# Bounds for randomized rumour spreading protocols

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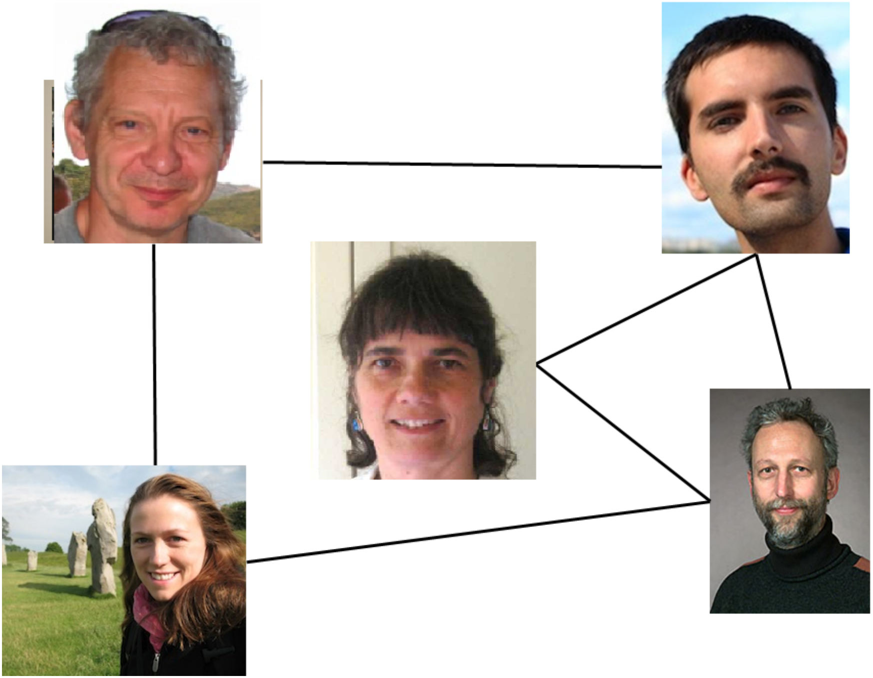
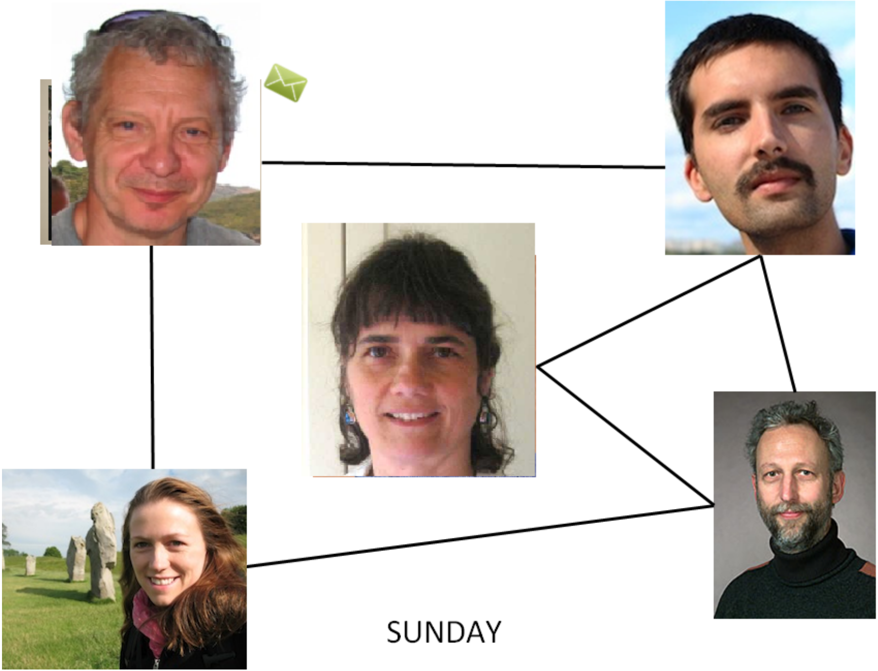
joint work with H. Acan, A. Collevecchio, A. Pourmiri, N. Wormald





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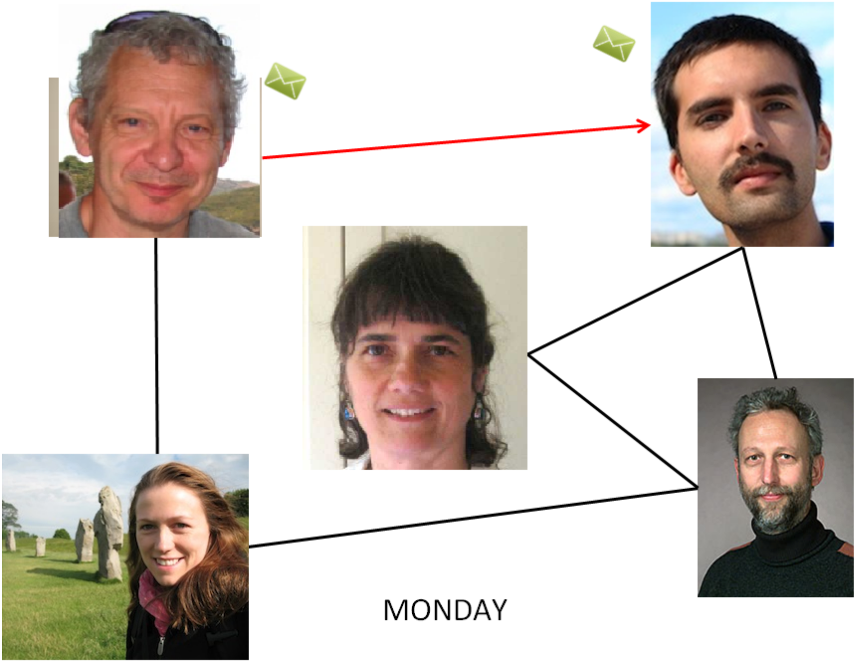
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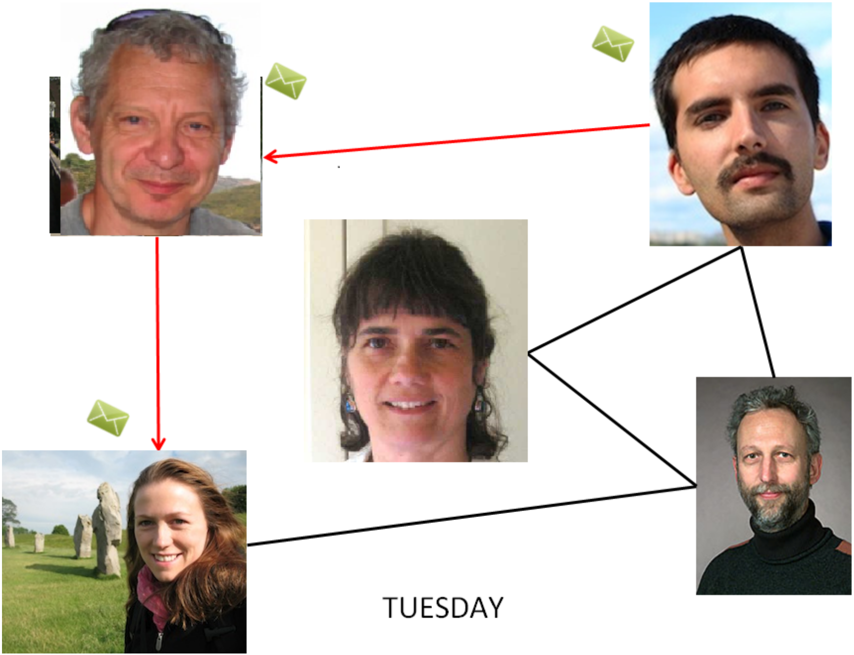
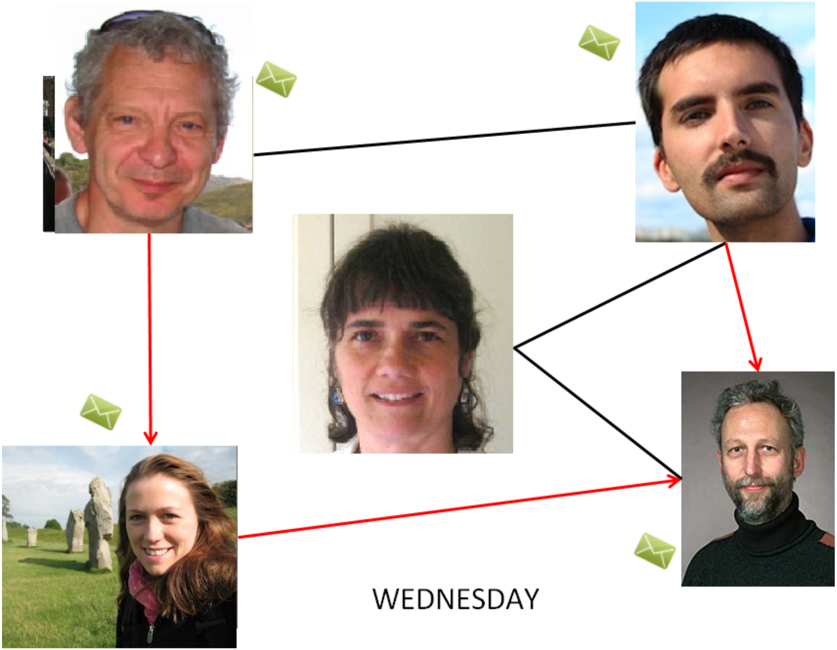


TUESDAY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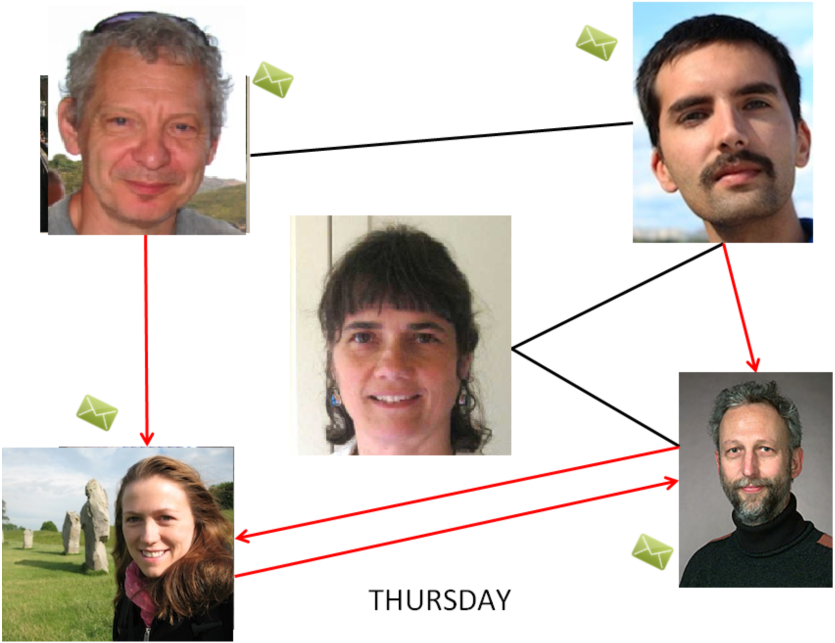
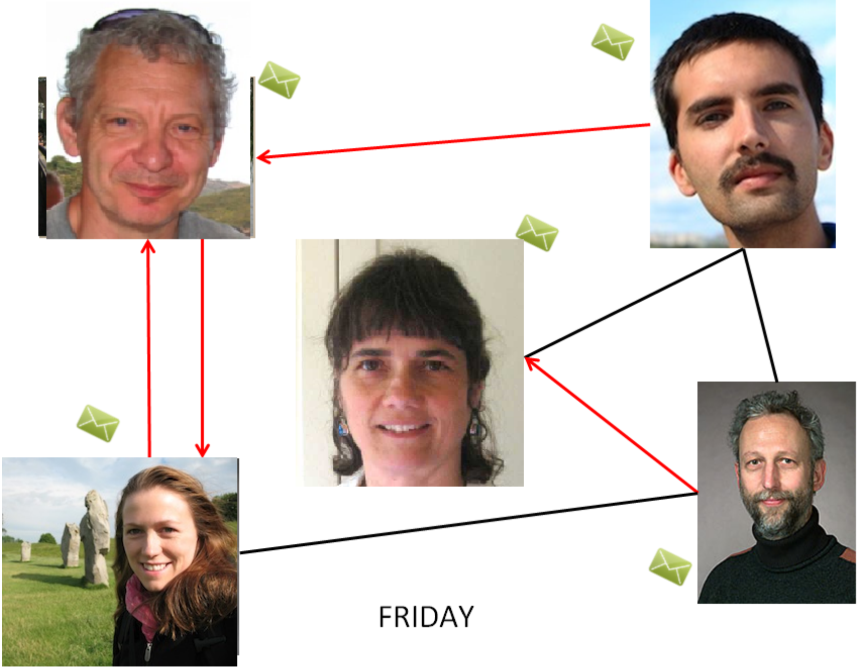

THURSDAY



FRIDAY

# Protocol definition

Demers, Gealy, Greene, Hauser, Irish, Larson, Manning, Shenker, Sturgis, Swinehart, Terry, Woods'87

1. The ground is a simple connected graph.
2. At time 0, one vertex knows a rumour.
3. At each time-step $1, 2, \ldots,$ every informed vertex tells the rumour to a random neighbour.

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Remark 2. If a vertex receives the rumour at time $t$, it starts passing it from time $t+1$.

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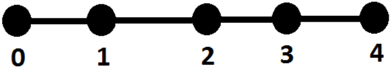
Remark 1. Informed vertex may call a neighbour in consecutive steps.
Remark 2. If a vertex receives the rumour at time $t$, it starts passing it from time $t + 1$.
inform-time($v$): the first time $v$ learns the rumour.
Spread Time: the first time everyone knows the rumour.

# Applications

- ✓ Replicated databases
- ✓ Broadcasting algorithms
- ✓ News propagation in social networks
- ✓ Spread of viruses on the Internet

# Example: a path



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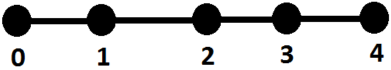


$$\mathrm{inform} - \mathrm{time}(0) = 0$$

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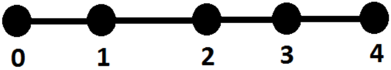


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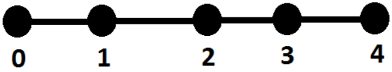


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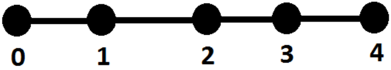


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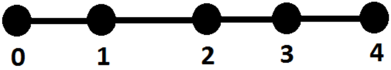


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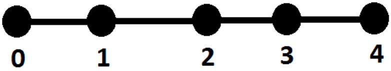


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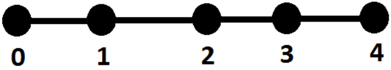


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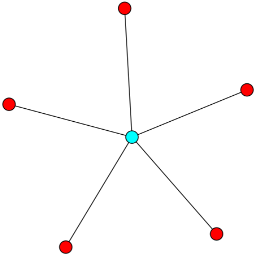
\mathbb{E}[\text{Spread Time}] &= 1 + 3 \times 2 = 7 \\
&= 2n - 3
\end{aligned}
$$

# Example: a star



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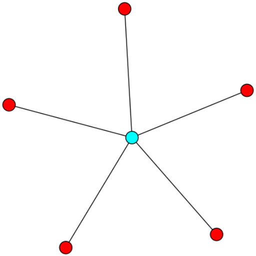

When $k+1$ vertices are informed and $n-1-k$ uninformed, after $\frac{n-1}{n-1-k}$ more rounds a new vertex will be informed.

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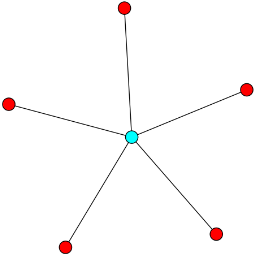
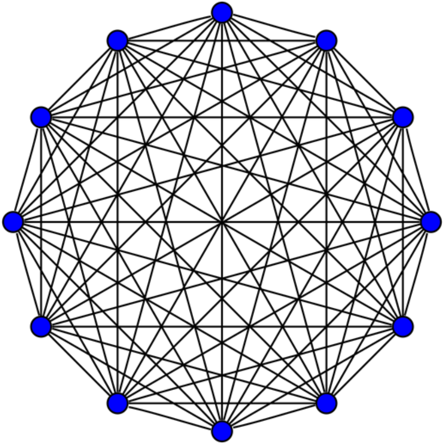


When $k + 1$ vertices are informed and $n - 1 - k$ uninformed, after $\frac{n-1}{n-1-k}$ more rounds a new vertex will be informed.

$$\mathbb{E}[\text{Spread Time}] = \frac{n-1}{n-1} + \frac{n-1}{n-2} + \cdots + \frac{n-1}{2} + \frac{n-1}{1} \approx n \ln n$$

# Example: a complete graph



$$\mathbb{E}[\text{Spread Time}] \approx \log_2 n + \ln n \qquad [\text{Pittel}'87]$$

# Other known results

With probability close to 1, for any starting vertex,

1. $\max\{\text{diameter}(G)/2, \log_2 n\} \leq \text{Spread Time} \leq n \ln n$

   [Elsässer and Sauerwald'06]

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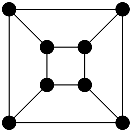
2. Spread Time of $\mathcal{H}_d = \Theta(d)$ [Feige, Peleg, Raghavan, Upfal'90]



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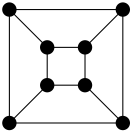

$\mathcal{H}_3$

3. If $pn \geq (1+\varepsilon)\ln n$ then Spread Time of $G(n,p) = \Theta(\ln n)$ [Feige et al.'90]

# Improving the protocol

Uninformed vertices ask the informed ones...

# The push-pull protocol

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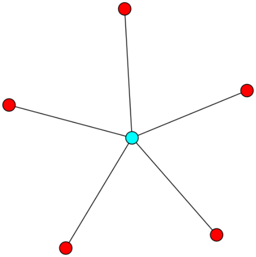
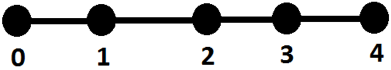
1. The ground is a simple connected graph.
2. At time 0, one vertex knows a rumour.
3. At each time-step $1, 2, \ldots,$
   every informed vertex sends the rumour to a random neighbour (PUSH);
   and every uninformed vertex queries a random neighbour about the rumour (PULL).

# Example: a star



push protocol: $n \ln n$ rounds
push-pull protocol: 1 or 2 rounds

# Example: a path



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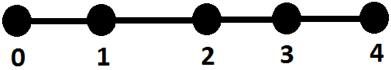


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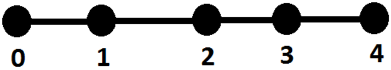


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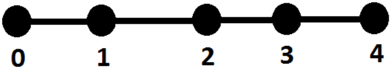
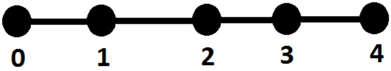



$$\begin{aligned}
\text{inform} - \text{time}(0) &= 0 \\
\text{inform} - \text{time}(1) &= 1 \\
\text{inform} - \text{time}(2) &= 1 + \min\{\text{Geo}(1/2), \text{Geo}(1/2)\} \\
&= 1 + \text{Geo}(3/4)
\end{aligned}$$

# Example: a path

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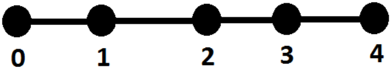


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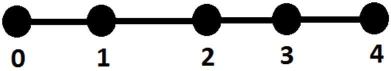


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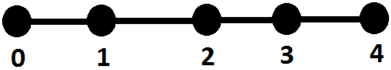


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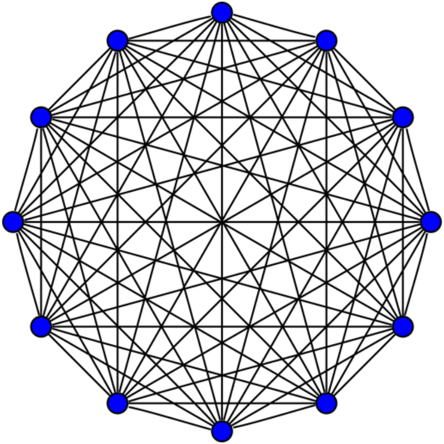
\mathrm{inform}-\mathrm{time}(4) &= 1+\mathrm{Geo}(3/4)+\mathrm{Geo}(3/4)+1\\
\mathbb{E}[\text{Spread Time}] &= 2+2\times 4/3 = 14/3\\
&= \frac{4}{3}n-2 \qquad (\text{versus } 2n-3 \text{ for push})
\end{aligned}
$$

# Example: a complete graph



push: $\log_2 n + \ln n$ [Pittel'87]

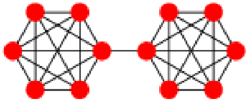
push-pull: $\log_3 n$ [Karp, Schindelhauer, Shenker, Vöcking'00]

# Other results on push-pull protocol

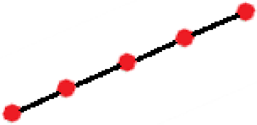
1. Barabasi-Albert preferential attachment graph has
   Spread Time $\Theta(\ln n)$,
   PUSH alone has Spread Time $poly(n)$.
   [Doerr, Fouz, Friedrich '11]
2. Random graphs with power-law expected degrees
   (a.k.a. the Chung-Lu model) with exponent $\in (2, 3)$ has
   Spread Time $\Theta(\ln n)$.
   [Fountoulakis, Panagiotou, Sauerwald '12]
3. If $\alpha$ is the vertex expansion (vertex isoperimetric number),
   and $\Phi$ is the conductance,
   Spread Time $\le C \max\{\Phi^{-1} \ln n, \alpha^{-1} \ln^2 n\}$. [Giakkoupis '11, '14]
   No bottleneck $\Rightarrow$ fast rumour spreading

# An extremal question

What's the maximum spread time of an $n$-vertex graph?



$n/4$



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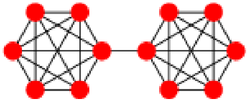

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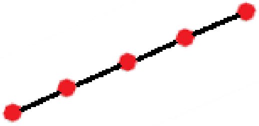

$4n/3$

**Theorem (Acan, Collevecchio, M, Wormald'14+)**

*On any $n$-vertex graph,* $\mathbb{E}\,[\text{Spread Time}] \leq 5n$

## ...


"Very well, Then — If there's no more 'old gossip', lets move on to 'new gossip'."

Push-Pull on Random $k$-trees

# Random $k$-trees

**Example ($k = 3$)**

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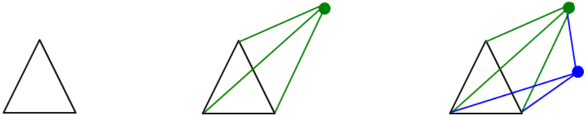

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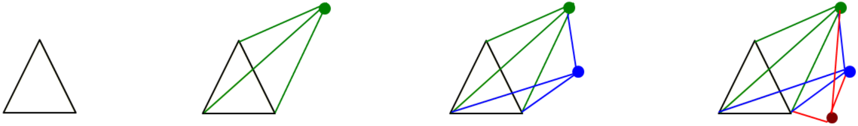
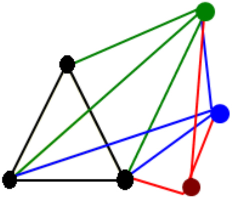


# Some properties of random $k$-trees



- ✓ Logarithmic diameter
- ✓ Power law degree sequence:
  fraction of vertices with degree $d \approx d^{-2-\frac{1}{k-1}}$
- ✓ Very small conductance and vertex expansion
  Lots of bottlenecks

# Our results

Push-Pull protocol on random $k$-trees ($k > 1$ fixed):

**Theorem (M, Pourmiri'14, upper bound)**

*If initially a random vertex knows the rumor,*
*after* $(\ln n)^{1+3/k}$ *rounds, 99 percent of vertices will know it.*

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Theorem (M, Pourmiri'14, lower bound)

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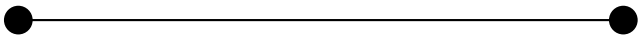
Exponential blow up if informing each and every vertex is required.

## Self-similarity of random $k$-trees



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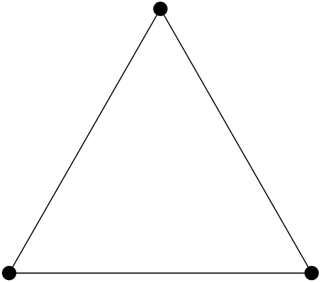
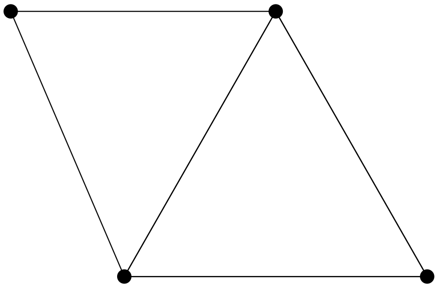


# Self-similarity of random $k$-trees



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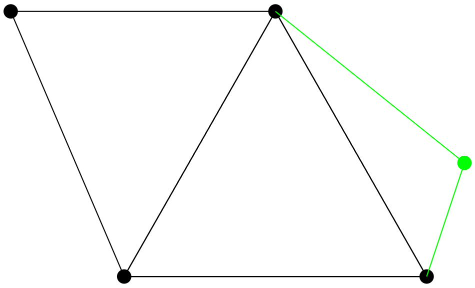

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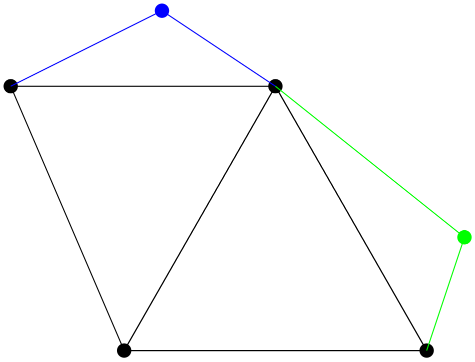

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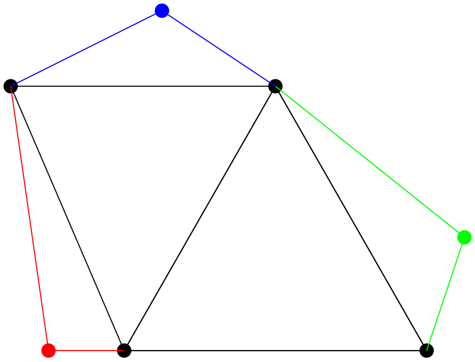

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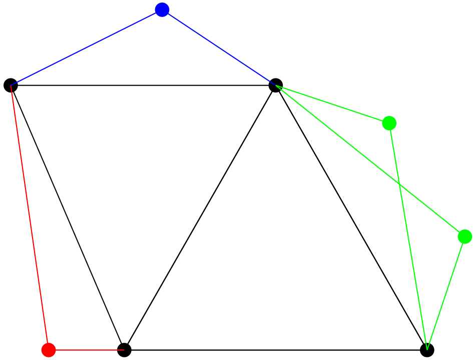

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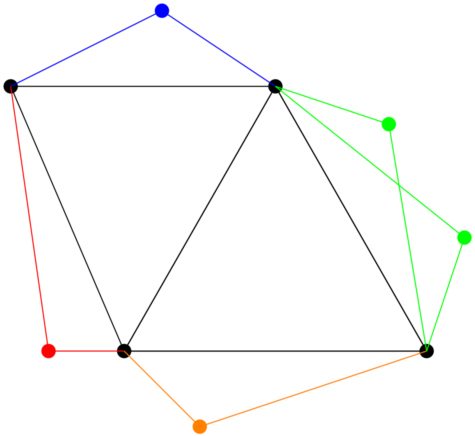

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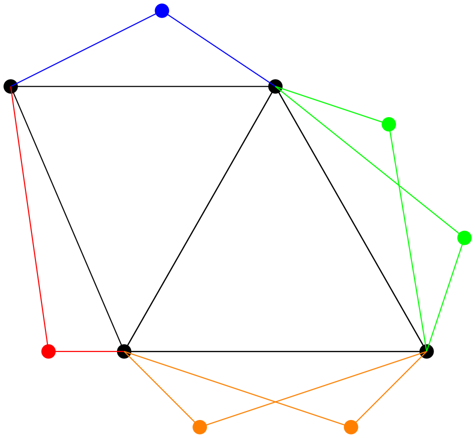

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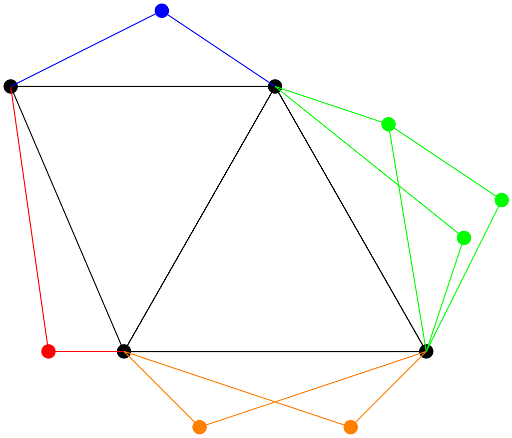

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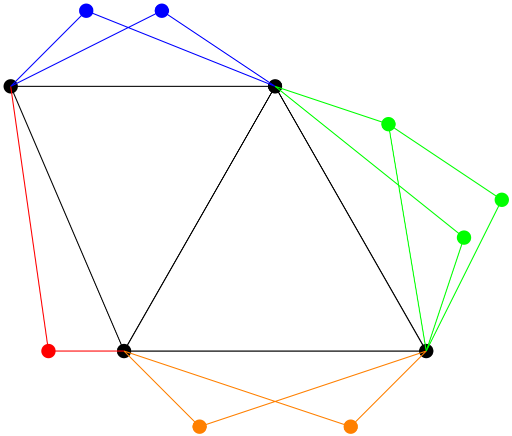

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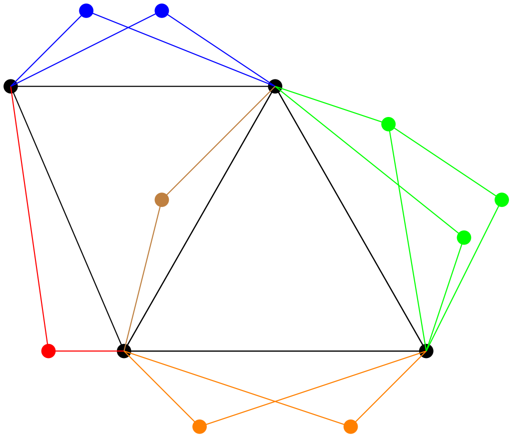

Proof sketch of the upper bound

# Fast edges

Let $d$ be some parameter.

**Definition (fast edge)**

An edge $uv \in E(H)$ is fast if $\deg(u) \leq d$ or $\deg(v) \leq d$ or $u$ and $v$ have a common neighbour with degree $\leq d$.

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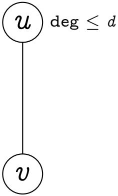

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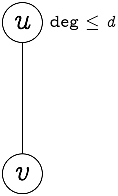


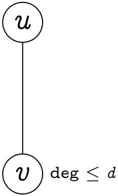

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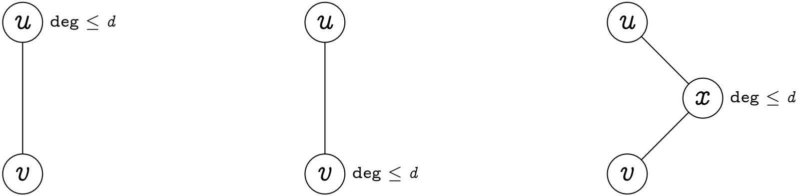

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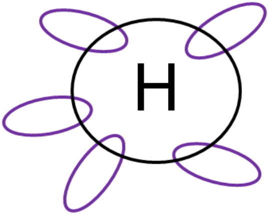
**Lemma**

*Transmission time along a path of length $\ell$ of fast edges is $\leq O(d(\ell + \ln n))$ with prob. $\geq 1 - n^{-3}$.*

# Proof of upper bound

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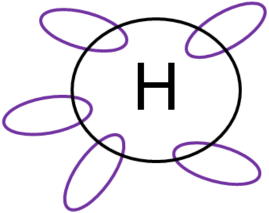


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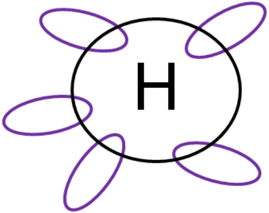


1. $H$ = graph at round $m \approx n / \ln n$
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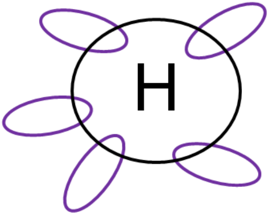


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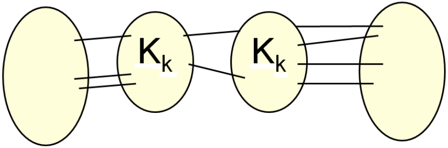
**Theorem (upper bound)**

*If initially a random vertex knows the rumor, after $O\left(\ln n \times (\ln n)^{3/k}\right)$ rounds, 99 percent of vertices will know it.*

# Proof sketch of the lower bound

# Proof of lower bound

Definition ($D$-barrier)




Vertices in the two $k$-cliques have degrees $\geq D$.

Observe: if a $D$-barrier exists, Spread Time is $\geq \Omega(D)$.

# Proof of lower bound

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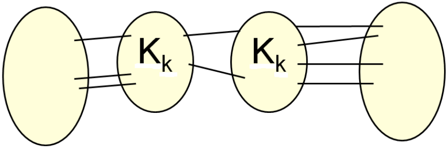


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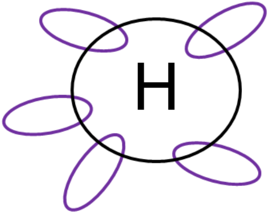
Observe: if a $D$-barrier exists, Spread Time is $\geq \Omega(D)$.

**Lemma**

*A random $k$-tree has a $\Omega\left(n^{1-1/k}\right)$-barrier with prob. $\geq \Omega\left(n^{-k}\right)$*

# Proof of lower bound

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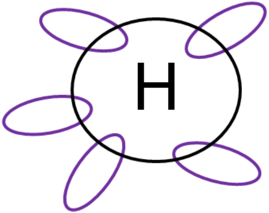


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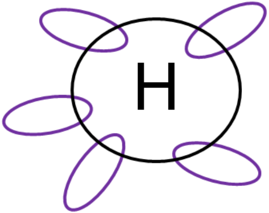


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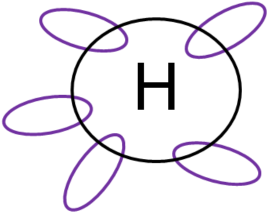


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Hence, w.h.p. Spread Time$> n^{1/3k}$

# Future research directions

1. Design a (deterministic) approximation algorithm for finding the Spread Time of a given graph.
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