On the Complexity of Closest Pair via Polar-Pair of Point-Sets*

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Abstract

Every graph G can be represented by a collection of equi-radii spheres in a d-dimensional metric Δ such that there is an edge uv in G if and only if the spheres corresponding to u and v intersect. The smallest integer d such that G can be represented by a collection of spheres (all of the same radius) in Δ is called the *sphericity* of G, and if the collection of spheres are non-overlapping, then the value d is called the *contact-dimension* of G. In this paper, we study the sphericity and contact dimension of the complete bipartite graph $K_{n,n}$ in various L^p-metrics and consequently connect the complexity of the monochromatic closest pair and bichromatic closest pair problems.

1 Introduction

This paper studies the geometric representation of a complete bipartite graph in L^p-metrics and consequently connects the complexity of the closest pair and bichromatic closest pair problems beyond certain dimensions. Given a point-set P in a d-dimensional L^p-metric, an *α-distance graph* is a graph $G = (V, E)$ with a vertex set $V = P$ and an edge set

$$E = \{uv : \|u - v\|_p \leq \alpha; u, v \in P; u \neq v\}.$$

In other words, points in P are centers of spheres of radius $\alpha/2$, and G has an edge uv if and only if the spheres centered at u and v intersect. The *sphericity* of a graph G in an L^p-metric, denoted by $\mathsf{sph}_p(G)$, is the smallest dimension d such that G is isomorphic to some α-distance graph in a d-dimensional L^p-metric, for some constant $\alpha > 0$. The sphericity of a graph in the L^∞-metric is

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known as *cubicity*. A notion closely related to sphericity is *contact-dimension*, which is defined in the same manner except that the spheres representing G must be non-overlapping. To be precise, an *α-contact graph* $G = (V, E)$ of a point-set P is an α-distance graph of P such that every edge uv of G has the same distance (i.e., $\|u - v\|_p = \alpha$). Thus, G has the vertex set $V = P$ and has an edge set E such that

$$\forall uv \in E, \quad \|u - v\|_p = \alpha \quad \text{and} \quad \forall uv \notin E, \quad \|u - v\|_p > \alpha.$$

The contact-dimension of a graph G in the L^p-metric, denoted by $\mathsf{cd}_p(G)$, is the smallest integer $d \geq 1$ such that G is isomorphic to a contact-graph in the d-dimensional L^p-metric. We will use distance and contact graphs to means 1-distance and 1-contact graphs.

We are interested in determining the sphericity and the contact-dimension of the biclique $K_{n,n}$ in various L^p-metrics. For notational convenience, we denote $\mathsf{sph}_p(K_{n,n})$ by $\mathsf{bsph}(L^p)$, the *biclique sphericity* of the L^p-metric, and denote $\mathsf{cd}_p(K_{n,n})$ by $\mathsf{bcd}(L^p)$, the *biclique contact-dimension* of the L^p-metric. We call a pair of point-sets (A, B) *polar* if it is the partition of the vertex set of a contact graph isomorphic to $K_{n,n}$. More precisely, a pair of point-sets (A, B) is polar in an L^p-metric if there exists a constant $\alpha > 0$ such that every inner-pair $u, u' \in A$ (resp., $v, v' \in B$) has L^p-distance greater than α while every crossing-pair $u \in A, v \in B$ has L^p-distance exactly α.

The biclique sphericity and contact-dimension of the L^2 and L^∞ metrics are well-studied in literature (see [Rob69, Mae84, Mae85, FM88, Mae91, BL05]). Maehara [Mae91, Mae84] showed that $n < \mathsf{bsph}(L^2) \leq (1.5)n$, and Maehara and Frankl & Maehara [Mae85, FM88] showed that $(1.286)n - 1 < \mathsf{bcd}(L^2) < (1.5)n$. For cubicity, Roberts [Rob69] showed that $\mathsf{bcd}(L^\infty) = \mathsf{bsph}(L^\infty) = 2\log_2 n$. Nevertheless, for other L^p-metrics, contact dimension and sphericity are not well-studied.

1.1 Our Results and Contributions

Our main conceptual contribution is connecting the complexity of the (monochromatic) closest pair problem (Closest Pair) to that of the bichromatic closest pair problem (BCP) through the contact dimension of the biclique. This is discussed in subsection 1.1.1. Our main technical contributions are bounds on the contact dimension and sphericity of the biclique for various L^p-metrics. This is discussed in subsection 1.1.2. Finally, as an application of the connection discussed in subsection 1.1.1 and the bounds discussed in subsection 1.1.2, we show computational equivalence between monochromatic and bichromatic closest pair problems.

1.1.1 Connection between Closest Pair and BCP

In Closest Pair, we are asked to find a pair of points in a set of m points with minimum distance. BCP is a generalization of Closest Pair, in which each point is colored red or blue, and we are asked to find a pair of red-blue points (i.e., bichromatic pair) with minimum distance. It is not hard to see that BCP is at least as hard as Closest Pair since we can apply an algorithm for BCP to solve Closest Pair with the same asymptotic running time. However, it is not clear whether the other direction is true. We will give a simple reduction from BCP to Closest Pair using a polar-pair of point-sets. First, take a polar-pair (A, B), each with cardinality $n = m/2$, in the L^p-metric. Next, pair up vectors in A and B to red and blue points, respectively, and then attach a vertex $u \in A$ (resp., $v \in B$) to its matching red (resp., blue) point. This reduction increases the distances between every pair of points, but by the definition of the polar-pair, this process has more effect on the distances of the *monochromatic* (i.e., red-red or blue-blue) pairs than that of *bichromatic pairs*, and the reduction, in fact, has no effect on the order of crossing-pair distances at all. By scaling the vectors in A and B appropriately, this gives an instance of CLOSEST PAIR whose closest pair of points is bichromatic. Consequently, provided that the polar-pair of point-sets (A, B) in a d-dimensional metric can be constructed within a running time at least as fast as the time for computing CLOSEST PAIR in the same metric, this gives a reduction from BCP to CLOSEST PAIR, thus implying that they have the same running time lower bound.

1.1.2 Bounds on Contact Dimension and Sphericity of Biclique

Our main technical results are lower and upper bounds on the biclique contact-dimension for the L^p-metric space where $p \in \mathbb{R}_{\geq 1} \cup \{0\}$.

Theorem 1. *The following are upper and lower bounds on biclique contact-dimension for the L^p-metric.*

$$\mathsf{bsph}(L^0) = \mathsf{bcd}(L^0) = n \tag{1}$$

$$n \leq \mathsf{bsph}(L^0_{\{0,1\}})) \leq \mathsf{bcd}(L^0_{\{0,1\}}) \leq n^2 \qquad \text{(i.e., } P \subseteq \{0,1\}^d\text{)} \tag{2}$$

$$\Omega(\log n) \leq \mathsf{bsph}(L^1) \leq \mathsf{bcd}(L^1) \leq n^2 \tag{3}$$

$$\Omega(\log n) \leq \mathsf{bsph}(L^p) \leq \mathsf{bcd}(L^p) \leq 2n \qquad \text{for } p \in (1,2) \tag{4}$$

$$\mathsf{bsph}(L^p) = \Theta(\mathsf{bcd}(L^p)) = \Theta(\log n) \qquad \text{for } p > 2 \tag{5}$$

Note that $\mathsf{bsph}(\Delta) \leq \mathsf{bcd}(\Delta)$ for any metric Δ. Thus, it suffices to prove a lower bound for $\mathsf{bsph}(\Delta)$ and prove an upper bound for $\mathsf{bcd}(\Delta)$.

We note that the bounds on the sphericity and the contact dimension of the L^1-metric in (3) are obtained from (5) and (1), respectively. We are unable to show a strong (e.g., linear) lower bound for the L^1-metric. However, we prove the weaker (average-case) result below for the L^1-metric which can be seen as a progress toward proving stronger lower bounds on the sphericity of the biclique in this metric (see Corollary 7 for more discussion on its applications).

Theorem 2. *For any integer $d > 0$, there exist no two finite-supported random variables X, Y taking values from $\mathbb{R}^d$such that the following hold.*

$$\mathop{\mathbb{E}}_{x_1,x_2 \in_R X}[\|x_1 - x_2\|_1] > \mathop{\mathbb{E}}_{x_1 \in_R X, y_1 \in_R Y}[\|x_1 - y_1\|_1]$$
$$\mathop{\mathbb{E}}_{y_1,y_2 \in_R Y}[\|y_1 - y_2\|_1] > \mathop{\mathbb{E}}_{x_1 \in_R X, y_1 \in_R Y}[\|x_1 - y_1\|_1] .$$

For an overview on the known bounds on bsph and bcd (including the results in this paper), please see Table 1.

In Appendix A, we give an alternate proof of the linear lower bound on $\mathsf{bsph}(L^2)$ using spectral analysis similar to that in [BL05]. While our lower bound is slightly weaker than the best known bounds [FM88, Mae91], our arguments require no heavy machinery and thus are arguably simpler than the previous works [FM88, Mae91, BL05].

Alman and Williams [AW15] showed the subquadratic-time hardness for BCP in L^p-metrics, for all $p \in \mathbb{R}_{\geq 1} \cup \{0\}$, under the *Orthogonal Vector Hypothesis* (OVH). From Theorem 1 and the connection between BCP and CLOSEST PAIR described in subsection 1.1.1, we have the following hardness of CLOSEST PAIR.

Metric	Bound	From
L^0	$\mathsf{bsph}(L^0) = \mathsf{bcd}(L^0) = n$	This paper
L^1	$\Omega(\log n) \leq \mathsf{bsph}(L^1) \leq \mathsf{bcd}(L^1) \leq n^2$	This paper
L^p, $p \in (1,2)$	$\Omega(\log n) \leq \mathsf{bsph}(L^p) \leq \mathsf{bcd}(L^p) \leq 2n$	This paper
L^2	$n < \mathsf{bsph}(L^2) \leq \mathsf{bcd}(L^2) < 1.5 \cdot n$	[Mae91, FM88]
L^p, $p > 2$	$\mathsf{bsph}(L^p) = \Theta(\mathsf{bcd}(L^p)) = \Theta(\log n)$	This paper
L^∞	$\mathsf{bsph}(L^\infty) = \mathsf{bcd}(L^\infty) = 2\log_2 n$	[Rob69]

Table 1: Known Bounds on Sphericity and Contact Dimension of Biclique

Theorem 3. *Let $p > 2$. For any $\varepsilon > 0$ and $d = \omega(\log n)$, the closest pair problem in the d-dimensional L^p-metric admits no $(n^{2-\varepsilon})$-time algorithm unless the Orthogonal Vectors Hypothesis is false.*

We remark here that showing conditional hardness for Closest Pair in the L^p metric for $p \leq 2$ remains an outstanding open problem*. Recently, Rubinstein [Rub18] showed that the subquadratic-time hardness holds even for approximating BCP: Assuming OVH, for every $p \in \mathbb{R}_{\geq 1} \cup \{0\}$ and every $\varepsilon > 0$, there is a constant $\gamma(\varepsilon, p) > 0$ such that there is no $(1+\gamma)$-approximation algorithm running in time $O(n^{2-\varepsilon})$ for BCP in the L^p-metric. By using the connection between BCP and Closest Pair described in subsection 1.1.1 and the bounds in Theorem 1 (to be precise we need the efficient construction with appropriate gap as given by Theorem 17), the hardness of approximation result can be extended to Closest Pair for L^p metrics where $p > 2$.

Theorem 4. *Let $p > 2$. For every $\varepsilon > 0$ and $d = \omega(\log n)$, there exists a constant $\gamma = \gamma(p, \varepsilon) > 0$ such that the closest pair problem in the d-dimensional L^p-metric admits no $(n^{2-\varepsilon})$-time $(1+\gamma)$-approximation algorithm unless the Orthogonal Vectors Hypothesis is false.*

We remark that the hardness for the case of the L^∞-metric does not follow (at least directly) from [AW15] or [Rub18]. For independent interest, we show the subquadratic-time hardness of BCP and Closest Pair in the L^∞-metric.

Theorem 5. *For any $\varepsilon > 0$ and $d = \omega(\log n)$, the closest pair problem in the d-dimensional L^∞-metric admits no $(n^{2-\varepsilon})$-time $(2 - o(1))$-approximation algorithm unless the Orthogonal Vectors Hypothesis is false.*

We note that the lower bounds on bsph act as barriers for gadget reductions from BCP to Closest Pair. This partially explains why there has been no progress in showing conditional hardness for Closest Pair in the Euclidean metric for $d = \omega(\log n)$ dimensions (as $\mathsf{bsph}(L^2) = \Omega(n)$). In addition, Rubinstein noted in [Rub18] that one obstacle in proving inapproximability results for Closest Pair is due to the triangle inequality – any two point-sets A and B in *any metric space* cannot have distinct points $a, a' \in A$ and $b \in B$ such that $\|a - a'\| > 2 \cdot \max\{\|a - b\|, \|a' - b\|\}$ (as otherwise it would violate the triangle inequality). This rules out the possibility of obtaining the conditional hardness for 2-approximating Closest Pair for any metric via simple gadget reductions. We note that the inapproximability factor of Theorem 5 matches the triangle inequality barrier (for the L^∞ metric).

*The subquadratic-time hardness of Closest Pair in the L^p-metric for $p \in \mathbb{R}_{\geq 1} \cup \{0\}$ was claimed in [ARW17a] but later retracted [ARW17b].

1.2 Related Works

While our paper studies sphericity and contact-dimension of the complete bipartite graph, determining the contact-dimension of a complete graph in L^p-metrics has also been extensively studied in the notion of *equilateral dimension*. To be precise, the equilateral dimension of a metric Δ which is the maximum number of equidistant points that can be packed in Δ. An interesting connection is in the case of the L^1-metric, for which we are unable to establish a strong lower bound for $\mathsf{bsph}(L^1)$. The equilateral dimension of L^1 is known to be at least $2d$, and this bound is believed to be tight [Guy83]. This is a notorious open problem known as *Kusner's conjecture*, which is confirmed for $d = 2, 3, 4$ [BCL98, KLS00], and the best upper bound for $d \geq 5$ is $O(d \log d)$ by Alon and Pavel [AP03]. If Kusner's conjecture is true for all d, then $\mathsf{sph}_1(K_n) = n/2$.

The complexity of Closest Pair has been a subject of study for many decades. There have been a series of developments on Closest Pair in the Euclidean space (see, e.g., [Ben80, HNS88, KM95, SH75, BS76]), which culminates in a deterministic $O(2^{O(d)} n \log n)$-time algorithm [BS76] and a randomized $O(2^{O(d)} n)$-time algorithm [Rab76, KM95]. For low (i.e., constant) dimensions, these algorithms are tight as the matching lower bound of $\Omega(n \log n)$ was shown by Ben-Or [Ben83] and Yao [Yao91] for the *algebraic decision tree* model, thus settling the complexity of Closest Pair in low dimensions. For high dimensions (i.e., $d = \omega(\log n)$), there is no known algorithm that runs in time significantly better than a trivial $O(n^2 d)$-time algorithm for general d except for the case that $d \geq \Omega(n)$ whereas there are subcubic-time algorithms in L^1 and L^∞ metrics [GS16, ILLP04].

In the last few years, there has been a lot of progress in our understanding of BCP, Closest Pair, and related problems. Alman and Williams [AW15] showed subquadratic time hardness for BCP in $d = \omega(\log n)$ dimensions under OVH in the L^p metric for every $p \in \mathbb{R}_{\geq 1} \cup \{0\}$. Williams [Wil17] extended the result of [AW15] and showed the above subquadratic-time hardness for BCP even for dimensions $d = \omega((\log \log n)^2)$ under OVH. In a recent breakthrough on hardness of approximation in P, Abboud et al. [ARW17b] showed the subquadratic-time hardness for approximating the Bichromatic Maximum Inner Product problem under OVH in the L^p metric for every $p \in \mathbb{R}_{\geq 1} \cup \{0\}$, and the result holds for almost polynomial approximation factors. More recently, building upon the ideas in [ARW17b], Rubinstein [Rub18] showed under OVH the inapproximablility of BCP for every L^p-metric for $p \in \mathbb{R}_{\geq 1} \cup \{0\}$.

1.3 Organization

This paper is organized as follows. In Section 2, we briefly describe the notations and problems of interest. In Section 3, we discuss the case of the L^1-metric. Although we are unable to prove either a strong lower or upper bound for this case, we do make progress towards proving lower bound for $\mathsf{bsph}(L^1)$ and $\mathsf{bcd}(L^1)$. In Section 4, we prove *tight* bounds for $\mathsf{bsph}(L^0)$ and $\mathsf{bcd}(L^0)$ ((1) in Theorem 1). In Section 5, we prove upper bound for the case of $\mathsf{bsph}(L^p)$ and $\mathsf{bcd}(L^p)$, for $p \in (1, 2)$ ((4) in Theorem 1). In Section 6, we prove logarithmic bounds on $\mathsf{bsph}(L^p)$ and $\mathsf{bcd}(L^p)$, for $p > 2$ ((5) in Theorem 1). In Section 7, we show the subquadratic-time hardness for Closest Pair in L^∞. Finally, in Section 8, we conclude our paper by giving open problems and highlighting some directions for future research.

2 Preliminaries

We use the following standard terminologies and notations.

Distance Measures. For any vector $x \in \mathbb{R}^d$, the L^p-norm of x is denoted by $||x||_p = \left(\sum_{i=1}^{d} |x_i|^p\right)^{1/p}$. The L^∞-norm of x is denoted by $||x||_\infty = \max_{i \in [d]} \{|x_i|\}$, and the L^0-norm of x is denoted by $||x||_0 = |\{x_i \neq 0 : i \in [d]\}|$, i.e., the number of non-zero coordinates of x. These norms define distance measures in $\mathbb{R}^d$. The distance of two points x and y w.r.t. the L^p-norm, say *L^p-distance*, is thus $||x - y||_p$. The distance measures that are well studied in literature are the *Hamming distance* L^0-norm, the *Rectilinear distance* L^1-norm, the *Euclidean distance* L^2-norm, the *Chebyshev distance* (a.k.a, *Maximum-norm*) L^∞-norm.

Problems. Here we give formal definitions of CLOSEST PAIR and BCP. In CLOSEST PAIR, we are given a collection of points $P \subseteq \mathbb{R}^d$ in a d-dimensional L^p-metric, and the goal is find a pair of distinct points $a, b \in P$ that minimizes $\|u - v\|_p$. In BCP, the input point-set is partitioned into two color classes (the collections of red and blue points) A and B, and the goal is find a pair of points $u \in A$ and $v \in B$ that minimizes $\|u - v\|_p$.

Fine-Grained Complexity and Conditional Hardness. Conditional hardness is the current trend in proving running-time lower bounds for polynomial-time solvable problems. This has now developed into the area of *Fine-Grained Complexity*. Please see, e.g., [Wil15, Wil16] and references therein.

The *Orthogonal Vectors Hypothesis* (OVH) is a popular complexity theoretic assumption in Fine-Grained Complexity. OVH states that in the Word RAM model with $O(\log n)$ bit words, any algorithm requires $n^{2-o(1)}$ time in expectation to determine whether collections of vectors $A, B \subseteq \{0,1\}^d$ with $|A| = |B| = n/2$ and $d = \omega(\log n)$ contain an orthogonal pair $u \in A$ and $v \in B$ (i.e., $\sum_{i=1}^{d} u_i \cdot v_i = 0$).

Another popular conjecture is the *Strong Exponential-Time Hypothesis* for SAT (SETH), which states that, for every $\varepsilon > 0$, there exists an integer k_ε such that k_ε-SAT on n variables cannot be solved in $O(2^{(1-\varepsilon)n})$-time. It was shown by Williams that SETH implies OVH [Wil05].

3 Geometric Representation of Biclique in L^1

In this section, we discuss the case of the L^1-metric. As discussed in the introduction, this is the only case where we are unable to prove neither strong lower bound nor linear upper bound. A weak lower bound $\mathsf{bsph}(L^1) \geq \Omega(\log n)$ follows from the proof for the L^p-metric with $p > 2$ in Section 6.1 (Theorem 16), and a quadratic upper bound $\mathsf{bcd}(L^1) \leq n^2$ follows from the proof for the L^0-metric in Section 4.2 (Corollary 12). However, we cannot prove any upper bound smaller than $\Omega(n^2)$ or any lower bound larger than $O(\log n)$. Hence, we study an average case relaxation of the question.

We show in Theorem 2 that there is no distribution whose expected distances simulate a polar-pair of point-sets in the L^1-metric. Consequently, even though we could not prove the bi-clique sphericity lower bound for the L^1-metric, we are able to refute an existence of a geometric representation with *large gap* for any dimension as shown in Corollary 7. (A similar result was shown in [DM94] for the L^2-metric.)

Definition 6 (L^1-distribution)**.** *For any $d > 0$, let X, Y be two random variables taking values from $\mathbb{R}^d$. An L^1-distribution is constructed by X, Y if the following holds.*

$$\begin{aligned}\mathop{\mathbb{E}}_{x_1,x_2\in_R X}\left[\|x_1 - x_2\|_1\right] &> \mathop{\mathbb{E}}_{x_1\in_R X,\, y_1\in_R Y}\left[\|x_1 - y_1\|_1\right], \\ \mathop{\mathbb{E}}_{y_1,y_2\in_R Y}\left[\|y_1 - y_2\|_1\right] &> \mathop{\mathbb{E}}_{x_1\in_R X,\, y_1\in_R Y}\left[\|x_1 - y_1\|_1\right].\end{aligned} \tag{6}$$

Theorem 2 (Restated)**.** *For any two finite-supported random variables X, Y that are taking values from $\mathbb{R}^d$, there is no L^1-distribution.*

Proof. Assume towards a contradiction that there exist two finite-supported random variables X, Y that are taking values in $\mathbb{R}^d$ and that are satisfying Eq. 6 of Definition 6. Given a vector $x \in \mathbb{R}^d$, we denote by $x(i)$ the value of the i-th coordinate of x. Hence the following inequalities hold,

$$\begin{aligned} 0 &> \mathop{\mathbb{E}}_{x_1\in_R X,\, y_1\in_R Y}\left[\|x_1 - y_1\|_1\right] - \mathop{\mathbb{E}}_{x_1,x_2\in_R X}\left[\|x_1 - x_2\|_1\right] \\ &= \mathop{\mathbb{E}}_{x_1,x_2\in_R X,\, y_1\in_R Y}\left[\|x_1 - y_1\|_1 - \|x_1 - x_2\|_1\right] \\ &= \frac{1}{d}\cdot \mathop{\mathbb{E}}_{x_1,x_2\in_R X,\, y_1\in_R Y}\left[\mathop{\mathbb{E}}_{i\in_R[1\ldots d]}\left[|x_1(i) - y_1(i)| - |x_1(i) - x_2(i)|\right]\right] \\ &= \frac{1}{d}\cdot \mathop{\mathbb{E}}_{i\in_R[1\ldots d]}\left[\mathop{\mathbb{E}}_{x_1,x_2\in_R X,\, y_1\in_R Y}\left[|x_1(i) - y_1(i)| - |x_1(i) - x_2(i)|\right]\right]. \end{aligned}$$

Thus for some $i^\star \in [d]$ the following holds,

$$0 > \mathop{\mathbb{E}}_{x_1,x_2\in_R X,\, y_1\in_R Y}\left[|x_1(i^\star) - y_1(i^\star)| - |x_1(i^\star) - x_2(i^\star)|\right]. \tag{7}$$

Fix $i^\star \in [d]$ satisfying the above inequality. For the sake of clarity, we assume that the random variables X, Y are taking values in $\mathbb{R}$ (i.e., projection on the $i^{\star\text{th}}$ coordinate). We can assume that the size of $\mathsf{supp}(X) \cup \mathsf{supp}(Y)$ is greater than 1 because if $\mathsf{supp}(X) \cup \mathsf{supp}(Y)$ contains a single point, then $\mathop{\mathbb{E}}_{x_1\in_R X,\, y_1\in_R Y}\left[\|x_1 - y_1\|_1\right] = \mathop{\mathbb{E}}_{x_1,x_2\in_R X}\left[\|x_1 - x_2\|_1\right] = 0$, contradicting Eq. 7. Let $\mathsf{supp}(X) \cup \mathsf{supp}(Y)$ contains $t \geq 2$ points. We prove by induction on t, that there are no X, Y over $\mathbb{R}$ satisfying Eq. 7. The base case is when $t = 2$. By Eq. 7, there exists 3 points $\tilde{x}_1, \tilde{x}_2, \tilde{y}_1$ in $\mathbb{R}$ such that

$$0 > \|\tilde{x}_1 - \tilde{y}_1\|_1 - \|\tilde{x}_1 - \tilde{x}_2\|_1. \tag{8}$$

Since $\mathsf{supp}(X) \cup \mathsf{supp}(Y)$ contains exactly two points, $\tilde{x}_1, \tilde{x}_2, \tilde{y}_1$ are supported by two distinct points in $\mathbb{R}$. Hence, there are two cases either that $x_1 = x_2$ (and $y_1 \neq x_1$) or that $x_1 \neq x_2$ (and either $\tilde{y}_1 = \tilde{x}_1$ or $\tilde{y}_1 = \tilde{x}_2$). It is easy to see that none of these cases satisfy Eq. 8, a contradiction.

Assume the induction hypothesis that there are no X, Y taking values from $\mathbb{R}$ satisfying Eq. 7 when the size of $\mathsf{supp}\,(X) \cup \mathsf{supp}\,(Y)$ is equal to $k \geq 2$. Then consider the case when $t = k+1 \geq 3$. Sort the points in $\mathsf{supp}\,(X) \cup \mathsf{supp}\,(Y)$ by their values, and denote by s_i the value of the i-th point of $\mathsf{supp}\,(X) \cup \mathsf{supp}\,(Y)$. For the sake of simplicity, we say that we *change* the value of s_{t-1} to $\tilde{s}_{t-1}$, where $s_{t-2} \leq \tilde{s}_{t-1} \leq s_t$, if after changing its value we change the values of (at least one of) X, Y to $\tilde{X}, \tilde{Y}$ in such a way that the value of the $(t-1)$-th point (after sorting) of $\mathsf{supp}\left(\tilde{X}\right) \cup \mathsf{supp}\left(\tilde{Y}\right)$ is equal to $\tilde{s}_{t-1}$ (if $s_{t-2} = \tilde{s}_{t-1}$, then the value of the $(t-2)$-th point of $\mathsf{supp}\left(\tilde{X}\right) \cup \mathsf{supp}\left(\tilde{Y}\right)$ is equal to $\tilde{s}_{t-1}$). Define the function $f : [s_{t-2}, s_t] \to \mathbb{R}$ as follows:

$$f(x) = \mathop{\mathbb{E}}_{x_1 \in_R \tilde{X}, y_1 \in_R \tilde{Y}} [\|x_1 - y_1\|_1] - \mathop{\mathbb{E}}_{x_1, x_2 \in_R \tilde{X}} [\|x_1 - x_2\|_1] \ ,$$

where $\tilde{X}, \tilde{Y}$ are obtained after changing s_{t-1} to $x \in [s_{t-2}, s_t]$. The crucial observation is that the function f is linear. Hence, either $f(s_{t-2}) \geq f(s_{t-1})$ or $f(s_t) \geq f(s_{t-1})$, and we can reduce the size of $\mathsf{supp}\,(X) \cup \mathsf{supp}\,(Y)$ by 1. However, this contradicts our induction hypothesis. □

The following corollary refutes the existence of a polar-pair of point-sets with large gap in any dimension.

Corollary 7 (No Polar-Pair of Point-Sets in L^1 with Large Gap)**.** *For any $\alpha > 0$, there exist no subsets $A, B \subseteq \mathbb{R}^d$ of $n/2$ vectors with $d < n/2$ such that*

- *For any u, v both in A, or both in B, $\|u - v\|_1 \geq \frac{1}{1-2/n} \cdot \alpha$.*
- *For any $u \in A$ and $v \in B$, $\|u - v\|_1 < \alpha$.*

Proof. Assume towards a contradiction that there exist a polar-pair of point-sets (A, B) in the L^1-metric that satisfies the conditions above. We can create a distribution X and Y such that

$$\mathop{\mathbb{E}}_{x_1, x_2 \in_R X} [\|x_1 - x_2\|_1] = \mathop{\mathbb{E}}_{y_1, y_2 \in_R Y} [\|y_1 - y_2\|_1] > \mathop{\mathbb{E}}_{x \in_R X, y \in_R Y} [\|x - y\|_1]$$

To see this, we create a uniform random variable X (resp., Y) for the set A (resp., B). Now the expected distance of two independent copies of X (resp., Y) is at least $\frac{1}{1-2/n}\alpha \cdot \left(1 - \frac{1}{n/2}\right) = \alpha$, which follows because we may pick the same point twice. Since the expected distance of the crossing pair $u \in A$ and $v \in B$ is less than α. This contradicts Theorem 2. □

We can show similar results that there are no polar-pairs of point-sets with large gap in the L^0 and L^2 metrics. The case of the L^0-metric follows directly from Theorem 2 when the alphabet set is $\{0, 1\}$. (Please also see Lemma 9 for an alternate proof.) The case of the L^2-metric follows from the fact that $\mathsf{bsph}(L^2) = \Omega(n)$ [FM88, Mae91] and that we can reduce the dimension of a polar-pairs of point-sets with constant gap to $O(\log n)$ using dimension reduction [JL84].

4 Geometric Representation of Biclique in L^0

In this section, we prove a lower bound on $\mathsf{bsph}(L^0)$ and an upper bound on $\mathsf{bcd}(L^0)$. We start by providing a real-to-binary reduction below. Then we proceed to prove the lower bound on $\mathsf{bsph}(L^0)$ in Section 4.1 and then the upper bounds on $\mathsf{bcd}(L^0)$ in Section 4.2.

Real to Binary Reduction. First we prove the following (trivial) lemma, which allows mapping from vectors in $\mathbb{R}^d$ to zero-one vectors.

Lemma 8 (Real to Binary Reduction). *Let $S \subseteq \mathbb{R}$ be a finite set of real numbers. Then there exists a transformation $\phi : S^d \to \{0,1\}^{d|S|}$ such that, for any $x, y \in S^d$,*

$$\|x - y\|_0 = \frac{1}{2} \cdot \|\phi(x) - \phi(y)\|_0$$

Proof. First, we order the elements in S in an arbitrary order and write it as $S = \{r_1, r_2, \ldots, r_{|S|}\}$. Next we define $\psi : S \to \{0,1\}^{|S|}$ so that the i-th coordinate of $\psi(r_i)$ is 1, and the rest are zeroes. That is,

$$\psi(r_i)_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

Then we define $\phi(x) = (\psi(x_1), \psi(x_2), \ldots, \psi(x_d))$. Clearly, $\|\psi(r_i) - \psi(r_j)\|_0 = 2$ if and only if $r_i \neq r_j$. Therefore, we conclude that for any $x, y \in P$,

$$\|\phi(x) - \phi(y)\|_0 = 2 \cdot \|x - y\|_0.$$

□

4.1 Lower Bound on the Biclique-Sphericity

Now we will show that $\mathsf{bsph}(L^0) \geq n$. Our proof requires the following lemma, which rules out a randomized algorithm that generates a polar-pair of point-sets.

Lemma 9 (No Distribution for L^0). *For any $\alpha > \beta \geq 0$, regardless of dimension, there exist no distributions $\mathcal{A}$ and $\mathcal{B}$ of points in $\mathbb{R}^d$ with finite supports such that*

- $\mathbb{E}_{x,x' \in_R \mathcal{A}}[\|x - x'\|_0] \geq \alpha$.
- $\mathbb{E}_{y,y' \in_R \mathcal{B}}[\|y - y'\|_0] \geq \alpha$.
- $\mathbb{E}_{x \in_R \mathcal{A}, y \in_R \mathcal{B}}[\|x - y\|_0] \leq \beta$.

Proof. We prove by contradiction. Assume to a contrary that such distributions exist. Then

$$\mathbb{E}_{x,x' \in_R \mathcal{A}}[\|x - x'\|_0] + \mathbb{E}_{y,y' \in_R \mathcal{B}}[\|y - y'\|_0] - 2\mathbb{E}_{x \in_R \mathcal{A}, y \in_R \mathcal{B}}[\|x - y\|_0] > 0. \quad (9)$$

Let A and B be supports of $\mathcal{A}$ and $\mathcal{B}$, respectively. By Lemma 8, we may assume that vectors in A and B are binary vectors. Observe that each coordinate of vectors in A and B contribute to the expectations independently. In particular, Eq. (9) can be written as

$$2\sum_i \rho^A_{0,i}\rho^A_{1,i} + 2\sum_i \rho^B_{0,i}\rho^B_{1,i} + 2\sum_i \left(\rho^A_{0,i}\rho^B_{1,i} + \rho^B_{0,i}\rho^A_{1,i}\right) > 0 \quad (10)$$

where $\rho^A_{0,i}$, $\rho^A_{1,i}$, $\rho^B_{0,i}$ and $\rho^B_{1,i}$ are the probability that the i-th coordinate of $x \in A$ (resp., $y \in B$) is 0 (resp., 1). Thus, to show a contradiction, it is sufficient to consider the coordinate which contributes the most to the summation in Eq. (10). The contribution of this coordinate to the summation is

$$2\rho_0^A\rho_1^A + 2\rho_0^B\rho_1^B - 2(\rho_0^A\rho_1^B + \rho_1^A\rho_0^B) = 2(\rho_0^A(\rho_1^A - \rho_1^B) + 2(\rho_0^B(\rho_1^B - \rho_1^A) = 2(\rho_0^A - \rho_0^B)(\rho_1^A - \rho_1^B) \quad (11)$$

Since $\rho_0^A + \rho_1^A = 1$ and $\rho_0^B + \rho_1^B = 1$, the summation in Eq.(11) can be non-negative only if $\rho_0^A = \rho_0^B$ and $\rho_1^A = \rho_1^B$. But, then this implies that the summation in Eq.(11) is zero. We have a contradiction since this coordinate contributes the most to the summation in Eq. (10) which we assume to be positive. □

The next Theorem shows that $\mathsf{bsph}(L^0) \geq n$.

Theorem 10 (Lower Bound for L^0 with Arbitrary Alphabet)**.** *For any integers $\alpha > \beta \geq 0$ and $n > 0$, there exist no subsets $A, B \subseteq \mathbb{R}^d$ of n vectors with $d < n$ such that*

- *For any $a, a' \in A$, $\|a - a'\|_0 \geq \alpha$.*
- *For any $b, b' \in B$, $\|b - b'\|_0 \geq \alpha$.*
- *For any $a \in A$ and $b \in B$, $\|a - b\|_0 \leq \beta$.*

Proof. Suppose for a contradiction that such subsets A and B exist with $d < n$. We build uniform distributions $\mathcal{A}$ and $\mathcal{B}$ by uniformly at random picking a vector in A and B, respectively. Then it is easy to see that the expected value of inner distance is

$$\mathbb{E}_{x,x' \in_R \mathcal{A}}[\|x - x'\|_0] \geq \alpha - \frac{\alpha}{n}$$

The inner distance of B is similar. We know that $\alpha - \beta \geq 1$ because they are integers and so are L^0-distances. But, then if $\alpha < n$, we would have distributions that contradict Lemma 9. Note that α and β are at most d (dimension). Therefore, we conclude that $d \geq n$. □

4.2 Upper Bound on the Biclique Contact-Dimension

Now we show that $\mathsf{bcd}(L^0) \leq n$.

Theorem 11 (Upper Bound for L^0 with Arbitrary Alphabet)**.** *For any integer $n > 0$ and $d = n$, there exist subsets $A, B \subseteq \mathbb{R}^d$ each with n vectors such that*

- *For any $a, a' \in A$, $\|a - a'\|_0 = d$.*
- *For any $b, b' \in B$, $\|b - b'\|_0 = d$.*
- *For any $a \in A$ and $b \in B$, $\|a - b\|_0 = d - 1$.*

Proof. First we construct a set of vectors A. For $i = 1, 2, \ldots, n$, we define the i-th vector a of A so that a is an all-i vector. That is,

$$a = (i, i, \ldots, i).$$

Next we construct a set of vectors B. The first vector of B is $(1, 2, \ldots, n)$. Then the $(i+1)$-th vector of B is the left rotation of the i-th vector. Thus, the i-th vector of B is

$$b = (i, i+1, \ldots, n, 1, 2, \ldots, i-1).$$

It can be seen that the L^0-distance between any two vectors from the same set is d because all the coordinates are different. Any vectors from different set, say $a \in A$ and $b \in B$, must have at least one common coordinate. Thus, their L^0-distance is $d-1$. This proves the lemma. □

Below is the upper bound for zero-one vectors, which is a corollary of Theorem 11.

Corollary 12 (Upper Bound for L^0 with Binary Vectors)**.** *For any integer $n > 0$ and $d = n^2$, there exist subsets $A, B \subseteq \mathbb{R}^d$ each with n vectors such that*

- *For any $a, a' \in A$, $\|a - a'\|_0 = n$.*
- *For any $b, b' \in B$, $\|b - b'\|_0 = n$.*
- *For any $a \in A$ and $b \in B$, $\|a - b\|_0 = n - 1$.*

Proof. We take the construction from Theorem 11. Denote the two sets by A' and B', and denote their dimensions by $d' = n$.

We transform A' and B' to sets A and B by applying the transformation ϕ in Lemma 8. That is,

$$A = \{\phi(a) : a \in A'\} \quad \text{and} \quad B = \{\phi(b) : b \in B'\}.$$

Since the alphabet set in Lemma 8 is $[n]$, we have a construction of A and B with dimension $d = n^2$. □

5 Geometric Representation of Biclique in L^p for $p \in (1, 2)$

In this section, we prove the upper bound on $\mathtt{bcd}(L^p)$ for $p \in (1, 2)$. We are unable to show any lower bound for these L_p-metrics except for the lower bound of $\Omega(\log n)$ obtained from the ϵ-net lower bound in Theorem 16 (which will be proven in the next Section).

Theorem 13 (Upper Bound for L^p with $1 < p < 2$)**.** *For every $1 < p < 2$ and for all integers $n \geq 1$, there exist two sets $A, B \subseteq \mathbb{R}^{2n}$ each of cardinality n such that the following holds:*

1. *For every distinct points $u, v \in A$, $\|u - v\|_p = 2^{1/p}$.*
2. *For every distinct points $u, v \in B$, $\|u - v\|_p = 2^{1/p}$.*
3. *For every points $u \in A$ and $v \in B$, $\|u - v\|_p < 2^{1/p}$.*

Proof. We will construct point-sets as claimed in the theorem for given p and n. Let α be a parameter depending on p and n, which will be set later. For each $i \in [n]$, we create a point $a \in A$ by setting

$$a_j = \begin{cases} 0 & \text{if } 1 \leq j \leq n \text{ and } j \neq i \\ 1 & \text{if } 1 \leq j \leq n \text{ and } i = j \\ \alpha & \text{if } n+1 \leq j \leq 2n \end{cases}$$

Similarly, for each $i \in [n]$, we create a point $b \in B$ by setting

$$b_j = \begin{cases} \alpha & \text{if } 1 \leq j \leq n \\ 0 & \text{if } n+1 \leq j \leq 2n \text{ and } j \neq n+i \\ 1 & \text{if } n+1 \leq j \leq 2n \text{ and } j = n+i \end{cases}$$

By construction, for every pair of points u, v both in A or both in B, their L^p-distance is $\|u - v\|_p = 2^{1/p}$, and for every pair of points from different sets, say $u \in A$ and $v \in B$, their L^p-distance is

$$\|u - v\|_p = 2^{1/p} \cdot ((1-\alpha)^p + (n-1) \cdot \alpha^p)^{1/p} \leq 2^{1/p} \cdot ((1-\alpha)^p + n \cdot \alpha^p)^{1/p} \tag{12}$$

Now let us choose $\alpha > n^{-1/(p-1)}$, and consider the term $(1-\alpha)^p + n \cdot \alpha^p$ in Eq. (12). Observe that $\alpha < n \cdot \alpha^p$ for $1 < p < 2$. Define a function $f(x) = (1-\alpha)^x + n \cdot \alpha^x$. We know that $f(x)$ is less than 1 as x goes from ∞ to 1 (i.e., $\lim_{x \to 1^+}((1-\alpha)^x + n \cdot \alpha^x) < 1$). Moreover, $f(x)$ is decreasing for $0 < \alpha < 1$, which means that $f(p) < 1$. Consequently, $\|u - v\|_p < 2^{1/p}$, and the theorem follows.

To finish the proof, we will show that $f(x)$ is decreasing for $x > 1$ provided that $0 < \alpha < 1$. It suffices to show that $f'(x) < 0$ for all values of x.

$$f'(x) = \frac{\partial}{\partial x}\left((1-\alpha)^x + n \cdot \alpha^x\right) = (1-\alpha)^x \ln(1-\alpha) + n \cdot \alpha^x \ln(\alpha) < 0.$$

The last inequality follows from the fact that $\ln(x) < 0$ for $0 < x < 1$ and that $0 < \alpha, 1 - \alpha < 1$. □

6 Geometric Representation of Biclique in L^p for $p > 2$

In this section, we show the lower bound on $\mathsf{bsph}(L^p)$ and an upper bound on $\mathsf{bcd}(L^p)$ for $p > 2$. Both bounds are logarithmic. The latter upper bound is constructive and efficient (in the sense that the polar-pair of point-sets can be constructed in $\widetilde{O}(n)$-time). This implies the subquadratic-time equivalence between CLOSEST PAIR and BCP.

6.1 Lower Bound on the Biclique Sphericity

Now we show the lower bound on the biclique sphericity of a complete bipartite graph in L^p-metrics with $p > 2$. In fact, we prove the lower bound for the case of a star graph on n vertices, denoted by S_n, and then use the fact that $\mathsf{bsph}(H) \leq \mathsf{bsph}(G)$ for all induced subgraph H of G (i.e., $\mathsf{bsph}(K_{n/2,n/2}, L^p) \geq \mathsf{bsph}(S_{n/2}, L^p)$).

In short, we show in Lemma 16 that $O(\log n)$ is the maximum number of L^p-balls of radius $1/2$ that we can pack in an L^p-ball of radius one so that no two of them intersect or touch each other. This upper bounds, in turn, implies the lower bound on the dimension. We proceed with the proof by volume arguments, which are commonly used in proving the minimum number of points in an *ϵ-net* that are sufficient to cover all the points in a sphere.

Definition 14 (ϵ-net)**.** *The unit L^p-ball in $\mathbb{R}^d$ centered at o is denoted by*

$$\mathbb{B}\left(L_p^d, o\right) = \left\{x \in \mathbb{R}^d \mid \|x - o\|_p \leq 1\right\}.$$

*For brevity, we write $\mathbb{B}\left(L_p^d\right)$ to mean $\mathbb{B}\left(L_p^d, o\right)$. Let (X, d) be a metric space and let S be a subset of X and ϵ be a constant greater than 0. A subset N_ϵ of X is called an ϵ-**net** of S under d if for every point $x \in S$ it holds for some point $y \in N_\epsilon$ that $d(x, y) \leq \epsilon$.*

The following lemma is well known in literature (see, e.g., [Ver10]). For the sake of completeness, we provide a proof below.

Lemma 15. *There exists an ϵ-net for $\mathbb{B}\left(L_p^d\right)$ under the L^p-metric of cardinality $\left(1 + \frac{2}{\epsilon}\right)^d$.*

Proof. Let us fix $\epsilon > 0$ and choose N_ϵ of maximal cardinality such that $\|x - y\|_p > \epsilon$ for all $x \neq y$ both in N_ϵ. We claim that N_ϵ is an ϵ-net of the $\mathbb{B}\left(L_p^d\right)$. Otherwise, there would exist a point $x \in \mathbb{B}\left(L_p^d\right)$ that is at least ϵ-far from all points in N_ϵ. Thus, $N_\epsilon \cup \{x\}$ contradicts the maximality of N_ϵ. After establishing that N_ϵ is an ϵ-net, we note that by the triangle inequality, we have that the balls of radii $\epsilon/2$ centered at the points in N_ϵ are disjoint. On the other hand, by the triangle inequality all such balls lie in $(1 + \epsilon/2)\,\mathbb{B}\left(L_p^d\right)$. Comparing the volumes gives us that $\text{vol}\left(\frac{\epsilon}{2}\mathbb{B}\left(L_p^d\right)\right) \cdot |N_\epsilon| \leq \text{vol}\left(\left(1 + \frac{\epsilon}{2}\right)\mathbb{B}\left(L_p^d\right)\right)$. Since $\text{vol}\left(r \cdot \mathbb{B}\left(L_p^d\right)\right) = r^d \cdot \text{vol}\left(\mathbb{B}\left(L_p^d\right)\right)$ for all $r \geq 0$, we conclude that $|N_\epsilon| \leq \frac{(1+\epsilon/2)^d}{(\epsilon/2)^d}| = \left(1 + \frac{2}{\epsilon}\right)^d$. □

Theorem 16. *For every $N, d \in \mathbb{N}$, for $p \geq 1$, and for any two sets $A, B \subseteq \mathbb{R}^d$, each of cardinality N, suppose the following holds for some non-negative real numbers α and β with $\alpha > \beta$.*

1. *For every u and v both in A, $\|u - v\|_p > \alpha$.*
2. *For every u and v both in B, $\|u - v\|_p > \alpha$.*
3. *For every u in A and v in B, $\|u - v\|_p \leq \beta$.*

Then the dimension d must be at least $\log_5(N)$.

Proof. Scale and translate the sets A, B in such a way that $\beta = 1$ and that $\vec{0} \in B$. It follows that $A \subseteq \mathbb{B}\left(L_p^d\right)$. By Lemma 15, we can fix a $1/2$-net $N_{1/2}$ for $\mathbb{B}\left(L_p^d\right)$ of size 5^d. Note that, for every $x \in N_{1/2}$, the ball $1/2 \cdot \mathbb{B}\left(L_p^d, x\right)$ contains at most one point from A. Note also that $N_{1/2}$ covers $\mathbb{B}\left(L_p^d\right)$. Thus, $|A| \leq 5^d$ which implies that $d \geq \log_5(N)$. □

6.2 Upper Bound on the Biclique Contact-Dimension

We first give a simple randomized construction that gives a logarithmic upper bound on the biclique contact-dimension of L^p. The construction is simple. We uniformly at random take a subset A of n vectors from $\{-1,1\}^{d/2} \times \{0\}^{d/2}$ and a subset B of n vectors from $\{0\}^{d/2} \times \{-1,1\}^{d/2}$. Observe that, for any $p > 2$, the L^p-distance of any pair of vectors $u \in A$ and $v \in B$ is exactly d while the *expected distance* between the inner pair $u, u' \in A$ (resp., $v, v' \in B$) is strictly larger than d. Thus, if we choose d to be sufficiently large, e.g., $d \geq 10 \ln n$, then we can show by a standard concentration bound (e.g., Chernoff's bound) that the probability that the inner-pair distance is strictly larger than d is at least $1 - 1/n^3$. Applying the union bound over all inner-pairs, we have that the d-neighborhood graph of $A \cup B$ is a bipartite complete graph with high probability. Moreover, the distances between any crossing pairs $u \in A$ and $v \in B$ are the same for all pairs. This shows the upper bound for the contact-dimension of a biclique in the L^p-metric for $p > 2$.

The above gives a simple proof of the upper bound on the biclique contact-dimension of the L^p-metric. Moreover, it shows a randomized construction of the polar-pair in the $O(\log n)$-dimensional L^p-metric, for $p > 2$, thus implying that CLOSEST PAIR and BCP are equivalent for these L^p-metrics.

For algorithmic purposes, we provide a deterministic construction. One way to derandomize the above process is to use *expanders*. We show it using appropriate codes.

Theorem 17. *For any $p > 2$, let $\zeta = 2^{p-3}$. There exist two sets $|A| = |B| = n$ of vectors in $\mathbb{R}^d$, where $d = 2\alpha \log_2 n$, for some constant $\alpha \geq 1$, such that the following holds.*

1. *For all $u, u' \in A$, $\|u - u'\|_p > ((\zeta + 1/2)d)^{1/p}$.*
2. *For all $v, v' \in B$, $\|v - v'\|_p > ((\zeta + 1/2)d)^{1/p}$.*
3. *For all $u \in A$, $v \in B$, $\|u - v\|_p = d^{1/p}$.*

Moreover, there exists a deterministic algorithm that outputs A and B in time $\widetilde{O}(n)$.

Proof. In literature, we note that for any constant $\delta > 0$, there is an explicit binary code of (some) constant *relative rate* and *relative distance* at least $\frac{1}{2} - \delta$ and the entire code can be listed in quasilinear time with respect to the size of the code (see Appendix E.1.2.5 from [Gol08], or Justesen codes [Jus72]). To be more specific, we can construct in $O(n \log^{O(1)} n)$-time a set $C \subseteq \{-1,1\}^{d'}$ such that (1) $|C| = n$, (2) $d' = d/2 = \alpha \log_2 n$ for some constant $\alpha \geq 1$ and (3) for every two vectors $x, y \in C$, x and y differ on at least $\left(\frac{1}{2} - \delta\right) d'$ coordinates, for some constant $\delta \in \left(0, \frac{1}{4} - \frac{1}{2^p}\right)$.

We construct the sets A and B as subsets of $\{-1,0,1\}^d$. For every $i \in [n]$, the i^{th} point of A is given by the concatenation of the i^{th} point of C with $0^{d'}$. Similarly, the i^{th} point of B is given by the concatenation of $0^{d'}$ with the i^{th} point of C (note the reversal in the order of the concatenation). In particular, points in A and B are of the form $(x_i, \vec{0})$ and $(\vec{0}, x_i)$, respectively, where x_i is the i^{th} point in C and $\vec{0}$ is the zero-vector of length $\alpha \log_2 n$.

First, consider any two points in the same set, say $u, u' \in A$ (resp., $v, v' \in B$). We have from the distance of C that on at least $\left(\frac{1}{2} - \delta\right) d'$ coordinates the two points differ by 2, thus implying that their L^p-distance is at least

$$\left(\left(\frac{1}{2}-\delta\right)d'2^p\right)^{1/p} > \left(\left(\frac{1}{4}+\frac{1}{2^p}\right)d'2^p\right)^{1/p} = \left(\left(2^{p-3}+\frac{1}{2}\right)d\right)^{1/p}.$$

This proves the first two items of the theorem. Next we prove the third item. Consider any two points from different sets, say $u \in A$ and $v \in B$. It is easy to see from the construction that u and v differ in every coordinate by exactly 1. Thus, the L^p-distance between any two points from different set is exactly

$$\left(2d'\right)^{1/p} = d^{1/p}. \qquad \square$$

7 Fine-Grained Complexity of Closest Pair in L^∞

In this section, we prove the quadratic-time hardness of Closest Pair in the L^∞-metric. Our reduction is from the *Orthogonal Vectors* problem (OV), which we phrase it as follows. Given a pair of collections of vectors $U, W \subseteq \{0,1\}^d$, the goal is to find a pair of vectors $u \in U$ and $w \in W$ such that $(u_i, w_i) \in \{(0,0),(0,1),(1,0)\}$ for all $i \in [d]$. Throughout, we denote by n the total number of vectors in U and W.

7.1 Reduction

Let $U, W \subseteq \{0,1\}^d$ be an instance of OV. We may assume that U and W have no duplicates. Otherwise, we may sort vectors in U (resp., W) in lexicographic order and then sequentially remove duplicates; this preprocessing takes $O(dn\log n)$-time.

We construct a pair of sets $A, B \subseteq \mathbb{R}^d$ of BCP from U, W as follows. For each vector $u \in U$ (resp., $w \in W$), we create a point $a \in A$ (resp., $b \in B$) such that

$$a_j = \begin{cases} 0 & \text{if } u_j = 0, \\ 2 & \text{if } u_j = 1. \end{cases} \qquad b_j = \begin{cases} 1 & \text{if } w_j = 0, \\ -1 & \text{if } w_j = 1. \end{cases}$$

Observe that, for any vectors $a \in A$ and $b \in B$, $|a_j - b_j| = 3$ only if $u_j = w_j = 1$; otherwise, $|a_j - b_j| = 1$. It can be seen that $\|a-b\|_p = d$ if and only if their corresponding vectors $u \in U$ and $w \in W$ are orthogonal. Thus, this gives an alternate proof for the quadratic-time hardness of BCP under OVH.

7.2 Analysis

Here we show that the reduction in Section 7.1 rules out both exact and 2-approximation algorithm for Closest Pair in L^∞ that runs in subquadratic-time (unless OVH is false). That is, we prove Theorem 5, which follows from the theorem below.

Theorem 18. *Assuming OVH, for any $\varepsilon > 0$ and $d = \omega(\log n)$, there is no $O\left(n^{2-\varepsilon}\right)$-time algorithm that, given a point-set $P \subseteq \mathbb{R}^d$, distinguishes between the following two cases:*

- *There exists a pair of vectors in P with L^∞-distance one.*

- *Every pair of vectors in P has L^∞-distance two.*

In particular, approximating Closest Pair *in the L^∞-metric to within a factor of two is at least as hard as solving the Orthogonal Vectors problem.*

Proof. Consider the sets A and B constructed from an instance of OV in Section 7.1.

First, observe that every inner pair has L^∞-distance at least 2. To see this, consider an inner pair $a, a' \in A$. Since all inner pairs are distinct, they must have at least one different coordinate, say $a_j \neq a'_j$ for some $j \in \{1, \ldots, n/2\}$. Consequently, $(a_j, a'_j) \in \{(0,2),(2,0)\}$, implying that the L^∞-distance of a and a' is at least 2. The case of an inner pair $b, b' \in B$ is similar. Thus, any pair of vectors with L^∞-distance less than two must be a crossing pair $a \in A, b \in B$.

Now suppose there is a pair of orthogonal vectors $u^* \in U, w^* \in W$, and let $a^* \in A$ and $b^* \in B$ be the corresponding vectors of u^* and w^* in the Closest Pair instance, respectively. Then we know from the construction that $(a^*_j, b^*_j) \in \{(0,1),(0,-1),(2,1)\}$ for all coordinates $j \in [n]$. Thus, the L^∞-distance of a^* and b^* must be one.

Next suppose that there is no orthogonal pair of vectors in $U \times W$. Then every vectors $u \in U$ and $w \in W$ must have one coordinate, say j, such that $u_j = w_j = 1$. So, the corresponding vectors a and b (of u and w, respectively) must have $a_i = 2, b_j = -1$. This means that a and b have L^∞-distance at least three. (Note that there might be an inner pair with L^∞-distance two.) Therefore, we conclude that every pair of points in $A \cup B$ has L^∞-distance at least two. □

8 Conclusion and Discussion

We have studied the sphericity and contact dimension of the complete bipartite graph in various metrics. We have proved lower and upper bounds on these measures for some metrics. However, biclique sphericity and biclique contact dimension in the L^1-metric remains poorly understood as we are unable to show any strong upper or lower bounds. However, we believe that both L^1 and L^2 metrics have linear upper and lower bounds. To be precise, we raise the following conjecture:

Conjecture 19 (L^1-Biclique Sphericity Conjecture)**.**

$$\mathsf{bsph}(L^1) = \Omega(n).$$

We have also shown conditional lower bounds for the Closest Pair problem in the L^p-metric, for all $p \in \mathbb{R}_{>2} \cup \{\infty\}$, by using polar-pair of point-sets. However, it is unlikely that our techniques could get to the regime of L^2, L^1, and L^0, which are popular metrics. An open question is thus whether there exists an alternative technique to derive a lower bound from OVH to the Closest Pair problem for these metrics. The answer might be on the positive side, i.e., there might exist an algorithm that performs well in the L^2-metric because there are more tools available, e.g., Johnson-Lindenstrauss' dimension reduction. Thus, it is possible that there exists a strongly subquadratic-time algorithm in the L^2-metric. This question remains an outstanding open problem.

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A Geometric Representation of Biclique in L^2

In this section we prove a lower bound on $\mathsf{bsph}(L^2)$ of $(n-3)/2$ using spectral analysis.

Theorem 20. *For every $n, d \in \mathbb{N}$, and any two sets $A, B \subseteq \mathbb{R}^d$, each of cardinality n, suppose the following holds for some non-negative real numbers α and β with $\alpha > \beta$.*

1. *For every u and v both in A, $\|u - v\|_2 > \alpha$.*
2. *For every u and v both in B, $\|u - v\|_2 > \alpha$.*
3. *For every u in A and v in B, $\|u - v\|_2 \leq \beta$.*

Then the dimension d must be at least $\frac{n-3}{2}$.

Proof. Let $|A| = |B| = n$ be *arbitrary* two sets of vectors in $\mathbb{R}^d$ that satisfy the above conditions. We will show that $d \geq \frac{n-3}{2}$. First, we scale all the vectors in $A \cup B$ so that the vector with the largest L^2-norm in $A \cup B$ has L^2-norm equal to 1 (by this scaling, the parameters α, β are scaled as well by, say s. For brevity, we will write α for α/s and similarly for β.). We modify A and B in two steps as follows. First, we add one new coordinate to all of the vectors with value $K \gg 1$ (to be determined exactly later) and obtain $A_1, B_1 \subseteq \mathbb{R}^{d+1}$. Note that each element in the new set of vectors A_1 and B_1 has L^2-norm roughly equal to K. More specifically, the square of the L^2-norm is bounded between K^2 and K^2+1 and the vector with the largest L^2-norm in $A_1 \cup B_1$ has L^2-norm equal to $\sqrt{K^2+1}$.

By adding to the last coordinate of each vector u in $A_1 \cup B_1$ a positive value c_u smaller than $1/K$, we can impose that all the vectors are with L^2-norm equal to $\sqrt{K^2+1}$. To see this, note that if we have a vector u_1 in $A_1 \cup B_1$ that has L^2-norm equal to K (namely, as small as possible), then by setting c_{u_1} to satisfy

$$(K + c_{u_1})^2 = K^2 + 1, \tag{13}$$

we have that the L^2-norm of u_1 is $\sqrt{K^2+1}$. So, any c_{u_1} that solves Eq. 13 is smaller than $1/K$. By assuming that u_1 has a larger L^2-norm, we would have a better bound on c_{u_1}.

Let $A_1' \cup B_1'$ be the set of vectors that was obtained by adding c_u's as described above. Let u, v be vectors in $A_1 \cup B_1$ and let u', v' be the corresponding vectors in $A_1' \cup B_1'$. By definition, the following holds:

$$\|u - v\|_2^2 \leq \|u' - v'\|_2^2 = \|u - v\|_2^2 + (c_u - c_v)^2 \leq \|u - v\|_2^2 + 1/K^2.$$

Hence, by choosing K to satisfy $1/K^2 \leq \frac{\alpha^2 - \beta^2}{2}$, it follows that $A_1' \cup B_1'$ satisfies the conditions of the theorem with $\alpha' = \alpha$ and $\beta' = \sqrt{\beta^2 + \frac{\alpha^2-\beta^2}{2}} < \alpha'$. Again, for brevity, we refer to α' as α and β' as β.

Given $A_1', B_1' \subseteq \mathbb{R}^{d+1}$, let $a_1, a_2, \ldots, a_n$ be the vectors from A_1', and $b_1, b_2, \ldots, b_n$ be the vectors from B_1'. Consider the following matrix in $\mathbb{R}^{2(d+1)\times 2n}$:

$$M = \begin{pmatrix} a_1, a_2, \ldots, a_n & b_1, b_2, \ldots, b_n \\ b_1, b_2, \ldots, b_n & a_1, a_2, \ldots, a_n \end{pmatrix} \tag{14}$$

Define the set A_2 to be the first n column vectors of M and define B_2 to be the last n column vectors of M. Note that $A_2 \cup B_2 \subseteq \mathbb{R}^{2(d+1)}$ and satisfies the conditions of the theorem with $\alpha'' = 2\alpha' > 2\beta' = \beta''$. Consider the inner product matrix $M^T M \in \mathbb{R}^{2n \times 2n}$ written in a block matrix form as follows:

$$M^T M = cI_{2n\times 2n} + \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix},$$

where $M_{1,1}, M_{1,2}, M_{2,1}, M_{2,2} \in \mathbb{R}^{n\times n}$ and c is such that the matrix $\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{2,1} & M_{2,2} \end{pmatrix}$ has the value 0 on the diagonal elements (recall that all the vectors have the same L^2-norm). By the definition of M (see Eq. 14), one can check that the following hold.

1. The matrices $M_{1,1}, M_{1,2}, M_{2,1}, M_{2,2}$ are all symmetric: for $M_{1,1}, M_{2,2}$ it follows since $M^T M$ is a symmetric matrix, and for $M_{1,2}, M_{2,1}$ it follows by the way M was defined; see Eq. 14.

2. $M_{1,1} = M_{2,2}$. This follows by Eq. 14.

3. $M_{1,2} = M_{2,1}$. This follows since $M_{1,2} = M_{2,1}^T = M_{2,1}$. Here the first equality follows since $M^T M$ is a symmetric matrix, and the last equality follows by item 1.

Hence, we can write $M^T M = cI_{2n\times 2n} + \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2} & M_{1,1} \end{pmatrix}$. In the rest of the proof, we analyze some of the eigenvectors of $M^T M$. To this end, we consider the matrix $M_{1,1} - M_{1,2}$. Since both $M_{1,1}$ and $M_{1,2}$ are symmetric, we have that $M_{1,1} - M_{1,2}$ is symmetric and has real eigenvalues. Moreover, by the conditions of the theorem, it holds that $M_{1,1} - M_{1,2}$ is strictly negative (i.e., all the entries of the matrix are negative). This follows because all the vectors have the same L^2-norm. Let $x_1, x_2, \ldots, x_n$ be the eigenvectors of $M_{1,1} - M_{1,2}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. By the Perron–Frobenius Theorem it follows that λ_1 is strictly smaller than $\lambda_2, \lambda_3, \ldots, \lambda_n$.

Let $x_i \in \mathbb{R}^n$ be an eigenvector of $M_{1,1} - M_{1,2}$ with eigenvalue λ_i. Then the following holds.

$$\begin{aligned} \begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2} & M_{1,1} \end{pmatrix} \begin{pmatrix} x_i \\ -x_i \end{pmatrix} &= \begin{pmatrix} (M_{1,1} - M_{1,2})\, x_i \\ -(M_{1,1} - M_{1,2})\, x_i \end{pmatrix} \\ &= \begin{pmatrix} \lambda_i x_i \\ -\lambda_i x_i \end{pmatrix} \\ &= \lambda_i \begin{pmatrix} x_i \\ -x_i \end{pmatrix}. \end{aligned}$$

Hence, the vectors $\begin{pmatrix} x_1 \\ -x_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ -x_2 \end{pmatrix}, \ldots, \begin{pmatrix} x_n \\ -x_n \end{pmatrix}$ are eigenvectors of $\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2} & M_{1,1} \end{pmatrix}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The operation of adding $cI_{2n\times 2n}$ to $\begin{pmatrix} M_{1,1} & M_{1,2} \\ M_{1,2} & M_{1,1} \end{pmatrix}$ shifts the eigenvalues of $M^T M$ to $\lambda_1 + c, \lambda_2 + c, \ldots, \lambda_n + c$.

Since $M^T M$ is a positive semidefinite matrix, $\lambda_1 + c, \lambda_2 + c, \ldots, \lambda_n + c \geq 0$. More specifically, $\lambda_1 + c \geq 0$ and $\lambda_2 + c, \ldots, \lambda_n + c > 0$ (since $\lambda_1 < \lambda_2, \lambda_3 \ldots, \lambda_n$). It follows that $M^T M$ has at least $n-1$ positive eigenvalues. Hence, the rank of $M^T M$ is at least $n-1$. By standard linear algebra arguments, it holds that the rank of M is at least the rank of $M^T M$, and the rank of M is at most $2\,(d+1)$. That is,

$$2(d+1) \geq \mathrm{rank}(M) \geq \mathrm{rank}(M^T M) \geq n-1. \qquad \square$$