On the Density of Nearly Regular Graphs with a Good Edge-Labelling

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What is a *Good Edge-Labelling*?

Graphs are simple and undirected and have $n$ vertices.

**Definition (Bermond, Cosnard, and Pérennes 2009)**

A good edge-labelling is a labelling of edges with integers such that for any ordered pair $(u, v)$, there is at most one increasing (non-decreasing) $(u, v)$-path.

Defined in the context of Wavelength Division Multiplexing problems.
Example
Example

$K_{2,3}$
What is a *good* graph?

**Definition**

A graph is **good** if it admits a good edge-labelling; otherwise it is **bad**.

**Example**

$C_4$ is good, $K_3$ and $K_{2,3}$ are bad.
Question

What is the maximum number of edges of a good graph?

Araújo, Cohen, Giroire, and Havet (2009):

$$\Omega(n \log n) \leq \gamma(n) \leq O(n^{3/2}).$$
Density of Good Graphs

Question

What is the maximum number of edges of a good graph?

Araújo, Cohen, Giroire, and Havet (2009):

\[ \Omega(n \log n) \leq \gamma(n) \leq O(n^{3/2}). \]

Today we will see that a good regular graph has \( n^{1+o(1)} \) edges.
Girth of Bad Graphs

Question [Bode, Farzad, and Theis 2011]
Is having a small girth the obstacle for being good?

NO! We will see there exist bad graphs with arbitrarily large girth.
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Is having a small girth the obstacle for being good?

NO! We will see there exist bad graphs with arbitrarily large girth.
The Main Result

Theorem (M 2012+)

For any integer $t$, there exists $\epsilon(t)$ such that any $d$-regular $n$-vertex graph with $\epsilon(t)d^t > n$ is bad.
The Approach

Consider an arbitrary labelling of the graph, and show there exist $\geq n^2$ increasing paths.
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**Definition**

A **nice** \( k \)-**walk** is an increasing non-backtracking walk of length \( k \).
Example

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Example

Example
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An Observation

If for some $k$ there exist $> n^2$ nice $k$-walks, then the labelling is not good.
The Strategy

cdba

cdbc

25 nice 3-walks

5 nice 2-walks

cdb
The Main Theorem

Writing and solving a recursive formula for the number of $k$-walks gives

Lemma

For any integer $t$, there exists $\epsilon(t) > 0$ such that any $d$-regular $n$-vertex graph has at least $\epsilon(t) d^{t}$ nice $t$-walks.

Theorem

Any $d$-regular $n$-vertex graph with $\epsilon(t) d^{t} > n$ is bad.

Can be extended to graphs with bounded $\Delta/d$. 
The Main Theorem

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Can be extended to graphs with bounded $\Delta / \overline{d}$.
Corollary

Let $\gamma_r(n)$ be the maximum number of edges of a good regular graph. Then

$$\gamma_r(n) \leq n^{1+o(1)}.$$
The First Corollary

**Corollary**

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**Proof.**

Consider a sequence of regular graphs with at least $n^{1+\frac{1}{k}}$ edges, $k$ fixed. Then their degree is going to infinity with $n$. 
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**Proof.**

Consider a sequence of regular graphs with at least $n^{1+\frac{1}{k}}$ edges, $k$ fixed. Then their degree is going to infinity with $n$.

Consider a $d$-regular graph in this sequence with $d > 1/\epsilon(k + 1)$. Then for this graph

$$\epsilon(k + 1)d^{k+1} > d^k = \left(\frac{2n^{1+\frac{1}{k}}}{n}\right)^k = 2^kn > n.$$ 

$\square$
The Second Corollary

Corollary

For any $g$, there exists a bad graph with girth $\geq g$. 

Proof. Lazebnik, Ustimenko, and Woldar (1997) proved that for any odd prime power $d$, there exists a $d$-regular graph with girth $g$ with $< 2d^{3/4}g$ vertices. Let $d > 2/\epsilon(g)$. Then $\epsilon(g) d^g > 2d^g - 1 > 2d^{3/4}g > n$. 

The Second Corollary

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Corollary

For any $g$, there exists a bad graph with girth $\geq g$.

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Let $d > 2/\epsilon(g)$. Then

$$\epsilon(g) d^g > 2d^{g-1} > 2d^{\frac{3}{4}g} > n.$$
A Result in The Other Direction

**Theorem (M 2012+)**

Any graph with max degree $\Delta$ and girth $\geq 2k$ such that

$$4ek^2(\Delta - 1)^{k-1} < k!$$

is good.

**Corollary**

Any graph with max degree $\Delta$ and girth $\geq 40\Delta$ is good.
Theorem

Any graph with max degree $\Delta$ and girth $\geq 2k$ such that

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1. The label of each edge: independent and uniform from $[0, 1]$. 
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2. Labelling not good $\Rightarrow \exists$ increasing path of length exactly $k$ ($k$-path).
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Proof.

1. The label of each edge: independent and uniform from $[0, 1]$.
2. Labelling not good $\Rightarrow \exists$ increasing path of length exactly $k$ ($k$-path).
3. For any $k$-path $P$, $\Pr[P$ increasing$] = \frac{2}{k!}$.
4. Any $k$-path intersects at most $2k^2(\Delta - 1)^{k-1}$ other $k$-paths.
Open Problems

We showed any graph with max degree $\Delta$ and girth $\geq 40\Delta$ is good.
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Open Problem 1
Improve the dependence on $\Delta$!
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Open Problem 1

Improve the dependence on $\Delta$!

Open Problem 2 (Araújo, Cohen, Giroire, and Havet)

Does every planar graph of girth at least 5 have a good edge-labelling?
Thanks for your attention :-}