1 Review

**Definition 1.1.** Let \((\Omega, \mathcal{F}, P)\) be a probability space. Random variables \(\{X_1, \ldots, X_n\}\) are called **jointly Gaussian** with variance matrix

\[
D = [D_{ij}] > 0
\]

if

\[
d\mu_{X_1 \ldots X_n} = (2\pi)^{-n/2} |\det D|^{-1/2} e^{-1/2 <x,D^{-1}x>} \, dx
\]

where \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \, dx = dx_1 \cdots dx_n\)

**Remark 1.2.**

1. \(\{X_1, \ldots, X_n\}\) are Gaussian with variance \(D = [D_{ij}]\) if and only if

\[
C_{X_1 \ldots X_n} = \mathbb{E}(e^{it_1 X_1 + \ldots + it_n X_n}) = e^{-1/2 <t, Dt>}
\]

\[
< t, Dt > = \sum_{i,j=1}^{n} t_i t_j D_{ij}, \quad t = (t_1, \ldots, t_n)
\]

2. \(D_{ij} = \mathbb{E}(X_i X_j), \, \mathbb{E}(X_i) = 0\)

3. \(\{X_1, \ldots, X_n\}\) are Gaussian if and only if

\[
\forall \alpha_i \in \mathbb{R}, \quad \alpha_1 X_1 + \ldots + \alpha_n X_n \text{ is Gaussian.}
\]

4. Let \(\{X_1, \ldots, X_n\}\) be Gaussian with variance \(I_{n \times n}\).

\[
\mathbb{E}(X_1^{m_1} X_2^{m_2} \cdots X_n^{m_n}) = \begin{cases} 
0 & \text{if } l \text{ is odd} \\
\frac{(2l')!}{2^l (l')!} & \text{if } l = 2l'
\end{cases}
\]

where \(l = m_1 + \ldots + m_n\)
2 Gaussian Random Fields

Definition 2.1. Let $G$ be a countable set. The family of random variables
\{\{X_n\}_{n \in G}\} is called a Gaussian Random Field (GRF), if for any finite subset
\{n_1, \ldots, n_k\} \subset G, the random variables
\{X_{n_1}, \ldots, X_{n_k}\}
are jointly Gaussian.

Remark 2.2. 1. $G$ could be finite, or $G$ could be a singleton, in which case we have a single random variable. However, we only care about the case when $G$ is infinite.

2. Same definition applies for an arbitrary $G$ (e.g. $G = [0, \infty)$ for Brownian motion, say). The only real difference is that some measure theoretic aspects are more delicate.

2.1 Uniqueness

Given a GRF \{\{X_n\}_{n \in G}\}, we have
\[ D_{nm} = \mathbb{E}(X_nX_m), \quad D : G \times G \to \mathbb{R}, \quad D(n, m) = D_{nm} \]

Theorem 2.3. Let \((\Omega, \mathcal{F}, P)\) and \((\Omega', \mathcal{F}', P')\) be two probability spaces, and \{\{X_n\}_{n \in G}\} and \{\{X'_n\}_{n \in G'}\} be two GRF on these spaces with variances $D_{nm}$ and $D'_{nm}$.

Suppose $D_{nm} = D'_{nm}$, and assume also, that $\mathcal{F}, \mathcal{F}'$ are minimal $\sigma$-fields with respect to which $\{X_n\}$ and $\{X'_n\}$ are measurable.

Then, \((\Omega, \mathcal{F}, P)\) and \((\Omega', \mathcal{F}', P')\) are isomorphic, and under this isomorphism, $X_j$ corresponds to $X'_j$.

Remark 2.4. Equivalent formulation:

\exists W, a unitary map, $W : L^2(\Omega, dP) \to L^2(\Omega', dP')$

such that

1. $W, W^{-1}$ map bounded functions to bounded functions.

2. $W(XY) = W(X) \cdot W(Y)$ for bounded $X, Y$.

3. $W(X_n) = X'_n$
2.2 Existence and Basic Model

Let \( D : G \times G \to \mathbb{R} \) be a map, \( D(n,m) = D_{nm} \), such that
\[
\forall n_1, \ldots, n_k \in G, [D_{n_i n_j}]_{1 \leq i,j \leq k} \text{ is strictly positive definite.}
\]

Set
\[
\Omega = \mathbb{R}^G = \prod_{n \in G} \mathbb{R}, \quad \omega \in \Omega, \{\omega(n)\}_{n \in G}, \omega(n) \in \mathbb{R}
\]
and let \( \mathcal{F} \) be the Borel \( \sigma \)-algebra generated by cylinders, i.e. sets of the form
\[
\{\omega : \omega(n_1) \in B_1, \ldots, \omega(n_k) \in B_k\}, \text{ where } B_i \text{ are Borel in } \mathbb{R}
\]
If \( a_n > 0, \sum_{n \in G} a_n < \infty \) and
\[
d(\omega, \omega') = \sum_{n \in G} a_n \frac{|\omega(n) - \omega'(n)|}{1 + |\omega(n) - \omega'(n)|}
\]
then \( d \) is a metric on \( \Omega \), and \( (\Omega, d) \) is a complete and separable metric space. Moreover, the Borel \( \sigma \)-field generated by the sets open with respect to \( d \) is precisely \( \mathcal{F} \).

Given a cylinder \( C = \{\omega : \omega(n_1) \in B_1, \ldots, \omega(n_k) \in B_k\} \), set
\[
P(C) = \mu_{n_1, \ldots, n_k}(C) = (2\pi)^{-k/2}(\det D_c)^{1/2} \int_{\prod_{i=1}^{k} B_i} e^{-\frac{1}{2} \langle x, D_c^{-1}x \rangle} dx
\]
where \( D_c = [D_{n_i n_j}]_{1 \leq i,j \leq k} \). This a good definition (does not depend on the way we write \( C \)) and if we take a field consisting of finite disjoint unions of cylinders, \( P \) is a countably additive set function. Hence, by Caratheodory Extension Theorem (or by Kolmogorov Theorem), \( P \) extends uniquely to a probability measure on \( (\Omega, \mathcal{F}) \).

Thus, \( (\Omega, \mathcal{F}, P) \) is a probability space such that, by our construction, \( X_n(\omega) = \omega(n), n \in G \) is a random variable, and the joint distribution of \( \{X_{n_1}, \ldots, X_{n_k}\} \) is \( \mu_{n_1, \ldots, n_k} \), and so, they are Gaussian with variance \( D = [D_{ij}] \).

From now on, we will work only with that model:
\[
\Omega \to \mathbb{R}^G, \quad \mathcal{F} \to \text{Borel } \sigma\text{-field}, \quad P \to \text{Gaussian measure induced by the marginals } \mu_{n_1, \ldots, n_k}
\]

2.3 Support properties of \( P \)

**Proposition 2.5.** Let \( A_n > 0, n \in G \) be a sequence such that
\[
\sum_{n \in G} A_n D_{nn} < \infty
\]
and let
\[ \Omega' = \{ \omega \in \Omega : \sum_{n \in G} A_n \omega^2(n) < \infty \} \]

Then the following holds:

1. \( \Omega' \) is measurable.
2. \( P(\Omega') = 1 \)

Proof. Property (1) is trivial. To prove (2), let
\[ F(\omega) = \sum_{n \in G} A_n \omega^2(n), \quad F : \Omega \rightarrow [0, \infty], \quad F \geq 0 \]
Then
\[ \int_\Omega F(\omega)dP(\omega) = \int_\Omega \sum_{n \in G} A_n \omega^2(n)dP(\omega) = \]
\[ = \sum_{n \in G} A_n \int_\Omega \omega^2(n)dP(\omega) = \sum_{n \in G} A_n D_{nn} < \infty \]
\[ \implies F(\omega) < \infty \quad P\text{-a.e. } \omega \]

2.4 Hilbert Spaces

Consider the Hilbert space
\[ l^2(G) = \{ \omega : \Omega \rightarrow \mathbb{R} | \sum_{n \in G} |\omega(n)|^2 < \infty \} \]
equipped with the usual inner product
\[ \langle \omega, \omega' \rangle = \sum_{n \in G} \omega(n)\omega'(n) \]

Remark 2.6. Usually, \( l^2(G) \) denotes the complex version of this Hilbert space. However, in this document, we are mostly considering \( l^2_\mathbb{R} \), and we decided to simplify our notation by dropping the subscript \( \mathbb{R} \). Hence, to avoid confusion, we will always write \( l^2_\mathbb{C} \) when referring to the complex space.

The matrix \( D = [D_{nm}] \) defines a linear map, which naturally extends to a (possibly unbounded) self-adjoint linear operator on \( l^2(G) \). We will assume that both \( D \) and \( D^{-1} \) are bounded. This amounts to say that, if
\[ (D\omega)(n) = \sum_{m \in G} D_{nm}\omega(m) \]
then the sum on the right hand side is convergent, and also assuming that for some \( M \geq 0 \)
\[
\sum_{n \in G} |(D\omega)(n)|^2 \leq M \sum_{n \in G} |\omega(n)|^2, \quad \forall \omega \in l^2(G)(G)
\]

This assures that \( D \) is bounded and self-adjoint, or more explicitly, that
\[
\left\{ \begin{array}{c}
\langle \omega, D\omega \rangle \leq \sqrt{M} \|\omega\|^2 \\
\langle \omega', D\omega \rangle = \langle D\omega', \omega \rangle
\end{array} \right.
\]

To insure the existence and boundedness of \( D^{-1} \), we further need to assume that
\[
\exists m > 0, \quad \langle \omega, D\omega \rangle \geq m \|\omega\|^2 \Rightarrow \|D^{-1}\|^2 \leq m
\]

In terms of \( \{D_{nm}\} \), \( D \) is bounded if
\[
\sup_{n \in G} \left\{ \sum_{m \in G} |D_{nm}| \right\} < \infty
\]

In particular, \( \forall n, \quad m \leq D_{nn} \leq \sqrt{M} \), so that
\[
\sum_{n \in G} A_n D_{nn} < \infty \iff \sum_{n \in G} A_n < \infty
\]

### 2.5 Explicit Computations with GRF

Let \( \alpha \in l^2(G) \), and suppose that only finitely many coordinates are non-zero.

The vector space of such \( \alpha \)'s is denoted \( f l^2_R(G) \).

\[
fl^2(G) = \{ \alpha \in l^2(G) : \alpha(n) \neq 0 \text{ for finitely many } n \}
\]

For \( \alpha \in fl^2(G) \), set
\[
\Phi_{\alpha}(\omega) = \sum_{n \in G} \alpha(n)\omega(n) = \sum_{n \in G} \alpha(n)X_n(\omega)
\]

Hence, \( \Phi_{\alpha}(\omega) \) is a Gaussian random variable and since
\[
\int_{\Omega} \Phi_{\alpha}(\omega)^2 dP(\omega) = \sum_{n,m \in G} \alpha(n)\alpha(m) \int_{\Omega} \omega(n)\omega(m)dP =
\]
\[
= \sum_{n,m \in G} \alpha(n)\alpha(m)D_{nm} = \langle \alpha, D\alpha \rangle
\]

\( \Phi_{\alpha}(\omega) \) has variance \( \langle \alpha, D\alpha \rangle \).
Remark 2.7. If $\delta_n(m)$ is the Kronecker delta on $l^2(G)$, i.e.

$$\delta_n(m) = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Then obviously, $\delta_n \in f l^2(G)$, and $\Phi \delta_n = X_n$.

The map

$$f l^2(G) \ni \alpha \mapsto \Phi \alpha \in L^2(\Omega, dP)$$

is linear (because $\Phi \alpha$ is a finite linear combination), and

$$\|\Phi \alpha\|_{L^2(\Omega, dP)}^2 = \int_{\Omega} \Phi \alpha(\omega)^2 dP(\omega) = \langle \alpha, D\alpha \rangle \leq \|D\| \cdot \|\alpha\|_{l^2(G)}^2$$

This implies that the map $\alpha \mapsto \Phi \alpha$ is uniformly continuous. Then, by Extension by Uniform continuity Theorem (MATH-354, Analysis 3), it extends uniquely to a bounded linear map

$$l^2(G) \to L^2(\Omega, dP)$$

and if $\alpha_n \in f l^2(G)$ is such that $\alpha_n \to \alpha$, then $\Phi \alpha_n \to \Phi \alpha$ in $L^2(\Omega, dP)$.

Claim 2.8. For all $\alpha \in l^2(G)$, $\Phi \alpha$ is a Gaussian random variable with variance $\langle \alpha, D\alpha \rangle$.

Proof. We know that $\forall \alpha_n \in f l^2(G)$,

$$\int_{\Omega} e^{it\Phi \alpha_n} dP = e^{-\frac{1}{2}\langle \alpha_n, D\alpha \rangle}$$

Furthermore, $\Phi \alpha_n \to \Phi \alpha$ in $L^2(\Omega, dP)$ implies that there exists a subsequence $\alpha_{nk} \to \alpha$, such that $\Phi \alpha_{nk} \to \Phi \alpha$ $P$-a.e. $\omega$.

Then, as $\Phi \alpha_{nk}$ is real, $|e^{it\Phi \alpha_{nk}}| = 1$, and thus, by Dominated Convergence Theorem,

$$\lim_{k \to \infty} \int_{\Omega} e^{it\Phi \alpha_{nk}} dP = \int_{\Omega} e^{it\Phi \alpha} dP$$

And obviously,

$$\lim_{k \to \infty} e^{-\frac{1}{2}\langle \alpha_{nk}, D\alpha_{nk} \rangle} = e^{-\frac{1}{2}\langle \alpha, D\alpha \rangle}$$

So that we have

$$\int_{\Omega} e^{it\Phi \alpha} dP = \lim_{k \to \infty} \int_{\Omega} e^{it\Phi \alpha_{nk}} dP = \lim_{k \to \infty} e^{-\frac{1}{2}\langle \alpha_{nk}, D\alpha_{nk} \rangle} = e^{-\frac{1}{2}\langle \alpha, D\alpha \rangle}$$

Note also that, by the same argument, $\int_{\Omega} \Phi \alpha dP = 0$.
Exercise 1. Show that
\[ \Phi_\alpha = \sum_{n \in G} \langle \delta_n, \alpha \rangle \Phi_{\delta_n} \]
where the sum on the right is converging in \( L^2 \)-sense.

Solution. If \( \alpha \in l^2(G) \), then the sum is finite, and the result holds trivially.
Now let \( \alpha \in l^2(G) \) be fixed and let \( \{g_1, g_2, \ldots\} \) be a numbering of elements of \( G \). Define
\[ \alpha_k(g_i) = \begin{cases} \alpha(g_i) & \text{if } i \leq k \\ 0 & \text{else} \end{cases} \]
Then clearly, \( \alpha_k \in f^2(G) \) and \( \alpha_k \to \alpha \). Hence, \( \Phi_\alpha \to \Phi_\alpha \) in \( L^2(\Omega, dP) \) and thus,
\[ \sum_{n \in G} \langle \delta_n, \alpha_k \rangle \Phi_{\delta_n} \to \Phi_\alpha \] in \( L^2(\Omega, dP) \)
Finally, notice that, the \( k \)th partial sum of \( \sum_{n \in G} \langle \delta_n, \alpha \rangle \Phi_{\delta_n} \) is
\[ \sum_{g_1, \ldots, g_k} \langle \delta_n, \alpha \rangle \Phi_{\delta_n} = \sum_{n \in G} \langle \delta_n, \alpha_k \rangle \Phi_{\delta_n} \]
Hence, the sequence of partial sums converges in \( L^2 \)-sense as required. \( \blacksquare \)

Proposition 2.9. If \( \{\alpha_1, \ldots, \alpha_n\} \) are linearly independent in \( l^2(G) \), then \( \{\Phi_{\alpha_1}, \ldots, \Phi_{\alpha_n}\} \) are jointly Gaussian with variance matrix
\[ [D_{ij}] = [\langle \alpha_i, D\alpha_j \rangle]_{1 \leq i, j \leq n} \]
Proof. The matrix \([\langle \alpha_i, D\alpha_j \rangle]_{1 \leq i, j \leq n}\) is strictly positive definite since
\[ \sum_{i,j=1}^n \gamma_i \gamma_j \langle \alpha_i, D\alpha_j \rangle = \left( \sum_{i=1}^n \gamma_i \alpha_i, \sum_{j=1}^n D\gamma_j \alpha_j \right) = \]
\[ \left( \sum_{i=1}^n \gamma_i \alpha_i, D \left( \sum_{j=1}^n \gamma_j \alpha_j \right) \right) \geq \|D\| \left\| \sum_{i=1}^n \gamma_i \alpha_i \right\| > 0 \]
unless \( \sum_i \gamma_i \alpha_i = 0 \) (because \( \|D\| \geq m > 0 \) by assumption). But, by the linear independence of \( \alpha_i \)'s, this is possible only if \( \gamma_i \equiv 0, \forall i \). Hence
\[ \int_{\Omega} e^{i\sum_n t_n \Phi_{\alpha_n}} dP(\omega) = \int_{\Omega} e^{i\Phi_{\sum_n t_n \alpha_n}} dP(\omega) = \]
\[ = e^{-\frac{1}{2} \langle \sum_n t_n \alpha_n, D \sum_n t_n \alpha_n \rangle} = e^{-\frac{1}{2} \sum_{i,j} t_it_j \langle \alpha_n, D\alpha_j \rangle} \]
\( \blacksquare \)

Fact 2.10. For any \( z \in \mathbb{C} \),
\[ e^{z\Phi_\alpha} \in L^2(\Omega, dP), \]
and
\[ \int_{\Omega} e^{z\Phi_\alpha} dP = \int_{\mathbb{R}} e^{zx} e^{-\frac{1}{2} \langle \alpha, D\alpha \rangle} dx = e^{-\frac{z^2}{2} \langle \alpha, D\alpha \rangle} \]
3 Trace Class

Background reading Barry and Simon, *Functional Analysis Volume I*, Chapter VI (last section).

3.1 Trace Class Operators

Definition 3.1. Let $\mathcal{H}$ be a real Hilbert space, and let

$$A : \mathcal{H} \to \mathcal{H}$$

be a bounded self-adjoint operator (as $\mathcal{H}$ is real, self-adjoint simply means symmetric). We say that $A$ is a *trace class operator* if

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n, \cdot \rangle f_n$$

where $\lambda_n \in \mathbb{R}$, $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, and $\{f_n\}$ form an orthonormal basis of $\mathcal{H}$

Note that from this definition it follows that $f_n$ is the eigenvector associated to eigenvalue $\lambda_n$, as

$$Af_n = \lambda_n f_n$$

Definition 3.2. Also define the trace of a trace class operator $A$:

$$\text{Tr}(A) = \sum_{n=1}^{\infty} \lambda_n$$

and the trace norm

$$\|A\|_1 = \sum_{n=1}^{\infty} |\lambda_n|$$

Note that we will continue to use the standard operator norm, and we will denote it $\|\cdot\|$ as usual, while $\|\cdot\|_1$ will now denote the trace norm.

Also, we say that $A$ is *Hilbert-Schmidt* if $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$.

$\rightarrow$ For a more detailed introduction to traces and trace norms, see *Trace ideals and its applications* by Barry Simon.

If $A$ is a trace class, the standard operator norm

$$\|A\| = \sup_n |\lambda_n|$$

Note that the supremum is actually achieved, as

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty \Rightarrow |\lambda_n| \to 0$$
Moreover, \( A \geq \inf_n \lambda_n \), meaning that \( \forall \psi \in \mathcal{H} \)

\[
\langle \psi, A\psi \rangle \geq \left( \inf_n \lambda_n \right) \langle \psi, \psi \rangle
\]

Now, let \( A \) be trace class such that \( A > -1 \), i.e. \( \inf_n \lambda_n > -1 \) or equivalently, \( \forall n, \lambda_n > -1 \). Then,

\[
\det(I + A) = \lim_{n \to \infty} \prod_{k=1}^{n} (1 + \lambda_k)
\]

**Claim 3.3.** The limit on the right hand side exists.

**Proof.** Write

\[
\prod_{k=1}^{n} (1 + \lambda_k) = e^{\sum_{k=1}^{n} \ln(1 + \lambda_k)}
\]

Now, as

\[
\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1 \quad \text{(by L'Hospital Rule)}
\]

For small enough \( |x| \), we have that

\[
\frac{1}{2} |x| \leq |\ln(1 + x)| \leq 2 |x|
\]

Now, as \( |\lambda_n| \to 0 \), we have that for large enough \( n \),

\[
\frac{1}{2} |\lambda_n| \leq |\ln(1 + \lambda_n)| \leq 2 |\lambda_n|
\]

and thus,

\[
\sum_{n=1}^{\infty} |\lambda_n| < \infty \Rightarrow \sum_{k=1}^{\infty} \ln(1 + \lambda_k) < \infty
\]

Hence, we have convergence, and we can write

\[
\det(I + A) = \prod_{k=1}^{\infty} (1 + \lambda_k)
\]

### 3.2 Variance Induced Inner Products

As before, consider \( l^2(G) \), and let \( D \) be self-adjoint and positive definite. Define

\[
\langle \cdot, \cdot \rangle_D = \langle \cdot, D(\cdot) \rangle = \left\langle D^{\frac{1}{2}}(\cdot), D^{\frac{1}{2}}(\cdot) \right\rangle
\]

Then, \( \langle \cdot, \cdot \rangle_D \) is an inner product on \( l^2(G) \), and \( (l^2(G), \langle \cdot, \cdot \rangle_D) \) is a Hilbert space. We will denote it by \( l^2_D(G) \).
Note that, if $D$ is bounded the resulting norm is equivalent to the Euclidean norm. Also, $\|I(G)\|_2 = \|I(G)\|_2$. Now, let $A$ be a trace class operator on $l^2_D(G)$, and write

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n, \cdot \rangle_D f_n$$

where $\{f_n\}$ is an orthonormal basis of $l^2_D(G)$, and $\sum_{n=1}^{\infty} |\lambda_n| < \infty$. Consider

$$D^{1/2}A D^{-1/2} = \sum_{n=1}^{\infty} \lambda_n \langle f_n, D^{-1/2} (\cdot) \rangle_D D^{1/2} f_n =$$

$$= \sum_{n=1}^{\infty} \lambda_n \langle f_n, D^{1/2} (\cdot) \rangle D^{1/2} f_n = \sum_{n=1}^{\infty} \lambda_n \langle D^{1/2} f_n, \cdot \rangle D^{1/2} f_n$$

Now, the set $\{D^{1/2} f_n\}$ is an orthonormal basis of $l^2(G)$, as

$$\langle D^{1/2} f_n, D^{1/2} f_m \rangle = \langle f_n, f_m \rangle_D = \delta_{nm}$$

And thus, $D^{1/2}A D^{-1/2}$ is trace class on $l^2(G)$ with eigenvalues $\lambda_n$ and eigenvectors $D^{1/2} f_n$. Conversely, if $A$ is trace class on $l^2(G)$, $D^{-1/2}AD^{1/2}$ is trace class on $l^2_D(G)$ with eigenvectors $D^{-1/2} f_n$.

### 3.3 Example

Consider $l^2_D(G)$ and let $A$ be a self-adjoint trace class operator such that $A > -1$. Write

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n, \cdot \rangle_D f_n$$

where $\{f_n\}$ are orthonormal w.r.t. $\langle \cdot, \cdot \rangle_D$. Now, set

$$F(\omega) = \sum_{n=1}^{\infty} \lambda_n \Phi_{f_n}^2(\omega)$$

Claim 3.4.

$$F \in L^1(\Omega, dP)$$

Proof. By Monotone convergence Theorem, we have

$$\int_{\Omega} |F(\omega)| dP(\omega) \leq \int_{\Omega} \sum_{n=1}^{\infty} |\lambda_n| \Phi_{f_n}^2(\omega) dP(\omega) = \sum_{n=1}^{\infty} |\lambda_n| \int_{\Omega} \Phi_{f_n}^2(\omega) dP(\omega) = \cdots$$

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\[ = \sum_{n=1}^{\infty} |\lambda_n| \langle f_n, Df_n \rangle = \sum_{n=1}^{\infty} |\lambda_n| < \infty \]

**Example 3.5.** Show that

\[ \int_{\Omega} e^{-\frac{1}{2} F(\omega)} dP(\omega) = \left[ \frac{1}{\det(I + A)} \right]^{1/2} \]

**Solution.** We wish to compute \( \int_{\Omega} e^{-\frac{1}{2} F(\omega)} dP(\omega) \). To this end, let us first compute

\[ \int_{\Omega} e^{-\frac{1}{2} \sum_{k=1}^{n} \lambda_k \Phi_k^2(\omega)} dP(\omega) \]

By Proposition 2.9, \( \{\Phi_k^j\}_{k=1}^{n} \) are jointly Gaussian with variance

\[ [\langle f_i, Df_j \rangle]_{1 \leq i, j \leq n} = I_{n \times n} \]

Hence,

\[ \int_{\Omega} e^{-\frac{1}{2} \sum_{k=1}^{n} \lambda_k \Phi_k^2(\omega)} dP(\omega) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{k=1}^{n} \lambda_k \Phi_k^2(\omega)} e^{-\frac{1}{2} \sum_{k=1}^{n} \lambda_k^2 \Phi_k^2(\omega)} dx \]

\[ = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{k=1}^{n} (\lambda_k + 1) x_k^2} dx = \prod_{k=1}^{n} (1 + \lambda_k)^{-1/2} \]

Taking \( n \to \infty \) in the above, we get that

\[ \lim_{n \to \infty} \int_{\Omega} e^{-\frac{1}{2} \sum_{k=1}^{n} \lambda_k \Phi_k^2(\omega)} dP(\omega) = \prod_{k=1}^{\infty} (1 + \lambda_k)^{-1/2} = [\det(I + A)]^{-1/2} \]

Now, since \( \lambda_n \to 0 \), there exists \( n_0 \) such that \( \forall n \geq n_0, |\lambda_n| < \frac{1}{2} \). So, set

\[ \tilde{F}(\omega) = \sum_{k=1}^{n_0-1} \lambda_k \Phi_k^2(\omega) - \sum_{k=n_0}^{\infty} |\lambda_k| \Phi_k^2(\omega) \]

As before, \( \tilde{F} \in L^1(\Omega, dP) \), and since \( \tilde{F}(\omega) \leq F(\omega) \), we have that

\[ \int_{\Omega} e^{-\frac{1}{2} \tilde{F}(\omega)} dP(\omega) \leq \int_{\Omega} e^{-\frac{1}{2} F(\omega)} dP(\omega) \]

As the second sum in \( \tilde{F}(\omega) \) is monotone, we can use the Monotone Convergence Theorem and we have

\[ \int_{\Omega} e^{-\frac{1}{2} \tilde{F}} dP = \int_{\Omega} \exp \left( -\frac{1}{2} \sum_{k=1}^{n_0-1} \lambda_k \Phi_k^2(\omega) \right) \exp \left( \sum_{k=n_0}^{\infty} |\lambda_k| \Phi_k^2(\omega) \right) dP(\omega) = \]
\[
\text{MCT} \quad \lim_{n \to \infty} \int_\Omega \exp \left( -\frac{1}{2} \sum_{k=1}^{n_0-1} \lambda_k \Phi^2_{f_k}(\omega) \right) \exp \left( \sum_{k=n_0}^{n} |\lambda_k| \Phi^2_{f_k}(\omega) \right) dP(\omega) = \\
= \lim_{n \to \infty} \left[ \prod_{k=1}^{n_0-1} (1 + \lambda_k) \right]^{-1/2} \left[ \prod_{k=n_0}^{n} (1 - |\lambda_k|) \right]^{-1/2}
\]

Since, \(|\lambda_k| < \frac{1}{2}\), \(\forall k \geq n_0\), the second product is non zero, and as \(\sum_{n=1}^{\infty} |\lambda_n| < \infty\) and \(|\lambda_n| < 1/2\), the limit exists by Claim 3.3. And thus,

\[
\int_\Omega e^{-\frac{1}{2} \tilde{F}(\omega)} dP(\omega) < \infty \implies \int_\Omega e^{-\frac{1}{2} F(\omega)} dP(\omega) < \infty
\]

making \(e^{-\frac{1}{2} F(\omega)}\) integrable.

Now, as \(F(\omega) = \lim_{N \to \infty} \sum_{n=1}^{N} \lambda_n \Phi^2_{f_n}(\omega)\), with the sum converging in \(L^1\)-sense, there exists a subsequence \(N_k \to \infty\) such that

\[
F(\omega) = \lim_{k \to \infty} \sum_{n=1}^{N_k} \lambda_n \Phi^2_{f_n}(\omega) \quad \text{P-a.e.} \ \omega
\]

But

\[
\forall k \quad e^{-\frac{1}{2} \sum_{n=1}^{N_k} \lambda_n \Phi^2_{f_n}(\omega)} \leq e^{-\frac{1}{2} \tilde{F}} \in L^1
\]

So, by Dominated Convergence Theorem,

\[
\int_\Omega e^{-\frac{1}{2} F(\omega)} dP(\omega) = \lim_{k \to \infty} \int_\Omega e^{-\frac{1}{2} \sum_{n=1}^{N_k} \lambda_n \Phi^2_{f_n}(\omega)} dP(\omega) = \left[ \frac{1}{\det(I + A)} \right]^{1/2}
\]

**Case D = I:** As seen before, in this case, \(A\) is trace class on \(l^2(G)\), and so,

\[
A = \sum_{n=1}^{\infty} \lambda_n \langle f_n, \cdot \rangle f_n
\]

Then, for \(\omega \in l^2(G)\), we have that

\[
\langle \omega, A \omega \rangle = \sum_{n=1}^{\infty} \lambda_n |\langle f_n, \omega \rangle|^2 = \sum_{n=1}^{\infty} \lambda_n \Phi^2_{f_n}(\omega)
\]

This leads us to define:

\[
\langle \omega, A \omega \rangle \overset{\text{def}}{=} \sum_{n=1}^{\infty} \lambda_n \Phi^2_{f_n}(\omega) = F(\omega)
\]

Hence, we can rewrite the result of Example 3.5 as

\[
\int_\Omega e^{-\frac{1}{2} \langle \omega, A \omega \rangle} dP(\omega) = \left[ \frac{1}{\det(I + A)} \right]^{1/2}
\]
3.4 Variance Induced Measures

Let $A$ be trace class on $l^2_D(G)$ and suppose $A > -1$. As usual, write,

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n, \cdot \rangle_D f_n, \quad \sum_{n=1}^{\infty} |\lambda_n| < \infty$$

where $\{f_n\}$ is an orthonormal basis of $l^2_D(G)$.

We were prompted to define

$$\langle \omega, A\omega \rangle_D \overset{\text{def}}{=} \sum_{n=1}^{\infty} \lambda_n \Phi^2_{f_n}(\omega)$$

And we showed that

1. $\langle \omega, A\omega \rangle_D \in L^1(\Omega, dP)$ (Claim 3.4)
2. $\int_{\Omega} \exp \left( -\frac{1}{2} \langle \omega, A\omega \rangle_D \right) dP(\omega) = [\det(I + A)]^{-1/2}$ (Example 3.5)

We can now introduce the probability measure

$$dQ = [\det(I + A)]^{1/2} e^{-\frac{1}{2} \langle \omega, A\omega \rangle_D} dP$$

and discuss its effect on the GRF.

**Claim 3.6.** $\{\Phi_{f_{n_1}}, \ldots, \Phi_{f_{n_K}}\}$ are jointly Gaussian w.r.t. $dQ$, with variance matrix

$$\begin{bmatrix} \delta_{ij} (1 + \lambda_{n_i})^{-1} \end{bmatrix}_{1 \leq i, j \leq K}$$

**Proof.** Consider

$$\mathbb{E}_{dQ} \left[ e^{i \left( t_1 \Phi_{f_{n_1}} + \ldots + t_K \Phi_{f_{n_K}} \right)} \right] = \int_{\Omega} e^{i \left( t_1 \Phi_{f_{n_1}} + \ldots + t_K \Phi_{f_{n_K}} \right)} dQ =$$

$$= [\det(I + A)]^{1/2} \int_{\Omega} \exp \left( i \sum_{k=1}^{K} t_k \Phi_{f_{n_k}} \right) \exp \left( -\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \Phi^2_{f_k}(\omega) \right) dP =$$

and $\forall m \in \mathbb{N},$

$$\left| \exp \left( i \sum_{k=1}^{K} t_j \Phi_{f_{n_k}} \right) \exp \left( -\frac{1}{2} \sum_{k=1}^{m} \lambda_k \Phi^2_{f_k}(\omega) \right) \right| = \left| \exp \left( -\frac{1}{2} \sum_{k=1}^{m} \lambda_k \Phi^2_{f_k}(\omega) \right) \right|$$

Recall that in Exercise 3.5, we have shown that there exists a subsequence $m_j \to \infty$ such that

$$\sum_{k=1}^{\infty} \lambda_k \Phi^2_{f_k}(\omega) = \lim_{j \to \infty} \sum_{k=1}^{m_j} \lambda_k \Phi^2_{f_k}(\omega) \quad P\text{-a.e. } \omega$$
and \( \forall j, \exp \left( -\frac{1}{2} \sum_{k=1}^{m_j} \lambda_k \Phi^2_{f_k}(\omega) \right) \leq \exp \left( -\frac{1}{2} \tilde{F}(\omega) \right) \in L^1 \)

Then obviously,
\[
\exp \left( i \sum_{k=1}^{K} t_k \Phi_{f_{nk}} \right) \exp \left( -\frac{1}{2} \sum_{k=1}^{\infty} \lambda_k \Phi^2_{f_k}(\omega) \right) = \lim_{j \to \infty} \left[ \exp \left( i \sum_{k=1}^{K} t_k \Phi_{f_{nk}} \right) \exp \left( -\frac{1}{2} \sum_{k=1}^{m_j} \lambda_k \Phi^2_{f_k}(\omega) \right) \right]
\]

Thus, by Dominated Convergence Theorem, we get that
\[
\int \Omega \exp \left( i \left( t_1 \Phi_{f_n} + \ldots + t_K \Phi_{f_{nk}} \right) \right) dQ = \lim_{j \to \infty} \int \Omega \exp \left( i \sum_{k=1}^{K} t_k \Phi_{f_{nk}} - \frac{1}{2} \sum_{k=1}^{m_j} \lambda_k \Phi^2_{f_k}(\omega) \right) dP = \lim_{j \to \infty} \left[ \prod_{x_{nk} \in \Phi_{f_n}} (1 + \lambda_k)^{1/2} \right]^{1/2} \int_{\mathbb{R}^{m_j}} \exp \left( i \sum_{k=1}^{K} t_k x_{nk} - \frac{1}{2} \sum_{k=1}^{m_j} (1 + \lambda_k)x^2_k \right) dx
\]

Now, we integrate out all the coordinates except \( \{x_{n1}, \ldots, x_{nk}\} \), and for all \( m_j \geq \max\{n_k : 1 \leq k \leq K\} \) we get
\[
\prod_{x_{nk} \in \Phi_{f_n}} (1 + \lambda_k)^{1/2} \int_{\mathbb{R}^{m_j}} \exp \left( i \sum_{k=1}^{K} t_k x_{nk} - \frac{1}{2} \sum_{k=1}^{m_j} (1 + \lambda_k)x^2_k \right) dx = \prod_{x_{nk} \in \Phi_{f_n}} (1 + \lambda_k)^{1/2} \int_{\mathbb{R}^K} \exp \left( \sum_{k=1}^{K} \left[ i t_k x_{nk} - \frac{1}{2} \lambda_n x^2_{nk} \right] \right) dx_{n1} \cdots dx_{nk}
\]

And thus, by uniqueness of characteristic function, \( \{\Phi_{f_{n1}}, \ldots, \Phi_{f_{nk}}\} \) are jointly Gaussian w.r.t. \( dQ \), with variance matrix
\[
[d_{ij}(1 + \lambda_n)]_{1 \leq i,j \leq k}
\]

Now consider the random variables \( \Phi_{\delta_n}(\omega) \) over \( dQ \).

**Proposition 3.7.** \( \{\Phi_{\delta_n}\}_{n=1}^{\infty} \) form a GRF over \((\Omega, \mathcal{F}, dQ)\), with variance \( D(I + A)^{-1} \).
Proof. First of all, notice that \( \{ \Phi_{\delta_n} \}_{n=1}^{\infty} \) form a GRF over \((\Omega, \mathcal{F}, dP)\) with variance \(D\), essentially by construction. It is obvious, when we apply Proposition 2.9 to any finite subcollection of \(\delta_n\)'s.

Now, over \(dQ\), consider

\[
t_1 \Phi_{\delta_{n_1}} + \ldots + t_K \Phi_{\delta_{n_K}}
\]

Expanding in \(f_n\), we have that

\[
\sum_{k=1}^{K} t_k \delta_{n_k} = \sum_{j=1}^{\infty} \left( f_j, \sum_{k=1}^{K} t_k \delta_{n_k} \right)_D f_j = \sum_{j=1}^{\infty} \langle f_j, \delta_t \rangle_D f_j
\]

where \(\delta_t = \sum_{k=1}^{K} t_k \delta_{n_k}\). And so,

\[
\sum_{k=1}^{K} t_k \Phi_{\delta_{n_k}} = \Phi_{\sum_{k=1}^{K} t_k \delta_{n_k}} = \Phi_{\delta_t} = \sum_{j=1}^{\infty} \langle f_j, \delta_t \rangle_D \Phi f_j
\]

with the sum on the right hand side converging in \(L^2(\Omega, dQ)\). Therefore, there exists a subsequence \(J_l \to \infty\) such that

\[
\Phi_{\delta_t} = \lim_{l \to \infty} \sum_{j=1}^{J_l} \langle f_j, \delta_t \rangle_D \Phi f_j \quad Q\text{-a.e.}
\]

Now, consider

\[
\mathbb{E}_{dQ} \left[ e^{i(t_1 \Phi_{\delta_{n_1}} + \ldots + t_K \Phi_{\delta_{n_K}})} \right] = \int_{\Omega} e^{i\left(t_1 \Phi_{\delta_{n_1}} + \ldots + t_K \Phi_{\delta_{n_K}}\right)} dQ = \int_{\Omega} e^{i\Phi_{\delta_t}} dQ
\]

By Dominated convergence Theorem, (bounded by 1), we get

\[
\int_{\Omega} e^{i\Phi_{\delta_t}} dQ = \lim_{l \to \infty} \int_{\Omega} e^{i\sum_{j=1}^{J_l} \langle f_j, \delta_t \rangle_D \Phi f_j} dQ
\]

But by previous claim, \(\Phi f_j\)'s have variance \([\delta_{ij}(1 + \lambda_{n_i})^{-1}]\) for \(1 \leq i, j \leq k\) w.r.t. \(dQ\),
so

\[
\lim_{l \to \infty} \int_{\Omega} e^{i \sum_{j=1}^{J_l} (f_j, \delta_t)} d\Phi f_j dQ = \\
= \lim_{l \to \infty} \exp \left( -\frac{1}{2} \sum_{j=1}^{J_l} \left[ (f_j, \delta_t)_D \right]^2 (1 + \lambda_j)^{-1} \right) = \\
= \lim_{l \to \infty} \exp \left( -\frac{1}{2} \sum_{j=1}^{J_l} \left[ \langle (I + A)^{-1/2} f_j, \delta_t \rangle_D \right]^2 \right) = \\
= \lim_{l \to \infty} \exp \left( -\frac{1}{2} \sum_{j=1}^{J_l} \left[ \langle f_j, (I + A)^{-1/2} \delta_t \rangle_D \right]^2 \right) = \\
= \exp \left( -\frac{1}{2} \sum_{j=1}^{\infty} \left[ \langle f_j, (I + A)^{-1/2} \delta_t \rangle_D \right]^2 \right) = \\
= \exp \left( -\frac{1}{2} \left\| (I + A)^{-1/2} \delta_t \right\|_D^2 \right) = \\
= \exp \left( -\frac{1}{2} \left\langle (I + A)^{-1/2} \delta_t, (I + A)^{-1/2} \delta_t \right\rangle_D \right) = \\
= \exp \left( -\frac{1}{2} \left\langle \delta_t, (I + A)^{-1} \delta_t \right\rangle_D \right) = \\
= \exp \left( -\frac{1}{2} \left\langle \delta_t, D(I + A)^{-1} \delta_t \right\rangle \right) = \\
= \exp \left( -\frac{1}{2} \sum_{i,j} t_i t_j \left\langle \delta_{n_i}, D(I + A)^{-1} \delta_{n_j} \right\rangle \right) = \\
\]

And thus, \( \{ \Phi_{\delta_n} \}_{n=1}^{\infty} \) form a GRF w.r.t. \( d\Phi \), with variance \( D(I + A)^{-1} \). \( \square \)

### 3.5 Mutual Absolute Continuity

Let \( D_1, D_2 \) be two variances under usual assumptions, and let \( P_1, P_2 \) be the corresponding Gaussian measures on \( (\Omega, \mathcal{F}) \). That is \( \{ \Phi_{\delta_n} \} \) are jointly Gaussian w.r.t. \( P_i \) with variance \( D_i, i = 1, 2 \).

**Question:** When are \( P_1 \) and \( P_2 \) mutually absolutely continuous?

**Theorem 3.8** (Shale Theorem). \( P_1 \) and \( P_2 \) are mutually absolutely continuous if and only if the operator \( (D_1^{-1} D_2 - I) \) is trace class on \( l_2^{D_1} \).

**Remark 3.9.** This is a fundamental result

**Remark 3.10.**

\[
\langle \omega, (D_1^{-1} D_2 - I) \omega \rangle_{D_1} = \langle \omega, D_1 (D_1^{-1} D_2 - I) \omega \rangle = \langle \omega, (D_2 - D_1) \omega \rangle_{\text{sym}} = \\
= \langle (D_2 - D_1) \omega, \omega \rangle = \langle D_1 (D_1^{-1} D_2 - I) \omega, \omega \rangle = \langle (D_1^{-1} D_2 - I) \omega, \omega \rangle_{D_1} \\
\]

That is \( (D_1^{-1} D_2 - I) \) is self-adjoint on \( l_2^{D_1} \), and the statement of the theorem makes sense.

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Remark 3.11. Trace class assumption means:

\[
D_1^{-1}D_2 - I = \sum_{n \in G} \lambda_n \langle f_n, \cdot \rangle_{D_1} f_n
\]

\[
D_1^{-1/2}D_2 - D_1^{1/2} = \sum_{n \in G} \lambda_n \langle f_n, \cdot \rangle_{D_1} D_1^{1/2} f_n
\]

\[
\left[D_1^{-1/2}D_2 - D_1^{1/2}\right] D_1^{-1/2} = \sum_{n \in G} \lambda_n \left\langle D_1^{1/2} f_n, \cdot \right\rangle D_1^{1/2} f_n
\]

As \(\{D_1^{1/2}f_n\}\) form an orthonormal basis of \(l^2(G)\), \((D_1^{-1/2}D_2D_1^{-1/2} - I)\) is trace class on \(l^2(G)\). This implies that \(D_1^{-1/2}(D_2 - D_1)D_1^{-1/2}\) is trace class. By properties of trace class operators, this yields that \((D_1 - D_2)\) is trace class.

So, the natural formulation of Shale Theorem is

“\(P_1\) and \(P_2\) are mutually absolutely continuous if and only if \((D_1 - D_2)\) is trace class on \(l^2(G)\).”

Exercise 2. Express the “\((D_1 - D_2)\) is trace class” condition in terms of matrix elements. In particular, show that

\[
\sum_{n, m \in G} \left| [D_1]_{nm} - [D_2]_{nm} \right|^2 < \infty
\]

is necessary.

Proof of Shale Theorem. We will prove only one direction, namely

If \((D_1^{-1}D_2 - I)\) is trace class on \(l^2(D_1)\), then \(P_1\) and \(P_2\) are mutually absolutely continuous.

So, write

\[
(D_1^{-1}D_2 - I) = \sum_{n=1}^{\infty} \lambda_n \langle f_n, \cdot \rangle_{D_1} f_n
\]

where \(\{f_n\}\) form an orthonormal basis of \(l^2(D_1)\), and \(\sum_{n=1}^{\infty} |\lambda_n| < \infty\).

Consider now the random variables \(\{\Phi_{f_n}\}\) on \(L^2(\Omega, dP_1)\).

\[
\int_{\Omega} \Phi_{f_n} \Phi_{f_m} dP_1 = \langle f_n, D_1 f_m \rangle = \langle f_n, f_m \rangle_{D_1} = \delta_{nm}
\]

Then, as \(\{\Phi_{f_n}\}\) are jointly Gaussian w.r.t. \(dP_1\), we have that

\[
\mathbb{E}_{dP_1} \left[ e^{i \sum_{n=1}^{N} t_n \Phi_{f_n}} \right] = \int_{\Omega} e^{i \sum_{n=1}^{N} t_n \Phi_{f_n}} dP_1 =
\]
\[
\begin{align*}
&= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \left[ e^{i \sum_{n=1}^N t_n x_n} e^{-\frac{1}{2} \sum_{n=1}^N x_n^2} \right] dx_1 \cdots dx_N \\
\text{Now, } (D_1^{-1}D_2 - I) \text{ is trace class, so}
\end{align*}
\]
\[
\lambda_n \delta_{nm} = \langle f_n, (D_1^{-1}D_2 - I) f_m \rangle_{D_1} = \langle f_n, D_1(D_1^{-1}D_2 - I) f_m \rangle =
\]
\[
= \langle f_n, D_2 f_m \rangle - \langle f_n, D_1 f_m \rangle = \langle f_n, D_2 f_m \rangle - \delta_{nm}
\]
\[
\implies \langle f_n, D_2 f_m \rangle = (1 + \lambda_n) \delta_{nm}
\]

So, if we consider \( \{ \Phi_{f_n} \} \) w.r.t. \( dP_2 \), they have variance \( D = [D_{nm}] \),
\( D_{nm} = (1 + \lambda_n) \delta_{nm} \). Also, notice that \( \langle f_n, D f_n \rangle > 0 \Rightarrow \lambda_n > -1, \forall n \). We can now compute
\[
\mathbb{E}_{dP_2} \left[ e^{i \sum_{n=1}^N t_n \Phi_{f_n}} \right] = \int_{\Omega} e^{i \sum_{n=1}^N t_n \Phi_{f_n}} dP_2 =
\]
\[
= \frac{1}{(2\pi)^{N/2}} \left[ \frac{1}{\prod_{n=1}^N (1 + \lambda_n)} \right]^{1/2} \int_{\mathbb{R}^N} \left[ e^{i \sum_{n=1}^N t_n x_n} e^{-\frac{1}{2} \sum_{n=1}^N x_n^2} \right] dx_1 \cdots dx_N
\]

Consider
\[
\left[ \frac{1}{\prod_{k=1}^K (1 + \lambda_{n_k})} \right]^{1/2} \int_{\Omega} e^{i \sum_{k=1}^K t_k \Phi_{f_{n_k}}} e^{-\frac{1}{2} \sum_{k=1}^K x_{n_k}^2} \Phi_{f_{n_k}}^2 \frac{1}{2} \sum_{k=1}^K \Phi_{f_{n_k}}^2 dP_1 =
\]
\[
= \frac{\mathcal{P}_K}{(2\pi)^{K/2}} \int_{\mathbb{R}^K} e^{i \sum_{k=1}^K t_k x_{n_k}} e^{-\frac{1}{2} \sum_{k=1}^K x_{n_k}^2} e^{\frac{1}{2} \sum_{k=1}^K x_{n_k}^2} d\mathbf{x} =
\]
\[
= \frac{\mathcal{P}_K}{(2\pi)^{K/2}} \int_{\mathbb{R}^K} e^{i \sum_{k=1}^K t_k x_{n_k}} e^{-\frac{1}{2} \sum_{k=1}^K x_{n_k}^2} d\mathbf{x} = \int_{\Omega} e^{i \sum_{k=1}^K t_k \Phi_{f_{n_k}}} dP_2
\]

Thus,
\[
\mathbb{E}_{dP_2} \left[ e^{i \sum_{k=1}^K t_k \Phi_{f_{n_k}}} \right] = \int_{\Omega} e^{i \sum_{k=1}^K t_k \Phi_{f_{n_k}}} dP_2 =
\]
\[
= \lim_{N \to \infty} \left[ \frac{1}{\prod_{n=1}^N (1 + \lambda_n)} \right]^{1/2} \int_{\Omega} e^{i \sum_{k=1}^K t_k \Phi_{f_{n_k}}} e^{\frac{1}{2} \sum_{n=1}^N \lambda_{n_k} \Phi_{f_{n_k}}^2} dP_1
\]

since for \( N > \max \{ n_k : 1 \leq k \leq K \} \) we simply integrate out all the extra coordinates. Now, by Claim 3.3,
\[
\lim_{N \to \infty} \frac{1}{\prod_{n=1}^N (1 + \lambda_n)}^{1/2} \quad \text{exists since} \quad \left\{ \sum_{n=1}^\infty |\lambda_n| < \infty \right\}, \lambda_n > -1
\]

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Hence, consider the function 

$$F(\omega) = e^{\frac{1}{2} \sum_{n=1}^{\infty} \frac{\lambda_n}{1 + \lambda_n} \Phi^2 f_n}$$

We will show that $F \in L^1(\Omega, dP_1)$, so that the limit and the integral can be interchanged by Dominated Convergence Theorem. So, as 

$$\int_{\Omega} \left| \sum_{n=1}^{\infty} \frac{\lambda_n}{1 + \lambda_n} \Phi^2 f_n \right| dP_1 \leq \sum_{n=1}^{\infty} \frac{|\lambda_n|}{1 + \lambda_n} \int_{\Omega} \Phi^2 f_n dP_1 = \sum_{n=1}^{\infty} \frac{|\lambda_n|}{1 + \lambda_n} < \infty$$

$F$ is well defined. Hence, set 

$$F_N = e^{\frac{1}{2} \sum_{n=N}^{\infty} \frac{\lambda_n}{1 + \lambda_n} \Phi^2 f_n}$$

As $\sum_{n=1}^{\infty} |\lambda_n| < \infty$, there is $N_0 > 0$ such that $\forall n \geq N_0$, $|\lambda_n| < \frac{1}{3}$. So for $N > N_0$, consider 

$$\tilde{F}_N = \exp \left( \frac{1}{2} \sum_{n=1}^{N_0} \frac{\lambda_n}{1 + \lambda_n} \Phi^2 f_n + \frac{1}{2} \sum_{n=N_0+1}^{N} \frac{|\lambda_n|}{1 + \lambda_n} \Phi^2 f_n \right)$$

Then, $\tilde{F}_N$ is monotonically increasing, and 

$$\int_{\Omega} \tilde{F}_N dP_1 = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{\frac{1}{2} \sum_{n=1}^{N_0} \frac{\lambda_n x_n^2}{1 + \lambda_n} + \frac{1}{2} \sum_{n=N_0+1}^{N} \frac{|\lambda_n| x_n^2}{1 + \lambda_n}} e^{-\frac{1}{2} \sum_{n=1}^{N} x_n^2} \, dx = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp \left( -\frac{1}{2} \sum_{n=1}^{N_0} \frac{1}{1 + \lambda_n} x_n^2 - \frac{1}{2} \sum_{n=N_0+1}^{N} \left( 1 - \frac{|\lambda_n|}{1 + \lambda_n} \right) x_n^2 \right) \, dx = \left[ \prod_{n=1}^{N_0} (1 + \lambda_n) \right]^{-1/2} \left[ \prod_{n=N_0+1}^{N} \left( 1 - \frac{|\lambda_n|}{1 + \lambda_n} \right) \right]^{-1/2}$$

And 

$$\lim_{N \to \infty} \left[ \prod_{n=N_0+1}^{N} \left( 1 - \frac{|\lambda_n|}{1 + \lambda_n} \right) \right]^{-1/2} \leq \lim_{N \to \infty} \left[ \prod_{n=N_0+1}^{N} \left( 1 - \frac{|\lambda_n|}{1 - 1/3} \right) \right]^{-1/2} = \lim_{N \to \infty} \left[ \prod_{n=N_0+1}^{N} \left( 1 - \frac{3|\lambda_n|}{2} \right) \right]^{-1/2} < \infty \quad (\text{by Claim 3.3})$$

as $\sum_{n=1}^{\infty} \frac{3|\lambda_n|}{2} < \infty$ and $|\lambda_n| < 1/3$. Thus, we have a uniform bound 

$$\int_{\Omega} \tilde{F}_N dP_1 \leq \left[ \prod_{n=1}^{N_0} (1 + \lambda_n) \right]^{-1/2} \left[ \prod_{n=N_0+1}^{N} \left( 1 - \frac{|\lambda_n|}{1 + \lambda_n} \right) \right]^{-1/2} < \infty$$
Hence, as $\tilde{F}_N$ is monotonically increasing, by Monotone Convergence Theorem, we have that:

$$\lim_{N \to \infty} \int_{\Omega} \tilde{F}_N dP_1 = \int_{\Omega} \left[ \lim_{N \to \infty} \tilde{F}_N \right] dP_1 = \int_{\Omega} \tilde{F} dP_1 < \infty$$

Now, as $F_N \leq \tilde{F}_N \leq \tilde{F} \in L^1(\Omega, dP_1)$ for all $N > N_0$, we apply Dominated Convergence Theorem to get

$$\lim_{N \to \infty} \int_{\Omega} F_N dP_1 = \int_{\Omega} F dP_1 \leq \int_{\Omega} \tilde{F} dP_1 < \infty \implies F \in L^1(\Omega, dP_1)$$

So that

$$\int_{\Omega} e^{i \sum_{k=1}^{K} t_k \Phi f_{nk}} dP_2 = \left[ \prod_{n=1}^{\infty} (1 + \lambda_n) \right]^{-1/2} \int_{\Omega} e^{i \sum_{k=1}^{K} t_k \Phi f_{nk}} F(\omega) dP_1$$

We will now show that

$$dP_2 = \frac{F(\omega)}{\left[ \prod_{n=1}^{\infty} (1 + \lambda_n) \right]^{1/2}} dP_1$$

First, consider $\delta_{nk} = \sum_{j=1}^{\infty} \langle f_j, \delta_{nk} \rangle_D f_j$, and

$$\Phi_{\delta_{nk}} = \sum_{j=1}^{\infty} \langle f_j, \delta_{nk} \rangle_D \Phi f_j$$

is converging in $L^2$-sense in both $L^2(\Omega, dP_1)$ and $L^2(\Omega, dP_2)$, because the expansion is in the same space, namely $l^2_D$. Then, by taking two successive subsequences, we get $J_m \to \infty$ such that

$$\sum_{k=1}^{K} t_k \Phi \delta_{nk} = \lim_{m \to \infty} \sum_{j=1}^{J_m} \left( f_j, \sum_{k=1}^{K} t_k \delta_{nk} \right)_D \Phi f_j$$

both $P_1$-a.e. $\omega$ and $P_2$-a.e. $\omega$, and thus, $\int_{\Omega} \exp \left( i \sum_{k=1}^{K} t_k \Phi \delta_{nk} \right) dP_2 =$

$$= \lim_{m \to \infty} \int_{\Omega} \exp \left[ i \sum_{j=1}^{J_m} \left( f_j, \sum_{k=1}^{K} t_k \delta_{nk} \right)_D \Phi f_j \right] dP_2 =$$

$$= \lim_{m \to \infty} \left[ \prod_{n=1}^{\infty} (1 + \lambda_n)^{-1/2} \int_{\Omega} \exp \left( i \sum_{j=1}^{J_m} \left( f_j, \sum_{k=1}^{K} t_k \delta_{nk} \right)_D \Phi f_j \right) F dP_1 \right] =$$

$$= \left[ \prod_{n=1}^{\infty} (1 + \lambda_n) \right]^{-1/2} \int_{\Omega} \exp \left( i \sum_{k=1}^{K} t_k \Phi \delta_{nk} \right) F dP_1 \left] = \cdots \right.$$
That is the characteristic functions of the marginals of the measures

$$(dP_2) \quad \text{and} \quad \left( \prod_{n=1}^{\infty} (1 + \lambda_n)^{-1/2} FdP_1 \right)$$

are the same, which implies that the marginals are the same. Thus, these measures coincide on the cylinders, and are therefore the same. $\square$

Exercise 3. Show that $F \in L^p(\Omega, dP_1)$ for some $p > 1$, and find the optimal $p$ in terms of $\lambda_n$'s.

4 Weak Convergence of Gaussian Measures

4.1 General Definition

Let $(M, d)$ be a complete and separable metric space, $\mathcal{F}$ - the Borel $\sigma$-algebra on $M$, and $\{P_t\}_{t>0}$ (or $\{P_n\}_{n \in \mathbb{N}}$) a family of Borel probability measures on $(M, \mathcal{F})$.

**Definition 4.1.** We say that $P_t$ converges weakly to $P$ as $t \to \infty$ if for any bounded continuous function $F : M \to \mathbb{R}$,

$$\lim_{t \to \infty} \int_M FdP_t = \int_M FdP$$

We then write $P_t \Rightarrow P$.

$\hookrightarrow$ For a more detailed discussion of weak convergence, see the first 18 pages of *Convergence of Probability Measures* by Patrick Billingsley.

4.2 The case of GRF

Suppose now, for our setting:

$\Omega \to \mathbb{R}^G$

$\mathcal{F} \to $ Borel $\sigma$-field

$P_t \to $ Gaussian measures with variances $D^{(t)}$

$\Phi_{\delta_n} \to $ GRF over $P_t$ with variance $D^{(t)}$ $\ (\Phi_{\delta_n} = X_n)$

For $\alpha \in f l^2(G)$, the map

$$\Phi_\alpha(\omega) = \sum_{n \in G} \alpha(n) \omega(n)$$

is continuous, and so, $e^{i\Phi_\alpha(\omega)}$ is continuous as well.
So, if $P_t \Rightarrow P$, we must have

$$\int_{\Omega} e^{i\Phi_\alpha(\omega)} dP_t \longrightarrow \int_{\Omega} e^{i\Phi_\alpha(\omega)} dP \Rightarrow \lim_{t \to \infty} e^{-1/2 \langle \alpha, D^{(t)} \alpha \rangle} = \int_{\Omega} e^{i\Phi_\alpha(\omega)} dP$$

We don’t want the integral on the right to be neither 0, nor $\infty$, so we assume that $\{D^{(t)}\}$ are uniformly bounded, i.e. that there exist $0 < m \leq M$ such that

$$m \leq D^{(t)} \leq M \iff m \|\alpha\|^2 \leq \left\langle \alpha, D^{(t)} \alpha \right\rangle \leq M \|\alpha\|^2$$

Then,

$$\lim_{t \to \infty} \left\langle \alpha, D^{(t)} \alpha \right\rangle \text{ exists}$$

By polarization,

$$\lim_{t \to \infty} \left\langle \alpha, D^{(t)} \beta \right\rangle$$

exists for $\alpha, \beta \in f L^2(G)$. By continuity and boundedness, the limit exists $\forall \alpha, \beta \in f L^2(G)$.

If $\alpha = \delta_n$ and $\beta = \delta_m$, $D_{nm} = \lim_{t \to \infty} D_{nm}^{(t)}$ exists. Let $D$ be an operator with matrix $[D_{nm}]$. Then, $D$ is self-adjoint and bounded, namely

$$m \leq D \leq M$$

Moreover,

$$\forall \alpha, \beta \in f L^2(G) \exists \lim_{t \to \infty} \left\langle \alpha, D^{(t)} \beta \right\rangle = \langle \alpha, D \beta \rangle$$

Hence, $D$ is a good variance, and

$$\lim_{t \to \infty} e^{-1/2 \langle \alpha, D^{(t)} \alpha \rangle} = e^{-1/2 \langle \alpha, D \alpha \rangle}$$

so that, $P$ is Gaussian with variance $D$.

**Theorem 4.2.** Let $P_t$ be a family of Gaussian measures with variances $D^{(t)}$ such that for some $M, m > 0$,

$$m \leq D^{(t)} \leq M$$

Then $P_t \Rightarrow P$ if and only if $D^{(t)} \to D$, and in that case, $P$ variance $D$.

**Proof.** We have just shown the $\Rightarrow$ direction.

Conversely, if $D^{(t)} \to D$,

$$\int_{\Omega} e^{i\Phi_\alpha(\omega)} dP_t = e^{-1/2 \langle \alpha, D^{(t)} \alpha \rangle} \longrightarrow e^{-1/2 \langle \alpha, D \alpha \rangle} = \int_{\Omega} e^{i\Phi_\alpha(\omega)} dP$$

From Billingsley, this implies that for every cylinder $C$

$$\lim_{t \to \infty} P_t(C) = P(C)$$

And this implies that $P_t \Rightarrow P$. 

Corollary 4.3. Let $\{P_t\}_{t>0}$ be a family of Gaussian measures with variances $D^{(t)}$ satisfying,

$$0 < m \leq D^{(t)} \leq M$$

Then there exists a subsequence $t_n \to \infty$ such that $P_t \Rightarrow P$ for a Gaussian $P$.

Proof. By diagonal argument, extract a subsequence $t_n \to \infty$ such that

$$D^{(t_n)} \longrightarrow D$$

and by theorem, we are done. \qed