# Cops and a Fast Robber on Planar and Random Graphs


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UBC Discrete Math seminar, 27 October 2015

joint work with Noga Alon

# Remarks

ANIMATE!

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1. perfect-information game
2. More than one cops can be at the same vertex.
3. Robber cannot jump over a cop.
4. Moves are deterministic.
5. When describing a strategy for the cops, we assume the robber is clever; and vice versa.
6. Interested in minimum number of cops to guarantee capture.

# What's known

- ✓ On a path/complete graph one cop suffices.
- ✓ On a cycle/grid, two cops suffice (bus problem).
- ✓ On a planar graph, three cops suffice. [Aigner,Fromme'84]
- ✓ Meyniel conjectured $\leq O\left(\sqrt{n}\right)$ cops suffice for any graph. [Frankl'87]
  We don't have a proof that $n^{0.99}$ cops suffice for all graphs!
  $n$ : number of vertices
- ✓ On a random graph $\leq O\left(\sqrt{n}\right)$ cops suffice with high prob. [Prałat,Wormald'15]

# The fast robber variant

ANIMATE!

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ANIMATE!

## Definition (The Game of Cops and Robber)

- ✓ In the beginning,
  - First, each cop chooses a starting vertex.
  - Then, the robber chooses a starting vertex.
- ✓ In each round,
  - First, each cop chooses to stay or go to an adjacent vertex.
  - Then, the robber chooses to stay, or move along a cop-free path.
- ✓ The cops capture the robber if, at some moment, a cop is at the same vertex with the robber.
- ✓ Cop number of $G = c(G)$

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- ✓ Computing $c(G)$ is NP-hard.
  [Fomin, Golovach, Kratochvíl'08]
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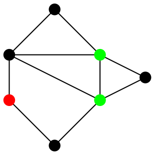
Today we study cop numbers of planar and random graphs.

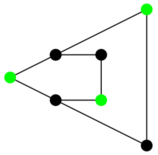
# Dominating Set

$N(S) :=$ (closed) neighbourhood of set $S$

$A$ is dominating set : $N(A) = V(G)$

Example





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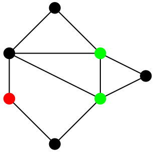
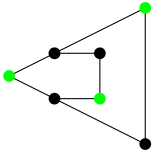


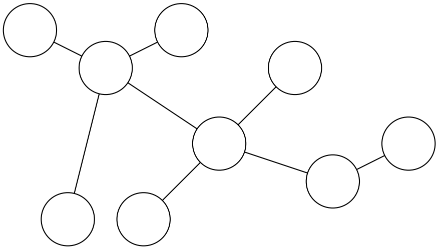


$c(G) \leq \gamma(G)$ = size of a minimum dominating set

(will be used for bounding cop number of random graphs)

# Cop number of trees

The cop number of any tree is one.



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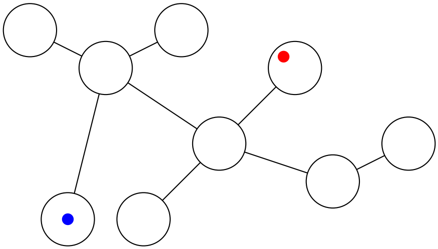

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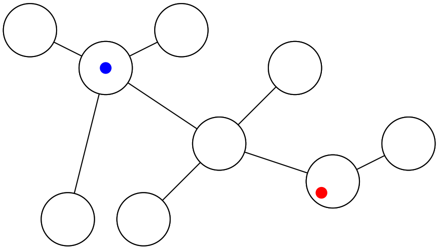

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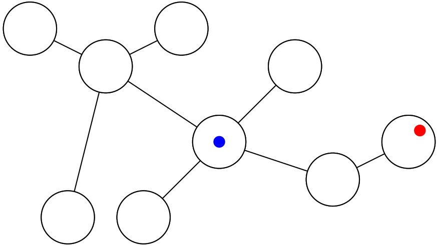

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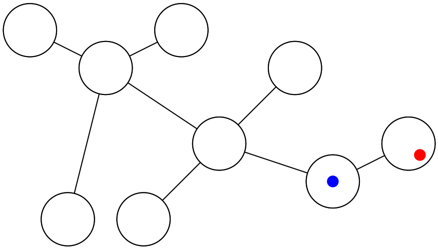
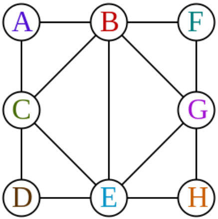
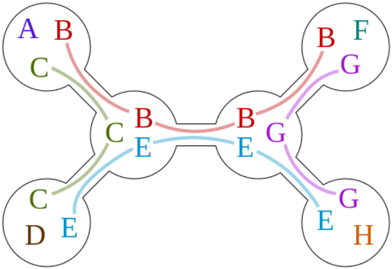


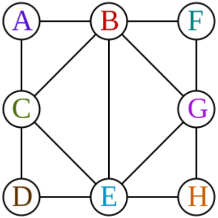
# Tree decompositions

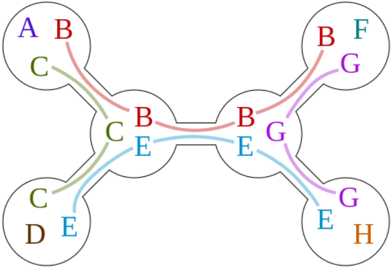



1. For every edge of graph there is a bag of tree containing both endpoints.
2. Each vertex of graph induces a connected subtree in the tree.

# Tree decompositions: treewidth



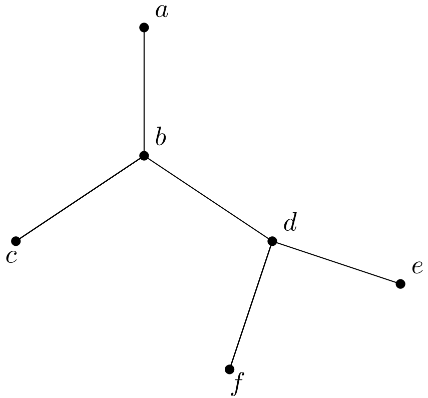




Width = maximum size of a bag $-1 = 2$

tw$(G)$ = minimum width of a tree decomposition for $G$

# Tree decomposition of a tree

Treewidth of a tree is 1



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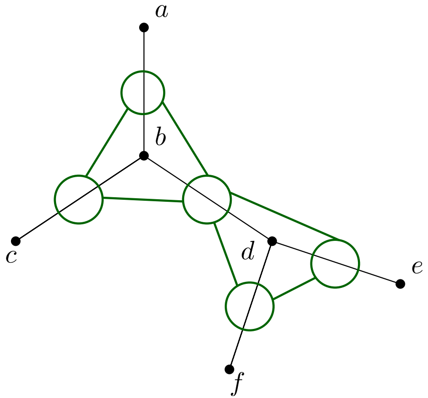

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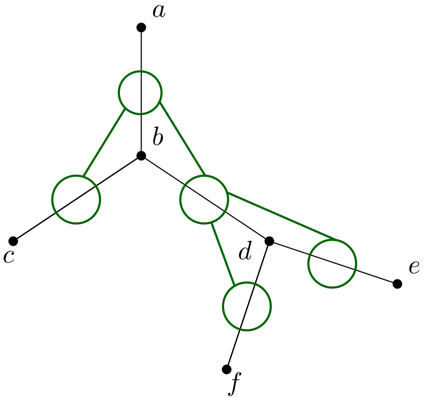

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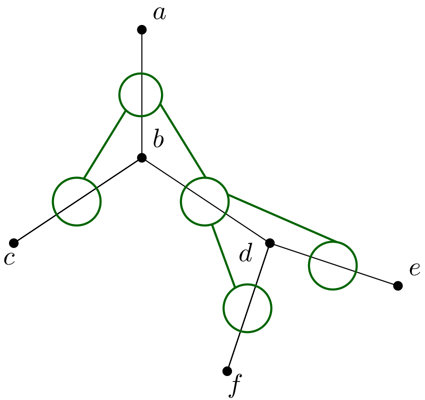
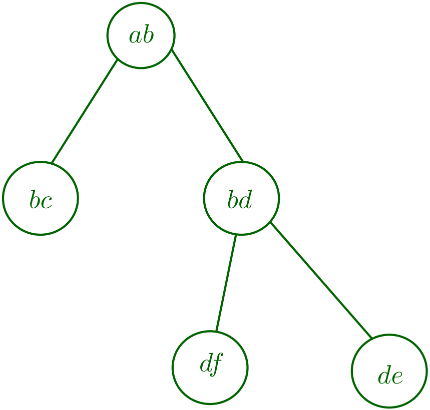
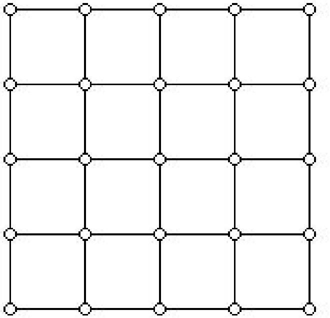


# Examples of treewidth

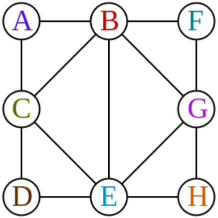
## Example

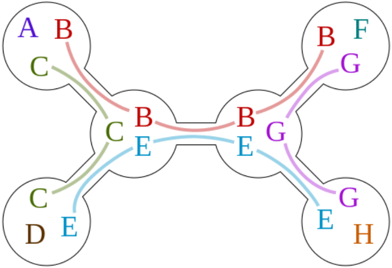
1. Treewidth of a complete graph is $n - 1$
2. Treewidth of a planar graph is $\leq O\left(\sqrt{n}\right)$
3. Treewidth of an $m \times m$ grid is $m$



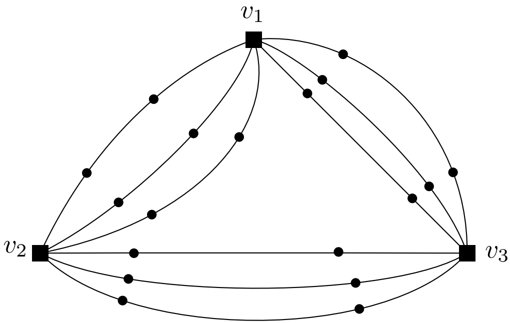
# The Relation Between Cop Number and Treewidth

For any $G$, $c(G) \leq \mathrm{tw}(G) + 1$



# $\mathrm{tw}(G) + 1$ cannot be improved!



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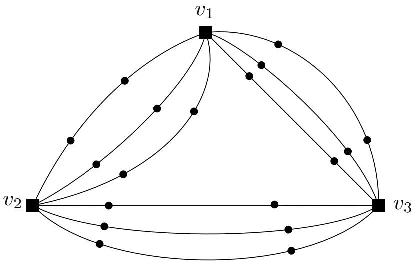


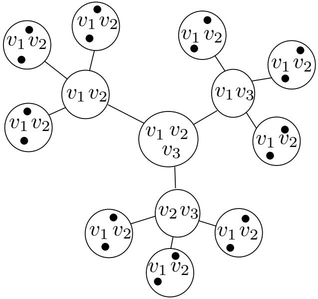


treewidth=$\max\{3, m-1\}$

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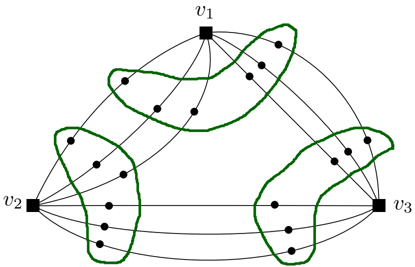
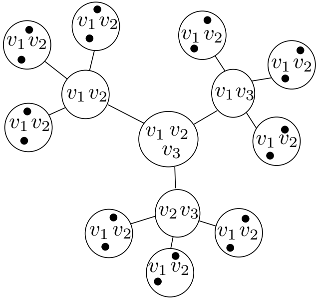



cop number $= 3$

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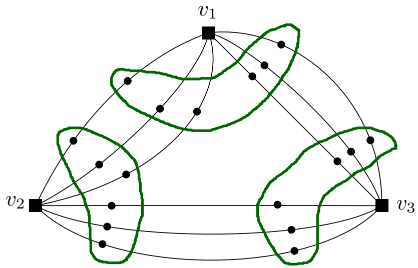


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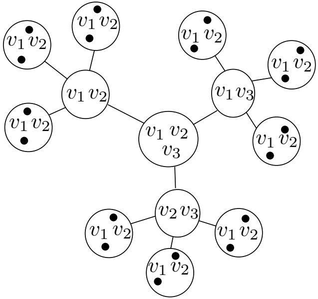


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For any graph $G$ we have

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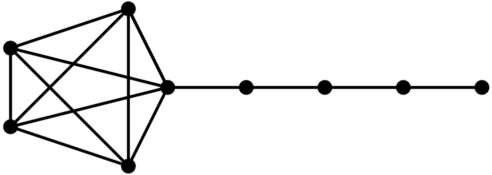

$$\mathrm{tw}(G) = \frac{n}{2} - 1, \gamma(G) \geq n/6$$
$$c(G) = 1$$

## Our main results

**Theorem (Alon, M'15)**

*There exists a constant* $\kappa > 0$ *such that for any planar* $G$

$$\kappa\,\mathrm{tw}(G) \leq c(G) \leq \mathrm{tw}(G) + 1$$

*There exists a constant* $\lambda > 0$ *such that for a random graph* $G = \mathcal{G}(n, p)$, *with high probability,*

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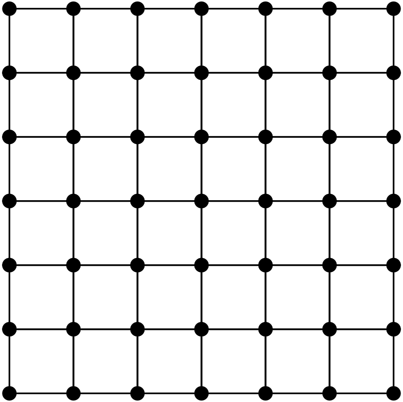
The two easy upper bounds are tight up to a constant factor, for two important classes of graphs.

Planar graphs

$$\kappa\,\mathrm{tw}(G) \leq c(G) \leq \mathrm{tw}(G) + 1$$

# Cop number of an $m \times m$ grid

$m/3 \leq$ cop number $\leq m$



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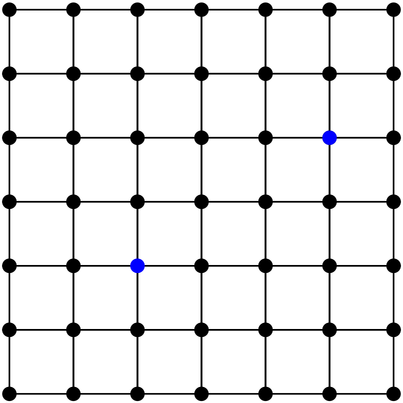

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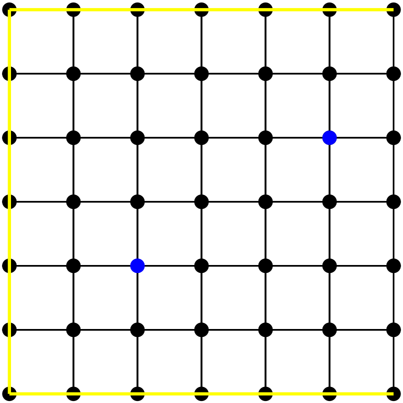

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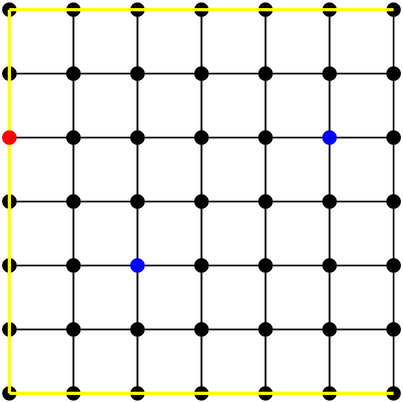

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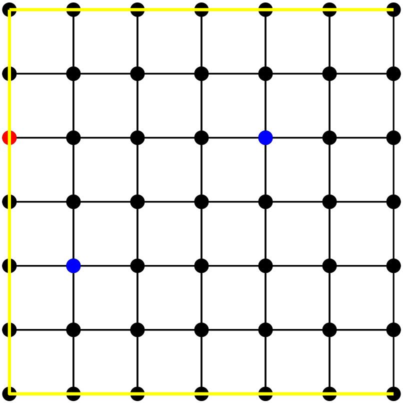

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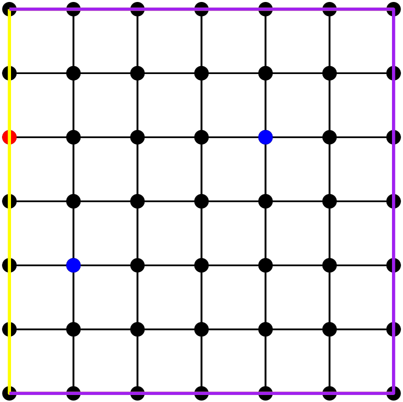

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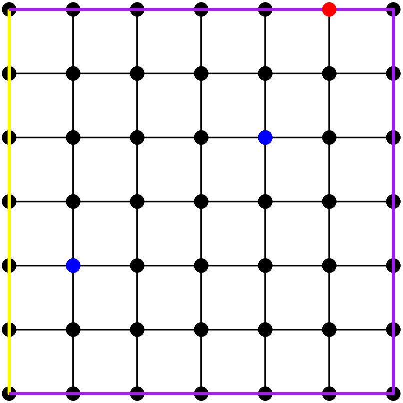
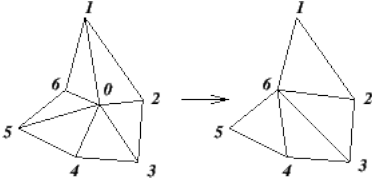


# Contraction and cop number




Observe: contracting an edge can only help the cops, and thus decrease the cop number!

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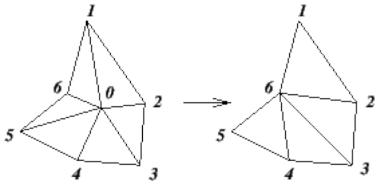


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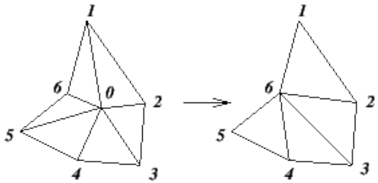


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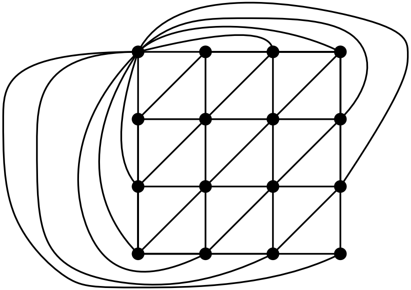
Obtain a “large” grid graph by contracting edges of a planar graph?

Theorem. Any planar graph $G$ contains an $\alpha\,\mathrm{tw}(G) \times \alpha\,\mathrm{tw}(G)$ grid as a minor. [Demaine-Hajiaghayi’08]

# A useful theorem

**Theorem** (Fomin, Golovach, Thilikos'11)

*Any planar graph $G$ contains an $\alpha\,\mathtt{tw}(G) \times \alpha\,\mathtt{tw}(G)$ grid-like graph as a contraction.*



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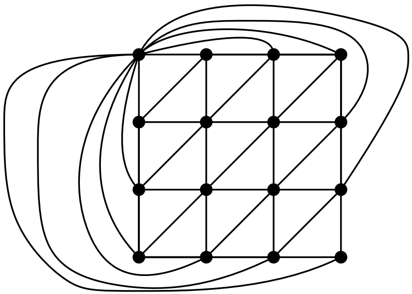
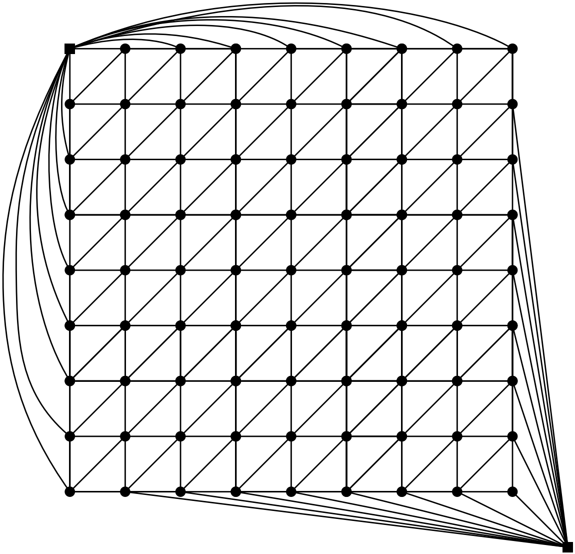


Hence for any planar $G$ we have

$$c(G) \geq c(\Gamma_{\alpha\,\mathrm{tw}(G)}) \geq \beta\alpha\,\mathrm{tw}(G)$$

# Cop number of a grid-like graph

$c(\Gamma_m) \geq (m-2)/3 > 2$



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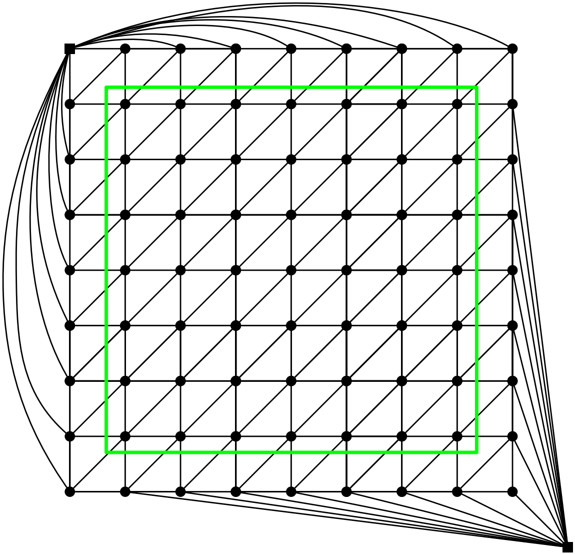

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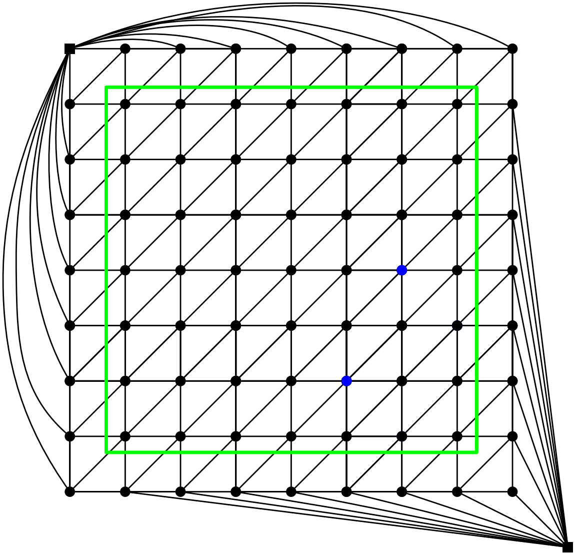

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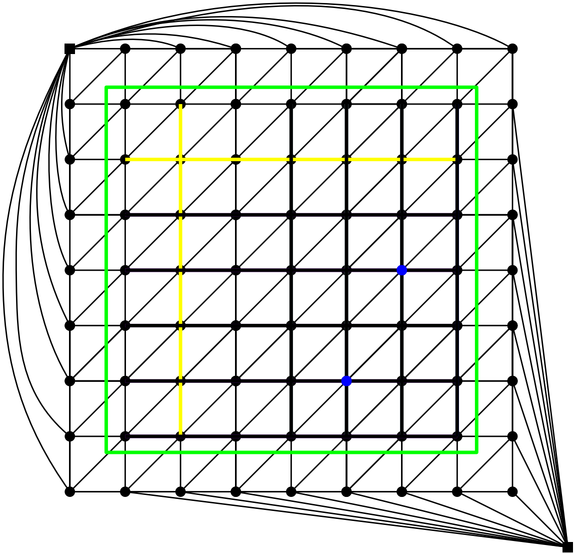

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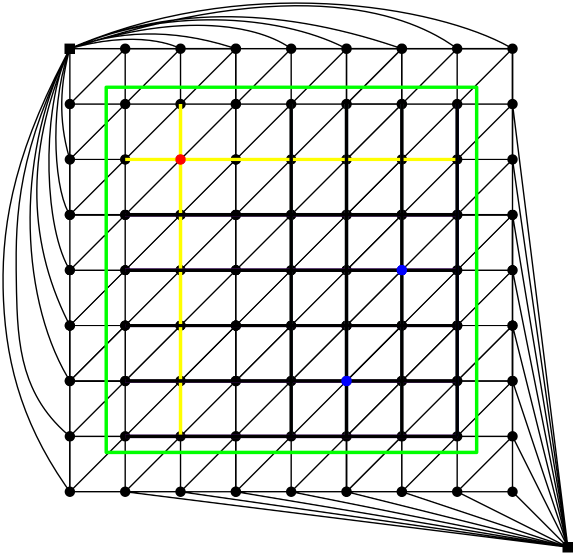

# What we've proved so far

**Theorem (Alon, M'15)**

*There exists a constant* $\kappa > 0$ *such that for any planar* $G$

$$\kappa\, \mathrm{tw}(G) \leq c(G) \leq \mathrm{tw}(G) + 1$$

1. $c(G) \leq O\left(\sqrt{n}\right)$
2. Constant-factor approximation algorithm for computing cop number of planar graphs.

Random graphs

$$\lambda\gamma(G) \leq c(G) \leq \gamma(G)$$

# The Random graph model

**Definition**

$\mathcal{G}(n, p)$ is a random graph on $n$ vertices,
each edge appears in $\mathcal{G}(n, p)$ with probability $p$
($p$ can depend on $n$).
For a graph property $\mathcal{A}$, we say $\mathcal{G}(n, p)$ asymptotically almost surely (a.a.s.) satisfies $\mathcal{A}$, if

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}(n, p) \text{ satisfies } \mathcal{A}\right] = 1$$

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We will show that for any $p = p(n)$, $\mathcal{G}(n,p)$ a.a.s. satisfies

$$\lambda\gamma(G) \le c(G) \le \gamma(G)$$

# The “small p” case

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$$\gamma(G) \geq c(G) \geq \text{number of isolated vertices} \geq ne^{-L}/2 \geq \gamma(G)e^{-L}/2$$

Next we analyze the case $\frac{1}{n} \ll p$, i.e. $pn \to \infty$

# Main lemma for random graphs

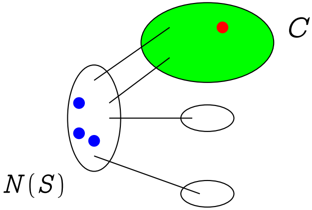
**Lemma**

*Suppose* $d := pn \to \infty$ *as* $n \to \infty$. *For any fixed* $\varepsilon > 0$ *we have*

$$(1-\varepsilon)\frac{\ln(d)}{d} \times n < c(G) \leq \gamma(G) < (1+\varepsilon)\frac{\ln(d)}{d} \times n$$

# Escaping strategy for the robber

$N(S) :=$ (closed) neighbourhood of $S$




Invariant: Robber in largest component of $G - N(S)$

$S =$ cops' position

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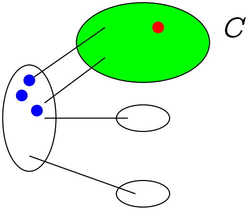

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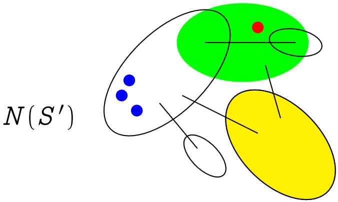

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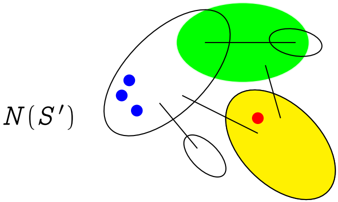


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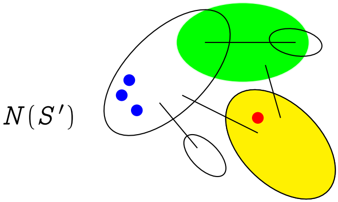


Invariant: Robber in largest component of $G - N(S')$
$S'$ = cops' position
Need 2 things:
(1) if $S$ is small, largest component of $G - N(S)$ is big,
(2) and there is an edge between any two big subsets of vertices

# Lower bound for cop number of random graphs

Principle: a.a.s. for all large sets $X$, $|V(G) \setminus N(X)|$ is very close to its expected value $= (1-p)^{|S|} \approx e^{-p|S|}$.

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Set $b := en \ln d/d$.
Fact 1: a.a.s. any two subsets of size $b$ are joined by an edge.
Bus problem: if you have some numbers summing to $\geq 3b$, one of the following is true:
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Fact 2: a.a.s. any $S$ with $|S| \leq (1-\varepsilon)n \ln d/d$ has $|V(G) \setminus N(S)| \geq 3b$.
Therefore, $c(G) > (1-\varepsilon)n \ln d/d$

# Upper bound for domination number of random graphs

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$$\begin{aligned}\gamma(G) &\leq \mathbb{E}\left[|X \cup (V(G) \setminus N(X))|\right] \\ &\leq \mathbb{E}\left[|X|\right] + \mathbb{E}\left[|V(G) \setminus N(X)|\right] \leq qn + (1-q)^{\delta} n\end{aligned}$$

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An adaptation of this proof gives a.a.s.

$$\gamma(\mathcal{G}(n,p)) < (1+\varepsilon)\frac{\ln(d)}{d} \times n$$

# What we've proved for random graphs

**Lemma**

*Suppose $d := pn \to \infty$ as $n \to \infty$. For any fixed $\varepsilon > 0$ we have*

$$(1-\varepsilon)\frac{\ln(d)}{d} \times n < c(G) \leq \gamma(G) < (1+\varepsilon)\frac{\ln(d)}{d} \times n$$

Combining with the analysis for the constant $d$ case, we conclude the following.

**Theorem (Alon, M'15)**

*There exists a constant $\lambda > 0$ such that for a random graph $G = \mathcal{G}(n, p)$, asymptotically almost surely,*

$$\lambda\gamma(G) \leq c(G) \leq \gamma(G)$$

# Conclusion

# Summary of our results

Theorem (Alon, M'15)

*There exists a constant* $\kappa > 0$ *such that for any planar* $G$

$$\kappa\,\mathrm{tw}(G) \le c(G) \le \mathrm{tw}(G) + 1$$

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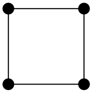
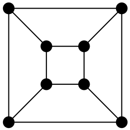
Question: find other graph classes for which these upper bounds are tight.

# Hypercube graph

1-cube

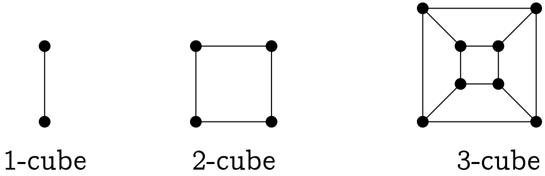


2-cube



3-cube

# Hypercube graph




### Proposition

*For any graph $G$ we have*

$$\frac{\mathrm{tw}(G)+1}{\Delta(G)+1} \leq c(G) \leq \gamma(G)$$

The $d$-cube graph has

$$\lambda_1 \times \frac{2^d}{d\sqrt{d}} \leq c(G) \leq \lambda_2 \times \frac{2^d}{d}$$