = \delta$ and $[\![\theta'_1]\!]\rho' = \rho_i$. By soundness lemma 5.5, $[\![\theta_2]\!]\rho_1 = \delta$, therefore for any $(N_i, \theta') \in (V,\ ((\Omega_1 \vdash \rho_1 \twoheadrightarrow C_1), \theta_2))$, where $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$ we have $[\![\theta']\!]\rho_i = \delta$.

Case.

$$\mathcal{D} = \frac{\begin{array}{ll}\Delta & \vdash C \sqcup \delta : \Omega \Longrightarrow (V, S) \\ \Delta, \Omega_1 & \vdash \rho_1 \sqcup \delta : \Omega \Longrightarrow \rho^*/(\Omega_2, \theta_1, \theta_2)\end{array}}{\Delta \vdash [(\Omega_1 \vdash \rho_1 \twoheadrightarrow C_1), C] \sqcup \delta : \Omega \Longrightarrow (V, (S\ ,\ ((\Omega_1 \vdash \rho_1 \twoheadrightarrow C_1), \Omega_2 \vdash \rho^*, \theta_1, \theta_2))}\ PC$$

By i.h., for any $(N_i, \theta'_2) \in V$, $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, we have $[\![\theta'_2]\!]\rho_i = \delta$ and for any $(N_i, (\Omega' \vdash \rho', \theta'_1, \theta'_2)) \in S$ where $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$, we have $[\![\theta'_2]\!]\rho' = \delta$ and $[\![\theta'_1]\!]\rho' = \rho_i$. By soundness lemma 5.5, $[\![\theta_2]\!]\rho^* = \delta$ and $[\![\theta_1]\!]\rho^* = \rho_1$, therefore for any $(N_i, \Omega' \vdash \rho', \theta'_1, \theta'_2) \in (S\ ,\ ((\Omega_1 \vdash \rho_1 \twoheadrightarrow C_1), \Omega_2 \vdash \rho^*, \theta_1, \theta_2))$, where $N_i = (\Omega_i \vdash \rho_i \twoheadrightarrow C_i)$ we have $[\![\theta'_1]\!]\rho' = \rho_i$ and $[\![\theta'_2]\!]\rho' = \delta$.

□

Theorem A.6 Soundness of instance algorithm for terms.
(previous Thm. 6.1 on page 28)

(1) *If* $\Delta_2; (\Delta_1, \Omega); \Gamma \vdash M_1 \doteq M_2 : A/(\theta, \rho)$
where $(\Delta_1, \Omega); \Gamma \vdash M_1 \Leftarrow A$ *and* $\Delta_2; \Gamma \vdash M_2 \Leftarrow A$ *then* $[\![\theta, \rho]\!]M_1 = M_2$.

(2) *If* $\Delta_2; (\Delta_1, \Omega); \Gamma \vdash R_1 \doteq R_2 : P/(\theta, \rho)$
where $(\Delta_1, \Omega); \Gamma \vdash R_1 \Rightarrow P$ *and* $\Delta_2; \Gamma \vdash R_2 \Rightarrow P$ *then* $[\![\theta, \rho]\!]R_1 = R_2$.

(3) *If* $\Delta_2; (\Delta_1, \Omega); \Gamma \vdash S_1 \doteq S_2 > A \Rightarrow P/(\theta, \rho)$
where $(\Delta_1, \Omega); \Gamma \vdash S_1 > A \Rightarrow P$ *and* $\Delta_2; \Gamma \vdash S_2 > A \Rightarrow P$ *then* $[\![\theta, \rho]\!]S_1 = S_2$.

Proof. Simultaneous structural induction on the first derivation. The proof

$$\textit{Case.}\ \mathcal{D} = \frac{}{\Delta_2; (\Delta_1, i{::}P[\Gamma]); \Gamma \vdash i[\mathsf{id}_\Gamma] \doteq R : P\ /\ (\cdot, (\hat{\Gamma}.R/i))}\ \textsf{mvar-1}$$

$(\Delta_1, i{::}P[\Gamma]); \Gamma \vdash i[\mathsf{id}_\Gamma] \Rightarrow P$ — by assumption
$\Delta_2; \Gamma \vdash R \Rightarrow P$ — by assumption
$R = R$ — by reflexivity
$[\![\hat{\Psi}.R/i]\!](i[\mathsf{id}_\Gamma]) = R$ — by substitution definition

$$\textit{Case.}\ \mathcal{D} = \frac{u{::}P[\Gamma] \in \Delta}{(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \doteq R : P\ /\ (\hat{\Gamma}.([\pi_\Gamma]^{-1}\ R/u), \cdot)}\ \textsf{mvar-2}$$

$\Delta_1; \Gamma \vdash u[\pi_\Gamma] \Rightarrow P$ where $u{::}P[\Gamma] \in \Delta_1$ — by assumption

$\Delta_2; \Gamma \vdash R \Rightarrow P$ by assumption
$[\pi_\Gamma]([\pi_\Gamma]^{-1} R) = R$ by property of inversion
$[\![\hat{\Gamma}.[\pi_\Gamma]^{-1} R/u]\!](u[\pi_\Gamma]) = R$ by substitution definition

Case. $\mathcal{D} = \dfrac{\Delta_2; (\Delta_1, \Omega); \Gamma, x{:}A_1 \vdash M_1 \doteq M_2 : A_2 \;/\; (\theta, \rho)}{\Delta_2; (\Delta_1, \Omega); \Gamma \vdash \lambda x.M_1 \doteq \lambda x.M_2 : A_1 \to A_2 \;/\; (\theta, \rho)}$ lam

$(\Delta_1, \Omega); \Gamma \vdash \lambda x.M_1 \Leftarrow A_1 \to A_2$ by assumption
$(\Delta_1, \Omega); \Gamma, x{:}A_1 \vdash M_1 \Leftarrow A_2$ by inversion
$\Delta_2; \Gamma \vdash \lambda x.M_2 \Leftarrow A_1 \to A_2$ by assumption
$\Delta_2, ; \Gamma, x{:}A_1 \vdash M_2 \Leftarrow A_2$ by inversion
$[\![\theta, \rho]\!] M_1 = M_2$ by i.h.
$[\![\theta, \rho]\!] \lambda x.M_1 = [\![\theta, \rho]\!] \lambda x.M_2$ by equality and contextual substitution definition

Case. $\mathcal{D} = \Delta_2; (\Delta_1, \Omega_1, \Omega_2); \Gamma \Vdash (M_1; S_1) \doteq (M_2; S_2) : A_1 \to A_2 > P \;/\; ((\theta_1, \theta_2),\ (\rho_1, \rho_2))$

$\Delta_2; (\Delta_1, \Omega_1); \Gamma \vdash M_1 \doteq M_2 : A_1 \;/\; (\theta_1, \rho_1)$
$\Delta_2; (\Delta_1, \Omega_2); \Gamma \Vdash S_1 \doteq S_2 : A_2 > P \;/\; (\theta_2, \rho_2)$ by premise
$(\Delta_1; \Omega_1, \Omega_2); \Gamma \vdash (M_1; S_1) < A_1 \to A_2 \Rightarrow P$ by assumption
$(\Delta_1; \Omega_1); \Gamma \vdash M_1 \Leftarrow A_1$ by inversion
$(\Delta_1; \Omega_2); \Gamma \vdash S_1 < A_2 \Rightarrow P$
$\Delta_2; \Gamma \vdash (M_2; S_2) < A_1 \to A_2 \Rightarrow P$ by assumption
$\Delta_2; \Gamma \vdash M_2 \Leftarrow A_1$ by inversion
$\Delta_2; \Gamma \vdash S_2 < A_2 \Rightarrow P$
$[\![\theta_1, \rho_1]\!] M_1 = M_2$ by i.h.
$[\![\theta_2, \rho_2]\!] S_1 = S_2$ by i.h.
$[\![\theta_1, \theta_2, \rho_1, \rho_2]\!] M_1 = M_2$ by weakening (using linearity condition)
$[\![\theta_1, \theta_2, \rho_1, \rho_2]\!] S_1 = S_2$ by weakening (using linearity condition)
$[\![\theta_1, \theta_2, \rho_1, \rho_2]\!](M_1\ S_1) = [\![\mathsf{id}_{\Delta_2}\theta_1, \theta_2, \rho_1, \rho_2]\!](M_2\ S_2)$ by rule and substitution definition

□

Theorem A.7 Completeness of instance algorithm for terms. (previous Thm. 6.2 on page 28)

(1) *If* $(\Delta_1, \Omega); \Gamma \vdash M_1 \Leftarrow A$ *and* $\Delta_2; \Gamma \vdash M_2 \Leftarrow A$ *and* $\Delta_2 \vdash \theta \Leftarrow \Delta_1$ *and* $\Delta_2 \vdash \rho \Leftarrow \Omega$ *and* $[\![\theta, \rho]\!] M_1 = M_2$ *then* $\Delta_2; (\Delta_1, \Omega); \Gamma \vdash M_1 \doteq M_2 : A/(\theta^*, \rho)$ *where* $\theta^* \subseteq \theta$.

(2) *If* $(\Delta_1, \Omega); \Gamma \vdash R_1 \Rightarrow P$ *and* $\Delta_2; \Gamma \vdash R_2 \Rightarrow P$ *and* $\Delta_2 \vdash \theta \Leftarrow \Delta_1$ *and* $\Delta_2 \vdash \rho \Leftarrow \Omega$ *and* $[\![\theta, \rho]\!] R_1 = R_2$ *then* $\Delta_2; (\Delta_1, \Omega); \Gamma \vdash R_1 \doteqdot R_2 : P/(\theta^*, \rho)$ *where* $\theta^* \subseteq \theta$.

(3) *If* $(\Delta_1, \Omega); \Gamma \vdash S_1 > A \Rightarrow P$ *and* $\Delta_2; \Gamma \vdash S_2 > A \Rightarrow P$ *and* $\Delta_2 \vdash \theta \Leftarrow \Delta_1$ *and* $\Delta_2 \vdash \rho \Leftarrow \Omega$ *and* $[\![\theta, \rho]\!] S_1 = S_2$ *then* $\Delta_2; (\Delta_1, \Omega); \Gamma \vdash S_1 \doteq S_2 : A > P/(\theta^*, \rho)$ *where* $\theta^* \subseteq \theta$.

Proof. Simultaneous structural induction on the first typing derivation.

Case. $\mathcal{D} = \dfrac{(\Delta_1, \Omega); \Gamma, x{:}A_1 \vdash M_1 \Leftarrow A_2}{(\Delta_1, \Omega); \Gamma \vdash \lambda x.M_1 \Leftarrow A_1 \to A_2}$

$\Delta_2; \Gamma \vdash \lambda x.M_2 \Leftarrow A_1 \to A_2$ by assumption
$\Delta_2; \Gamma, x{:}A_1 \vdash M_2 \Leftarrow A_2$ by inversion
$[\![\theta, \rho]\!](\lambda x.M_1) = \lambda x.M_2$ by assumption
$\lambda x.[\![\theta, \rho]\!](M_1) = \lambda x.M_2$ by substitution definition
$[\![\theta, \rho]\!](M_1) = M_2$ by syntactic equality
$\Delta_2; (\Delta_1, \Omega); \Gamma, x{:}A_1 \vdash M_1 \doteq M_2 : A_2/(\theta^*, \rho^*)$ by i.h.
$\theta^* \subseteq \theta$ and $\rho^* \subseteq \rho$
$\Delta_2; (\Delta_1, \Omega); \Gamma \vdash \lambda x.M_1 \doteq \lambda x.M_2 : A_1 \to A_2/(\theta^*, \rho^*)$ by rule

Case. $\mathcal{D} = \dfrac{}{(\Delta_1, i{::}P[\Gamma]); \Gamma \vdash i[\mathsf{id}_\Gamma] \Rightarrow P}$

$i{::}P[\Gamma]; \Gamma \vdash i[\mathsf{id}_\Gamma] \Rightarrow P$ by rule
$\Delta_2; \Gamma \vdash R_2 \Rightarrow P$ by assumption
$[\![\theta, \rho]\!](i[\mathsf{id}_\Gamma]) = R_2$ by assumption
$\hat{\Gamma}.R_2/i \in \rho$ by assumption
$\Delta_2; (\Delta_1, i{::}P[\Gamma]); \Gamma \vdash i[\mathsf{id}_\Gamma] \doteq R_2 : P/(\cdot, \hat{\Gamma}.R_2/i)$ by rule
$\cdot \subseteq \mathsf{id}_\Delta$ and $(\hat{\Gamma}.R_2/i) \subset \rho$

Case. $\mathcal{D} = \dfrac{u{::}P[\Gamma] \in \Delta_1}{(\Delta_1, \cdot); \Gamma \vdash u[\pi_\Gamma] \Rightarrow P}$

$u{::}P[\Gamma]; \Gamma \vdash u[\pi_\Gamma] \Rightarrow P$ by rule
$\Delta_1 = \Delta_1', u{::}P[\Gamma], \Delta_1''$
$\Delta_2; \Gamma \vdash R_2 \Rightarrow P$ by assumption
$\theta = (\theta_1, \hat{\Gamma}.R/u, \theta_2)$ by assumption
$[\![\theta, \rho]\!](u[\pi_\Gamma]) = R_2$ by assumption
$[\pi_\Gamma]R = R_2$ by substitution definition
$R = [\pi_\Gamma]^{-1} R_2$ and $[\pi_\Gamma]([\pi_\Gamma]^{-1} R_2) = R_2$ by inverse substitution property
$\Delta_2, u{::}P[\Gamma]; \Gamma \vdash u[\pi_\Gamma] \doteq R_2 : P/(\hat{\Gamma}.[\pi_\Gamma]^{-1} R_2/u, \cdot)$ by rule
$(\hat{\Gamma}.[\pi_\Gamma]^{-1} R_2/u) \subseteq \theta$ and $\cdot \subseteq \rho$

Case. $\mathcal{D} = \dfrac{(\Delta_1, \Omega); \Gamma \vdash M_1 \Leftarrow A_1 \qquad (\Delta_1, \Omega); \Gamma \Vdash S_1 > A \Rightarrow P}{(\Delta_1, \Omega); \Gamma \Vdash (M_1; S_1) > A_1 \to A \Rightarrow P}$

$[\![\theta, \rho]\!](M_1; S_1) = S'$ by assumption
$[\![\theta, \rho]\!](M_1)\ ;\ [\![\theta, \rho]\!](S_1) = S'$ by substitution definition
$S' = (M_2; S_2)$ by inversion
$[\![\theta, \rho]\!](M_1) = M_2$ by inversion
$[\![\theta, \rho]\!](S_1) = S_2$ by inversion
$\Delta_2; \Gamma \vdash (M_2; S_2) > A_1 \to A \Rightarrow P$ by assumption
$\Delta_2; \Gamma \vdash M_2 \Leftarrow A_1$ by inversion
$\Delta_2; \Gamma \vdash S_2 > A \Rightarrow P$

$\Delta_2; (\Delta_1, \Omega_1); \Gamma \vdash M_1 \doteq M_2 : A_1/(\theta_1^*, \rho_1)$ and $\theta_1^* \subseteq \theta$ by i.h.
$\Delta_2; (\Delta_1, \Omega_2); \Gamma \vdash S_1 \doteq S_2 : A > P/(\theta_2^*, \rho_2)$ and $\theta_2^* \subseteq \theta$ by i.h.
$(\Delta, \Omega); \Gamma \vdash (M_1; S_1) \doteq (M_2; S_2) : A_1 \to A > P/((\theta_1^*, \theta_2^*), (\rho_1, \rho_2))$ by rule
$(\theta_1^*, \theta_2^*) \subseteq \theta$ by subset property

□

Theorem A.8 Interaction between mslg and instance algorithm. (previous Thm. 6.4 on page 29)

(1) *If* $(\Delta_1, \Omega); \Gamma \vdash M_1 \Leftarrow A$ *and* $\Delta_2; \Gamma \vdash M_2 \Leftarrow A$ *and* $(\Delta_2, \Delta_1), \Omega; \Gamma \vdash M_1 \sqcup M_2 : A \Longrightarrow M/(\Omega', \rho_1, \rho_2)$ *then* $(\Delta_1; \Omega'; \Gamma \vdash M \doteq M_1 : A/(\cdot, \rho_1)$ *and* $\Delta_2; \Omega'; \Gamma \vdash M \doteq M_2 : A/(\cdot, \rho_2)$.

(2) *If* $(\Delta_1, \Omega); \Gamma \vdash R_1 \Rightarrow P$ *and* $\Delta_2; \Gamma \vdash R_2 \Rightarrow P$ *and* $(\Delta_2, \Delta_1), \Omega; \Gamma \vdash R_1 \sqcup R_2 : P \Longrightarrow R/(\Omega', \rho_1, \rho_2)$ *then* $\Delta_1; \Omega'; \Gamma \vdash R \doteqdot R_1 : P/(\cdot, \rho_1)$ *and* $\Delta_2; \Omega'; \Gamma \vdash R \doteqdot R_2 : P/(\cdot, \rho_2)$.

(3) *If* $(\Delta_1, \Omega); \Gamma \vdash S_1 > A \Rightarrow P$ *and* $\Delta_2; \Gamma \vdash S_2 > A \Rightarrow P$ *and* $(\Delta_2, \Delta_1), \Omega; \Gamma \vdash S_1 \sqcup S_2 : A > P \Longrightarrow S/(\Omega', \rho_1, \rho_2)$ *then* $\Delta_1; \Omega'; \Gamma \vdash S \doteq S_1 : A > P/(\cdot, \rho_1)$ *and* $\Delta_2; \Omega'; \Gamma \vdash S \doteq S_2 : A > P/(\cdot, \rho_2)$.

Proof. Simultaneous structural induction on the first derivation.
Let $\Delta = \Delta_2, \Delta_1$.

Case. $\mathcal{D} = \dfrac{(\Delta_2, \Delta_1, \Omega); \Gamma, x{:}A_1 \vdash M_1 \sqcup M_2 : A_2 \Longrightarrow M/(\Omega', \rho_1, \rho_2)}{(\Delta, \Omega); \Gamma \vdash \lambda x.M_1 \sqcup \lambda x..M_2 : A_1 \to A_2 \Longrightarrow \lambda x.M/(\Omega',\ \rho_1, \rho_2)}$

$\Delta_1; \Omega'; \Gamma, x{:}A_1 \vdash M \doteq M_1 : A_2/(\cdot, \rho_1)$ by i.h.
$\Delta_1; \Omega'; \Gamma \vdash \lambda x.M \doteq \lambda x.M_1 : A_1 \to A_2/(\cdot, \rho_1)$ by rule
$\Delta_2; \Omega'; \Gamma, x{:}A_1 \vdash M \doteq M_2 : A_2/(\cdot, \rho_2)$ by i.h.
$\Delta_2; \Omega'; \Gamma \vdash \lambda x.M \doteq \lambda x.M_2 : A_1 \to A_2/(\cdot, \rho_2)$ by rule

Case. $\mathcal{D} = \dfrac{u{::}(P[\Gamma]) \in \Delta}{(\Delta, \Omega); \Gamma \vdash u[\pi_\Gamma] \sqcup R : P \Longrightarrow i[\mathsf{id}_\Gamma]/(i{::}P[\Gamma], \hat{\Gamma}.u[\pi_\Gamma]/i, \hat{\Gamma}.R/i)}$

$\Delta_1; i{::}P[\Gamma]; \Gamma \vdash i[\mathsf{id}_\Gamma] \doteqdot R : P/(\cdot, \hat{\Gamma}.R/i)$ by rule meta-1
$\Delta_1; i{::}P[\Gamma]; \Gamma \vdash i[\mathsf{id}_\Gamma] \doteqdot u[\pi_\Gamma] : P/(\cdot, \hat{\Gamma}.u[\pi_\Gamma]/i)$ by rule meta-1

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash H_1 \cdot S_1 \sqcup H_2 \cdot S_2 : P \Longrightarrow i[\mathsf{id}_\Gamma]/ ((i{::}P[\Gamma]), (H_1 \cdot S_1/i), (H_2 \cdot S_2/i))$

$H_1 \neq H_2$ and i must be new by inversion
$\Delta_1; \Omega; \Gamma \vdash i[\mathsf{id}_\Gamma] \doteqdot H_1 \cdot S_1 : P/(\cdot, \hat{\Gamma}.H_1 \cdot S_1/i)$ by meta-1
$\Delta_2; \Omega; \Gamma \vdash i[\mathsf{id}_\Gamma] \doteqdot H_2 \cdot S_2 : P/(\cdot, \hat{\Gamma}.H_2 \cdot S_2/i)$ by meta-1

Case. $\mathcal{D} = (\Delta, \Omega); \Gamma \vdash (M_1; S_1) \sqcup (M_2; S_2) : A_1 \to A_2 > P \Longrightarrow (M; S)/((\Omega_1, \Omega_2), (\rho_1, \rho_2), (\rho_1', \rho_2'))$

$(\Delta, \Omega); \Gamma \vdash M_1 \sqcup M_2 : A_1 \Longrightarrow M/(\Omega_1,\ \rho_1, \rho_1')$ by inversion

$(\Delta, \Omega); \Gamma \vdash S_1 \sqcup S_2 : A_2 > P \Longrightarrow S/(\Omega_2,\ \rho_2, \rho_2')$
$(\Delta, \Omega); \Gamma \vdash (M_1; S_1) > A_1 \rightarrow A_2 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash M_1 \Leftarrow A_1$
$(\Delta, \Omega); \Gamma \vdash S_1 > A_2 \Rightarrow P$ by inversion
$(\Delta, \Omega); \Gamma \vdash (M_2; S_2) > A_1 \rightarrow A_2 \Rightarrow P$ by assumption
$(\Delta, \Omega); \Gamma \vdash M_2 \Leftarrow A_1$
$(\Delta, \Omega); \Gamma \vdash S_2 > A_2 \Rightarrow P$ by inversion
$\Delta_1; \Omega_1; \Gamma \vdash M \doteqdot M_1 : A_1/(\cdot, \rho_1)$ by i.h.
$\Delta_2; \Omega_1; \Gamma \vdash M \doteqdot M_2 : A_1/(\cdot, \rho_1')$ by i.h.
$\Delta_1; \Omega_2; \Gamma \vdash S \doteq S_1 : A_2 > P/(\cdot, \rho_2)$ by i.h.
$\Delta_2, \Omega_2; \Gamma \vdash S \doteq S_2 : A_2 > P/(\cdot, \rho_2)$ by i.h.
$\Delta_1; \Omega_1, \Omega_2; \Gamma \vdash (M; S) \doteq (M_1; S_1) : A_1/(\cdot, (\rho_1, \rho_1'))$ by rule
$\Delta_2, \Omega_1, \Omega_2; \Gamma \vdash (M; S) \doteq (M_2; S_2) : A_1/(\cdot, (\rho_2, \rho_2'))$ by rule □