COMP 426 Automated Reasoning
Homework 3, Exercise 1: Solutions
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Exercise 1: Prove subject reduction for the rules for natural numbers (see page 44 of the Constructive Logic course notes). You need to show that if $\cdot \vdash t \in \tau$ and $t \Longrightarrow t'$ then $\cdot \vdash t' \in \tau$.

Solution: 4 cases

1. $\textbf{case } 0 \textbf{ of } 0 \Rightarrow t_0 \mid \mathrm{s}(x) \Rightarrow t_s \Longrightarrow t_0$

   By assumption, we have that $(\textbf{case } 0 \textbf{ of } 0 \Rightarrow t_0 \mid \mathrm{s}(x) \Rightarrow t_s) \in \tau$
   We want to show that $\cdot \vdash t_0 \in \tau$
   By inversion of the $\mathbf{nat}E^x$ rule, we have that $\cdot \vdash t_0 \in \tau$, so we're done.

2. $\textbf{case } \mathrm{s}(n) \textbf{ of } 0 \Rightarrow t_0 \mid \mathrm{s}(x) \Rightarrow t_s \Longrightarrow [n/x]t_s$

   By assumption, we have that $(\textbf{case } \mathrm{s}(n) \textbf{ of } 0 \Rightarrow t_0 \mid \mathrm{s}(x) \Rightarrow t_s) \in \tau$
   We want to show that $\cdot \vdash [n/x]t_s \in \tau$
   By inversion of the $\mathbf{nat}E^x$ rule, we have that $\cdot \vdash \mathrm{s}(n) \in \mathbf{nat}$, and that $x \in \mathbf{nat} \vdash t_s \in \tau$.
   By inversion of the $\mathbf{nat}I_s$ rule, we have that $\cdot \vdash n \in \mathbf{nat}$.
   By the substitution property of hypothetical judgments, we may conclude that $\cdot \vdash [n/x]t_s \in \tau$

3. $\textbf{rec } 0 \textbf{ of } f(0) \Rightarrow t_0 \mid f(\mathrm{s}(x)) \Rightarrow t_s \Longrightarrow t_0$

   This case is very similar to Case 1. We arrive at the desired conclusion by inversion of the $\mathbf{nat}E^{f,x}$ rule.

4. $\textbf{rec } \mathrm{s}(n) \textbf{ of } f(0) \Rightarrow t_0 \mid f(\mathrm{s}(x)) \Rightarrow t_s$
   $\Longrightarrow [\textbf{rec } n \textbf{ of } f(0) \Rightarrow t_0 \mid f(\mathrm{s}(x)) \Rightarrow t_s/f(x)][n/x]t_s$

   By assumption, we have that $(\textbf{rec } \mathrm{s}(n) \textbf{ of } f(0) \Rightarrow t_0 \mid f(\mathrm{s}(x)) \Rightarrow t_s) \in \tau$
   We want to show that $\cdot \vdash [\textbf{rec } n \textbf{ of } f(0) \Rightarrow t_0 \mid f(\mathrm{s}(x)) \Rightarrow t_s/f(x)][n/x]t_s \in \tau$
   By inversion of the $\mathbf{nat}E^{f,x}$ rule, we have that $\cdot \vdash \mathrm{s}(n) \in \mathbf{nat}$, $x \in \mathbf{nat}, f(x) \in \tau \vdash t_s \in \tau$ and $\cdot \vdash t_0 \in \tau$. By inversion of the $\mathbf{nat}I_s$ rule, we have that $\cdot \vdash n \in \mathbf{nat}$.
   From all this, and by the $\mathbf{nat}E^{f,x}$ rule, we have that

$\cdot \vdash \textbf{rec}\ n\ \textbf{of}\ f(0) \Rightarrow t_0 \mid f(\mathrm{s}(x)) \Rightarrow t_s \in \tau.$
By the substitution property of hypothetical judgments (applied twice), we arrive at the desired conclusion.