Overcoming Performance Barriers: efficient proof search in logical frameworks

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Outline

- Logical frameworks and applications
- Efficient proof search in logical frameworks
  - Optimizing higher-order unification
  - Higher-order term indexing
- Conclusion and future work

Logical frameworks

- Meta-languages for deductive systems
  - High-level specification (e.g. logics, type systems)
  - Direct implementations (e.g. proof search, type checking)
  - Meta-reasoning (e.g. cut elim., type preservation)
- Examples:
  λProlog[Nadathur'99], Twelf[Pf'99], Isabelle[Paulson86]
- Other higher-order systems: Coq, PVS, Nuprl, HOL, ...

Higher-order logic programming

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- Dynamic program clauses: Clauses may contain nested universal quantifiers and implications
- Result of query execution: Evidence for a proof together with answer substitution
- Theoretical foundation based on uniform proofs [Miller et. al. 91], [Pf'91]
- Extensions to tabled higher-order logic programming [Pie'03, Pie'05]

Example

- Object logic: First-order logic formula

$$A \;::=\; P \mid A \supset A \mid A \vee A \mid \neg A \mid \forall x.A \mid \exists x.A \mid \ldots$$

- Specifying equivalence preserving transformations
- Sample rules:

$$\begin{array}{lcl} A \supset B & \leftrightarrow & \neg A \vee B \\ \forall x.(A(x) \vee B) & \leftrightarrow & (\forall x.A(x)) \vee B \\ \forall x.(A(x) \supset B) & \leftrightarrow & (\exists x.A(x)) \supset B \end{array}$$

if x is not free in B

Specification in LF

- Based on higher order abstract syntax:

i	:	type.	o	:	type
neg	:	o $\longrightarrow$ o			
imp	:	o $\longrightarrow$ o $\longrightarrow$ o.	all	:	(i $\longrightarrow$ o) $\longrightarrow$ o.
or	:	o $\longrightarrow$ o $\longrightarrow$ o.	exists	:	(i $\longrightarrow$ o) $\longrightarrow$ o.

- Transforming propositions:

$$A \supset B \quad \leftrightarrow \quad \neg A \vee B$$

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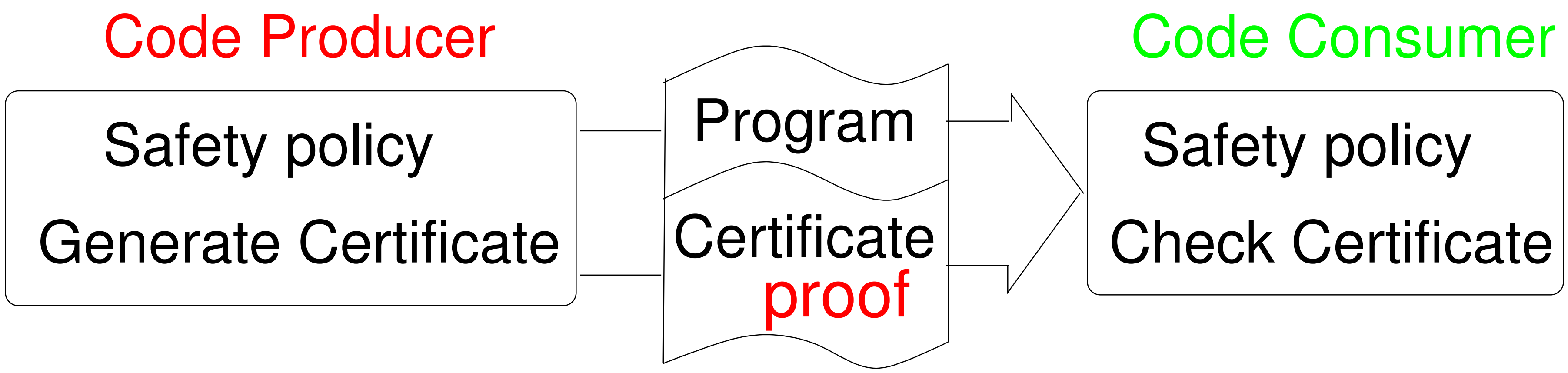
- A: i $\longrightarrow$ o and B: o are meta-variables

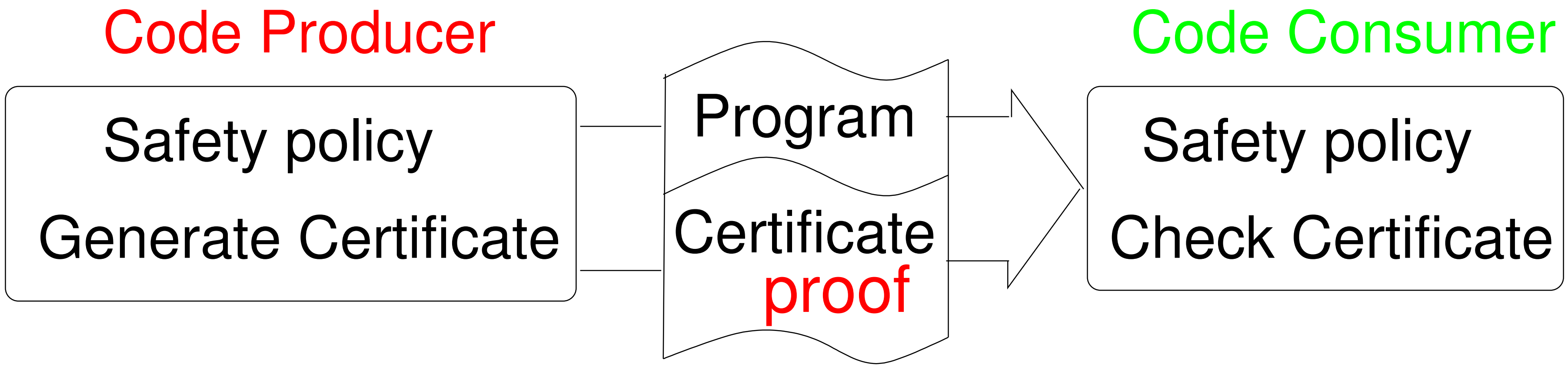
  also sometimes called *existential variables* or *logic variables*

Application: certified code



- Foundational proof-carrying code : [Appel, Felty 00]
- Temporal-logic proof carrying code [Bernard,Lee02]
- Foundational typed assembly language : [Crary 03]
- Distributed access control: [Bauer,Reiter'05]

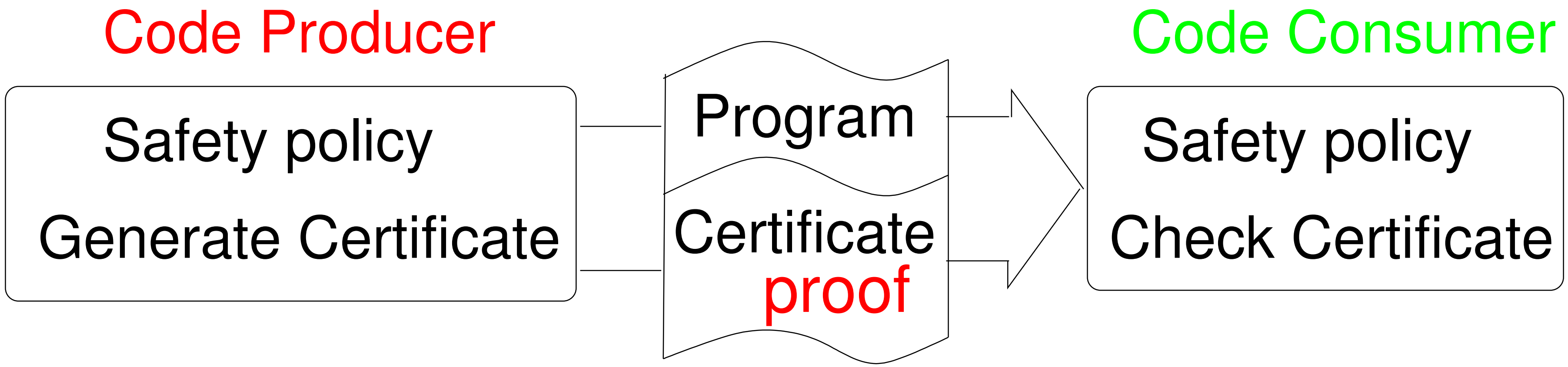
Application: certified code



Large-scale applications

- Typical code size: 70,000 – 100,000 lines
  includes data-type definitions and proofs
- Higher-order logic program: 5,000 lines
- Over 600 – 700 clauses

Application: certified code



Special-purpose logical frameworks :

- Efficient representation and validation of proofs [Necula,Lee98] [Reed'04]
- Proof checking via "higher-order" logic programming [Necula'01], [Wu'03]

Application: Verified Software

- Neglected aspect: language we write programs in
- We need tools to
  - Model and specify programming languages
  - Experiment easily with language extensions
  - Mechanically check their meta-theoretic properties
- POPLmark Challenge [Pierce et al 05]
  "Mechanically check every POPL paper by 2010!"

> Logical framework allows us to represent, execute, and reason about formal systems.

State of the art

- Logical frameworks are widely used.
- Many challenges remain:
  - Higher-order systems are not efficient enough in practice.
  - Complexity of higher-order issues poorly understood.
  - Higher-order systems lack automatic support.
  - ...

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This talk

Eliminating some performance problems

- Optimizing higher-order unification
- Higher-order term indexing

This is a significant step towards
efficient proof search in logical frameworks

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Higher-order unification is undecidable!

For decidable fragment [Miller91, Pfenning91]: at best linear [Qian93]!

Basic operation: unification

- Example 1:

eq	(A imp B)	((not A) or B)	Success
eq	(p imp q)	((not C) or q)	A = p, B = q, C=A

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- Example 2:

eq	(A imp B)	((not A) or B)	Failure(occurs-check!) C = (A imp B),
eq	C	((not C) or q)	A = C, B = q

Basic operation: unification

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  eq (p imp q) ((not C) or q) A = p, B = q, C=A

- Example 2:

  Failure(occurs-check!)

  eq (A imp B) ((not A) or B) C = (A imp B),

  eq C ((not C) or q) A = C, B = q

- Occurs check is expensive!

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eq	C	((not C) or q)	A = C, B = q

- Occurs check is expensive!
- No occurs check is necessary
  if every meta-variable occurs only once!

Higher-order pattern unification

- Meta-variables must be applied to some distinct bound variables

  (all λ x. ((A x) imp B)) – ok ((C T) imp B) – not ok!

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- Closed instantiation for meta-variables!

  eq (all λ y. $\underline{\text{((p y) imp (p y))}}$ imp $\underline{\text{q}}$) $\underline{\text{C}}$

  $\doteq$

  eq (all λ x. $\underline{\text{(A x)}}$ imp $\underline{\text{B}}$) $\underline{\text{((exists } \lambda\text{x. A x) imp B)}}$

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- Solution:

$$
\begin{array}{lcl}
A & = & \lambda \text{ z. (p z) imp (p z)} \\
B & = & \text{q} \\
C & = & \text{((exists (}\lambda\text{x. A x)) imp B)} \\
 & = & \text{(imp (exists (}\lambda\text{x. imp (p x) (p x))) q)}
\end{array}
$$

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$$\doteq$$

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- Failure

  A = λ z. (p z) imp (p z)

  B = ? There is no closed instantiation for B!

  C = . . .

Subtle issues due to bound variables

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  - A depends on bound variable x
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Linearization

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  - every meta-variable occurs only once
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- Example:

eq (A imp B) ((not A) or B)

$$\Longleftrightarrow$$

eq (A imp B) ((not A') or B') and A' $\doteq$ A and B' $\doteq$ B

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  - every meta-variable depends on all distinct bound variables
- Every clause head is transformed into a linear term and variable definitions
- Example:

eq (all λ x. (A x) imp B) ((exists λ x . (A x)) imp B)

$$\Longleftrightarrow$$

eq (all λ x. (A x) imp (B' x)) ((exists λ x. A' x) imp B)

A' $\doteq$ A and $\forall$ x. (B' x) $\doteq$ B

Why does linearization work?

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- Linearization is performed statically.
- Many problems are already linear.
  constant time assignment algorithm
- Unification often fails.
  Failure can be very expensive in higher-order unification, even in the decidable fragment.

Foundational PCC

example	standard	opt	reduction
mul2	9.52 sec	5.51 sec	42.86%
div2	153.61 sec	121.96 sec	20.63%
pack	1075.61 sec	197.07 sec	81.65%
divx	1133.15 sec	333.69 sec	70.50%
listsum	∞	1073.33 sec	100%

∞ = process does not terminate in 6h

Intel Pentium 1.6GHz, RAM 256MB,

SML New Jersey 110, Twelf 1.4.

Evaluation

- Performance improvement is substantial

  20% – 82% runtime improvement; in some case 100%!
  - 63% of the time there are no variable defs.
  - 80% of the calls to unification failed
- Benchmarks (simply typed):
  - Meta-interpreter for linear ordered logic: 60%
  - Classical natural deduction (NK): 42%
- Benchmarks (dependently typed):
  - Compiler translations : 99.95%, in some cases 100%
  - Translating proofs into cut-free proofs: 43% - 52%

Contribution and related work

- Foundation for meta-variables based on modal logic (joint work with F. Pfenning)(CADE'03)
  - Extends earlier work by [Dowek et al. 95]
  - Contextual modal type theory and applications (joint work with A. Nanevski, F. Pfenning, 2005)
- Related work: λProlog (Teyjus-compiler) [Nadathur, Mitchell 99]
  - General higher-order unification (highly non-deterministic)
  - WAM with special higher-order support

Optimizing unification further

- Eliminating redundant type arguments [IJCAR'06]
  - Dependently typed terms have implicit type arguments
  - Some implicit type arguments in a term M are uniquely determined by the overall type of M.
  - These implicit arguments can be skipped during unification!
- Early empirical study [Michaylov,Pfenning'92]

Experiments and evaluation

- Compiler translation:
  - Substantial number of redundant type arguments (up to 1496)
  - Substantial size of skipped arguments (av 30, max 185)
  - Run-time improvement: 11.19% - 21.87%
- Proof translations:
  - Substantial number of redundant type arguments (up to 264387)
  - Size of skipped arguments (av 7)
  - Run-time improvement: 3% - 10%

Contribution and related work

- Performance improvement up to 20%
- Numerous redundant type arguments
- Theoretical justification [IJCAR06]
- Related Work: λ-Prolog : redundant type arguments due to polymorphism [Nadathur, Qi'05]
  - incorporated into the WAM
  - no experimental comparison

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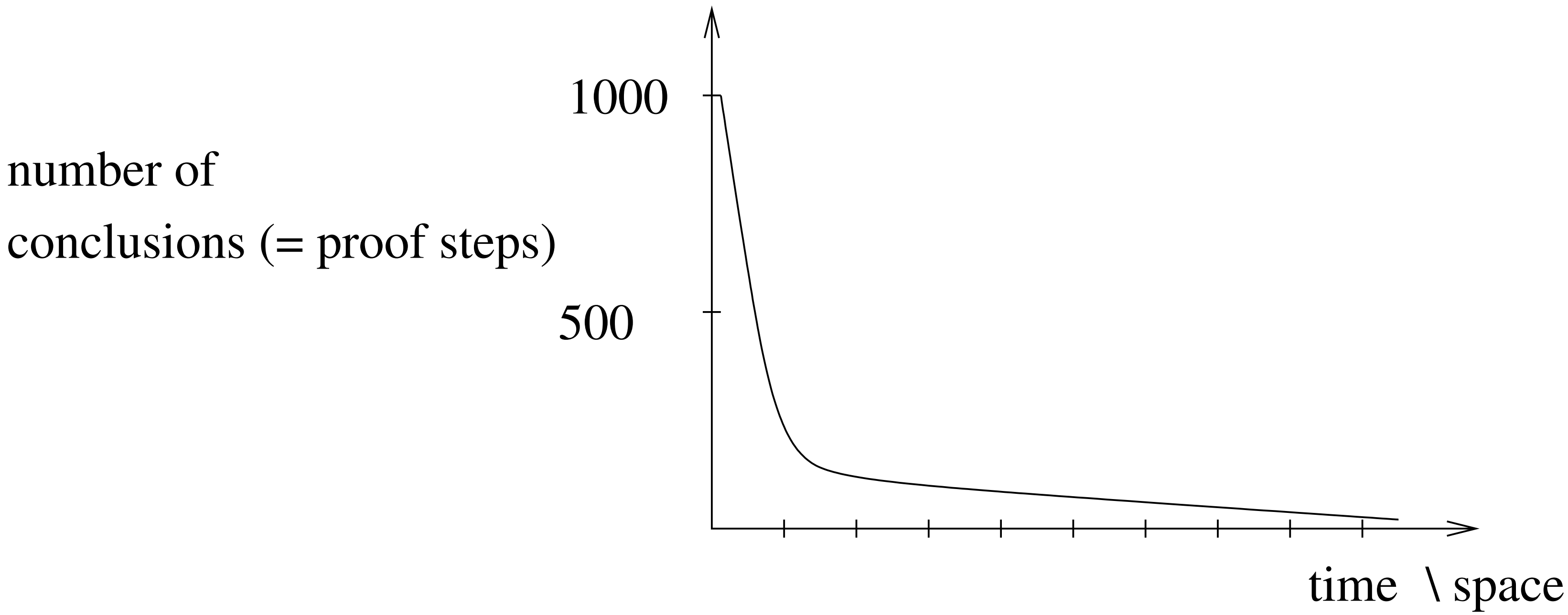

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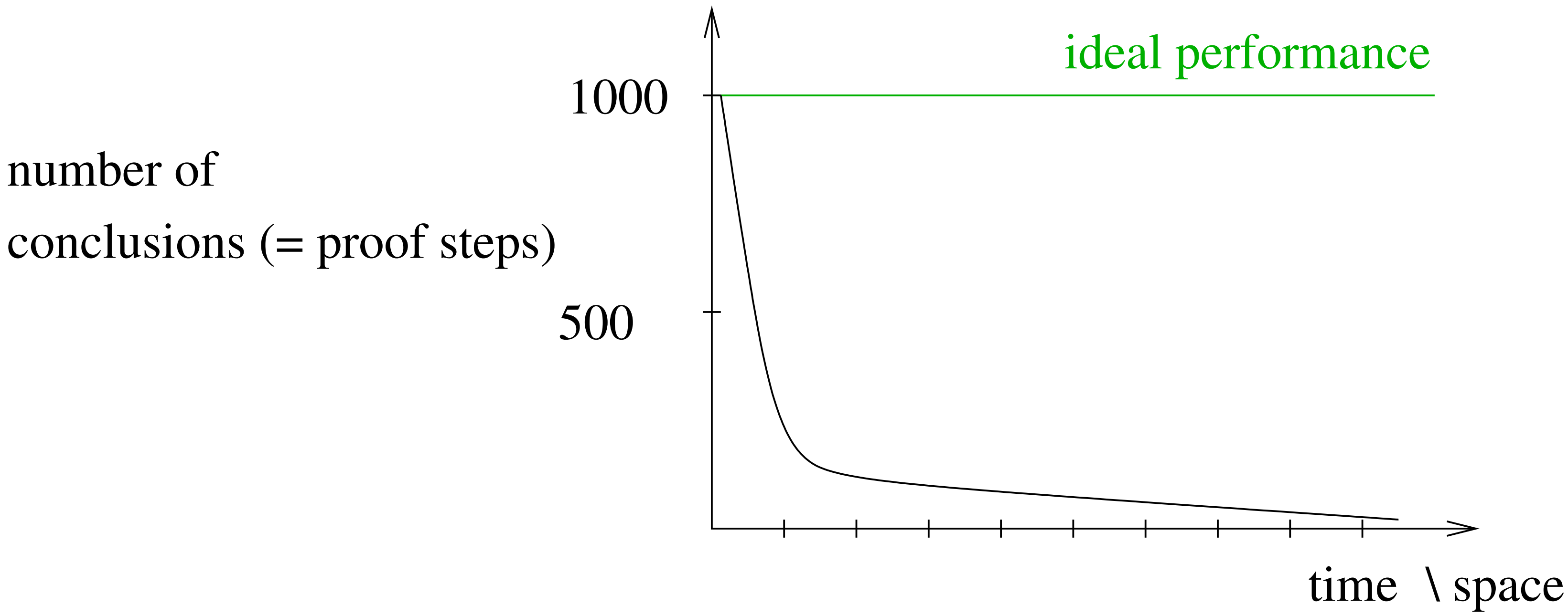

Indexing

Set of terms

eq (all λ x. ((A x) or B))	((all λ x. A x) or B)
eq (A imp B)	((not A) or B)
eq (not (A and B))	((not A) or (not B))

How can we efficiently store and retrieve data?

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Step 1: Linearization

Set of linear terms		Constraints
(1) eq (all λ x. ((A x) or (B' x)))	((all λ x. A' x) or B)	A = A', $\forall$ x. B' x $\doteq$ B
(2) eq (A imp B)	((not A') or B')	A' $\doteq$ A, B $\doteq$ B'
(3) eq (not (A and B))	((not A') or (not B'))	A' $\doteq$ A, B $\doteq$ B'

- Linearize every terms
  Factor out “hard” sub-expressions
- Uniform naming for variables

Step 2: Common sub-expression

Set of linear terms		Constraints
(1) eq (all λ x. ((A x) or (B' x)))	((all λ x. A' x) or B)	$\forall$ x. B' x $\doteq$ B, A = A'
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  $\boxed{\text{eq } i_1 \text{ ((not A') or } i_2\text{)}}$

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- In general the most specific common generalization does not exist!
  Key: linearization

Higher-order substitution trees

Set of linear terms

(1) eq (all λ x. ((A x) or (B' x))) ((all λ x. A' x) or B)
(2) eq (A imp B) ((not A') or B')
(3) eq (not (A and B)) ((not A') or (not B'))

eq i_1 (i_2 or i_3) /i_0

(not A') / i_2

all λ x. (A x) or (B' x) / i_1
all λ x. (A' x) / i_2
B / i_3

(1)

Compose substitutions!

B' / i_3,
A imp B / i_1

(2)

(not B') / i_3,
not (A and B) / i_1

(3)

Parser for formulas

	iterative	memoization		
#tok	deepening	noindex	index	reduction
20	0.98 sec	0.13 sec	0.07 sec	46%
58	∞	2.61 sec	1.25 sec	52%
117	∞	10.44 sec	5.12 sec	51%
235	∞	75.57 sec	26.08 sec	66%

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Refinement type-checking

	example	noindex	index	reduction	orig
First	sub	3.19 sec	0.46 sec	86%	
answer	mult	7.78 sec	0.89 sec	89%	
	square	9.02 sec	0.98 sec	89%	
Not	mult	2.38 sec	0.38 sec	84%	
provable	plus	6.48 sec	0.85 sec	87%	
	square	9.29 sec	1.09 sec	88%	
All	sub	6.88 sec	0.71 sec	90%	
answers	mult	9.06 sec	0.98 sec	89%	
	square	10.30 sec	1.08 sec	90%	

Refinement type-checking

	example	noindex	index	time red.	orig
First	sub	3.19 sec	0.46 sec	86%	0.15 sec
answer	mult	7.78 sec	0.89 sec	89%	0.15 sec
	square	9.02 sec	0.98 sec	89%	0.16 sec
Not	mult	2.38 sec	0.38 sec	84%	13.50 sec
provable	plus	6.48 sec	0.85 sec	87%	∞
	square	9.29 sec	1.09 sec	88%	∞
All	sub	6.88 sec	0.71 sec	90%	5.59 sec
answers	mult	9.06 sec	0.98 sec	89%	∞
	square	10.30 sec	1.08 sec	90%	∞

Contribution and related work

- Contribution:
  - Higher-order term indexing (key: linearization, η-longform)
  - Indexing substantially improves performance
    runtime reduced between 46% and 90% (ICLP'03)
  - Application: Small proof witness [ICLP'05]
  - Application: Propositional theorem proving [CADE'05]

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  - Application: Small proof witness [ICLP'05]
  - Application: Propositional theorem proving [CADE'05]
- Related Work:
  - Substitution trees for first-order terms [Graf95]
  - (Higher-order) automata-driven indexing [Necula,Rahul01]
    imperfect filter, full higher-order unification to check candidates

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Conclusion

- This is opens many new opportunities
  - to experiment and develop large-scale systems.
    for example: proof-carrying code, POPLmark
  - to explore the full potential of logical frameworks
    new applications: authentication, security

- Efficient proof search techniques are critical
  - to sustain performance.
  - to reduce response time to the developer.

Future work

Narrowing the performance gap further

- Mode, determinism, termination analysis [Schrijvers et al. 02]
- Exploiting properties of local theories (joint work with Xi Li(McGill))

Tabled higher-order logic programming [Pie'03, Pie'05]

- Strongly connected components (SCC) [Swift, Sagonas98]
- Model-checking over high-level specifications [Ramakrishnan'97]

Finally ...

if you want to find out more:

http://www.cs.mcgill.ca/~bpientka

email: bpientka@cs.mcgill.ca