On longest paths and diameter in random Apollonian networks

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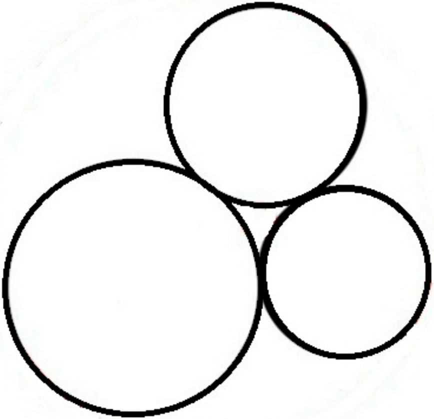
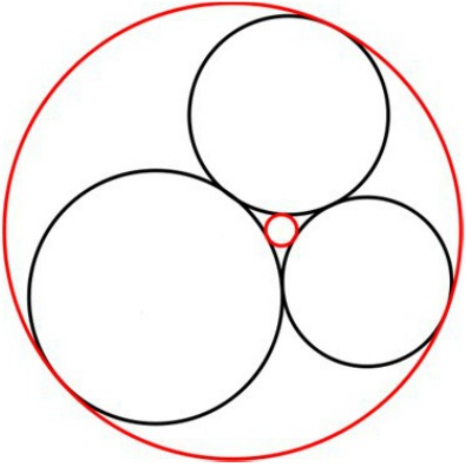
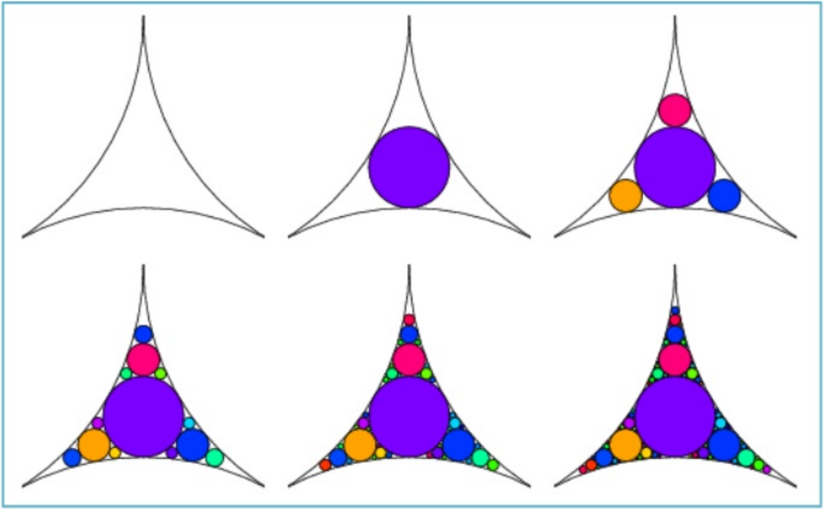
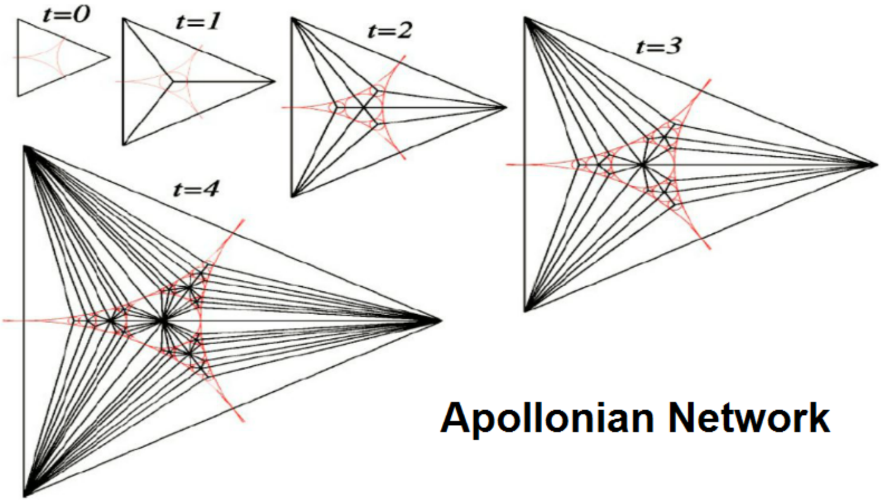
joint work with A. Collevecchio, E. Ebrahimzadeh, L. Farczadi, P. Gao, C. Sato, N. Wormald, and J. Zung



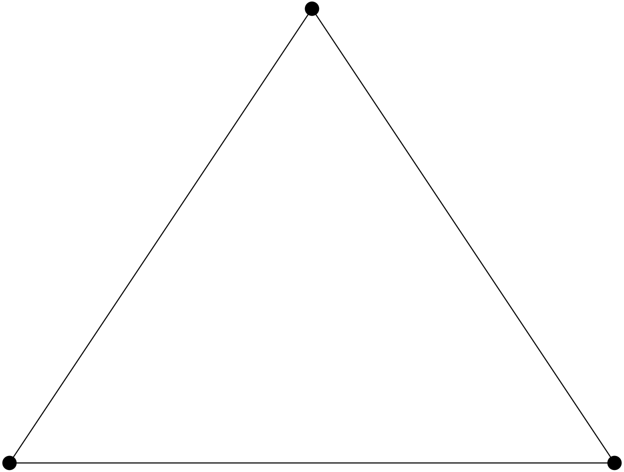




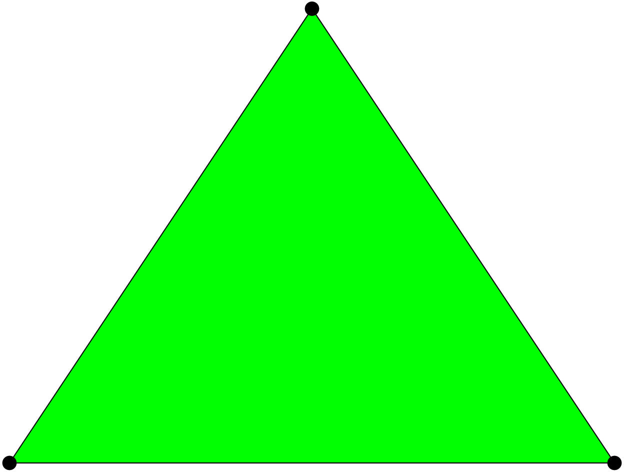
Apollonian Gasket



Apollonian Network



$t = 0$



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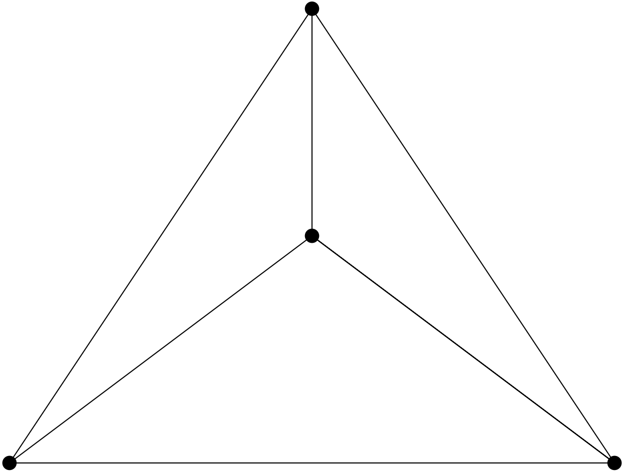
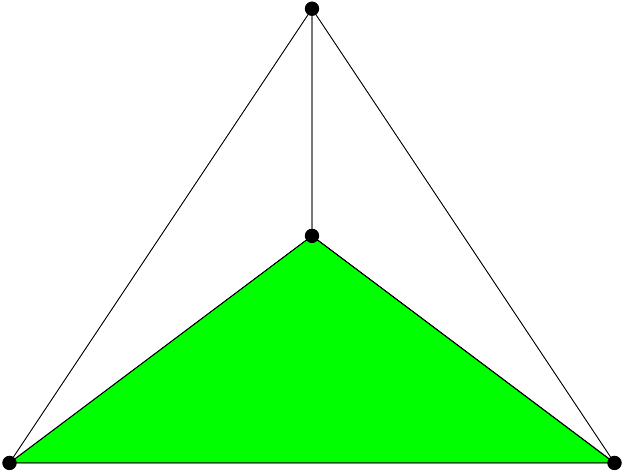


$t = 1$



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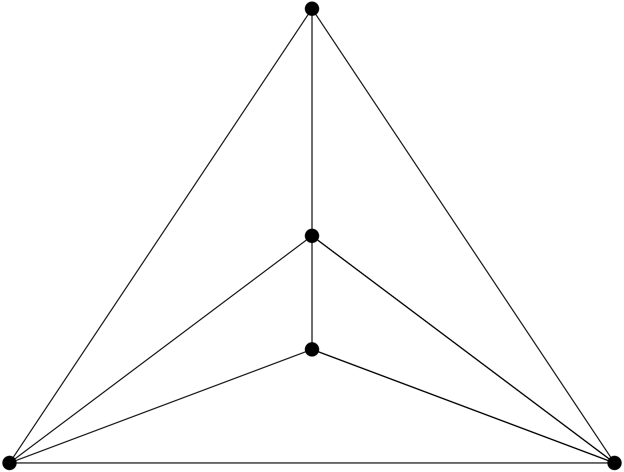
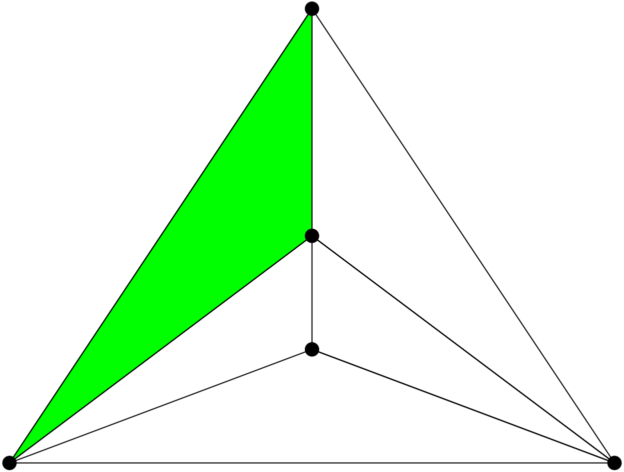


$t = 2$



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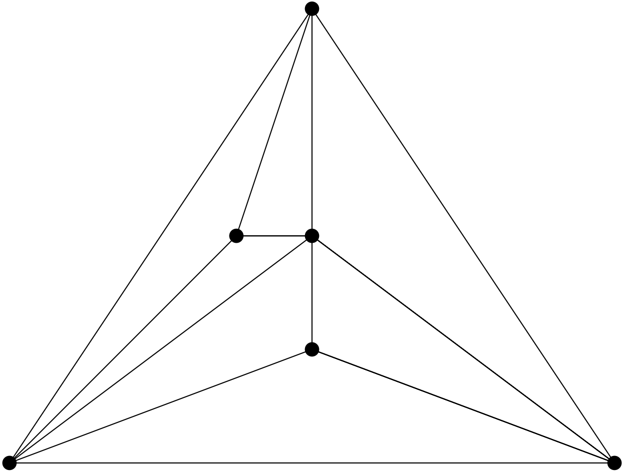
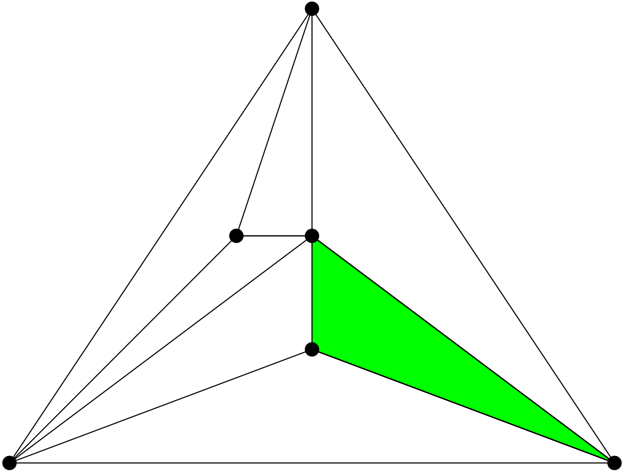


$t = 3$



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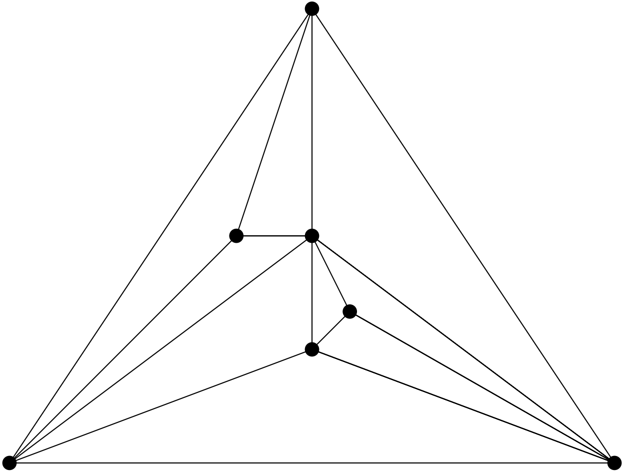


$t = 4$

After t steps,

- ✓ a random triangulated plane graph
- ✓ $n = t + 3$ vertices
- ✓ $3t + 3$ edges
- ✓ $2t + 1$ faces

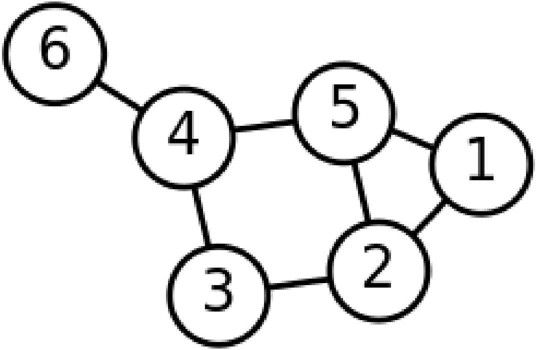
called a Random Apollonian Network (RAN).

Zhou, Yan, Wang'05: generating power-law planar graphs.

Theorem (Frieze and Tsourakakis'12)

For any fixed k, the fraction of vertices with degree k is concentrated around k^{-3}.

The Diameter of a Graph



Diameter = 3

Diameters of RANs

Theorem (Albenque and Marckert'08; Frieze and Tsourakakis'12)

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A similar result was proved independently by
Cooper, Frieze, Uehara'13 and Kolossváry, Komjáty, Vágó'13.

Longest paths in RANs

$\mathcal{L}_n :=$ length of a longest path (self-avoiding walk)

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Theorem (EFGMSWZ'13)

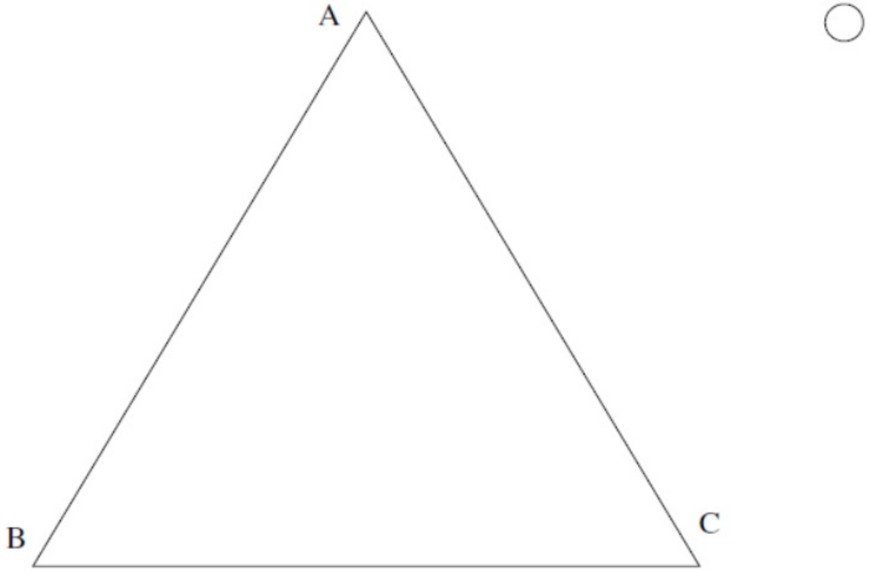
We have

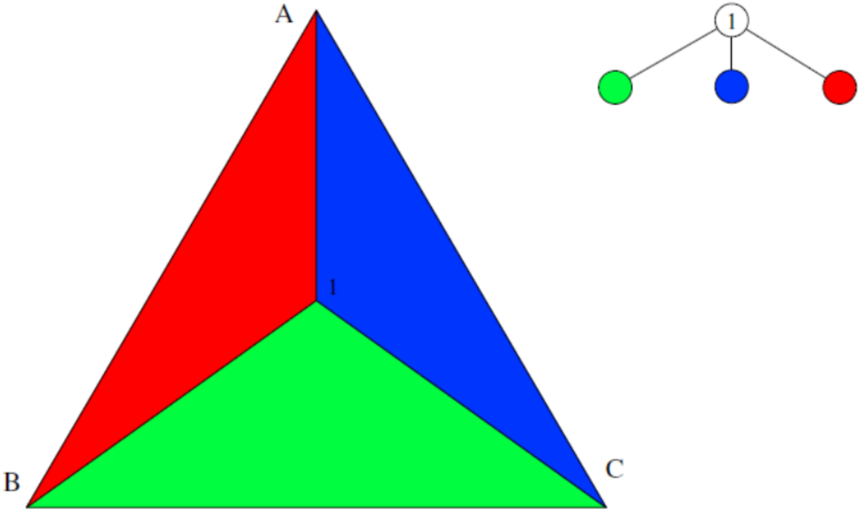
$$\mathcal{L}_n > n^{0.63}$$

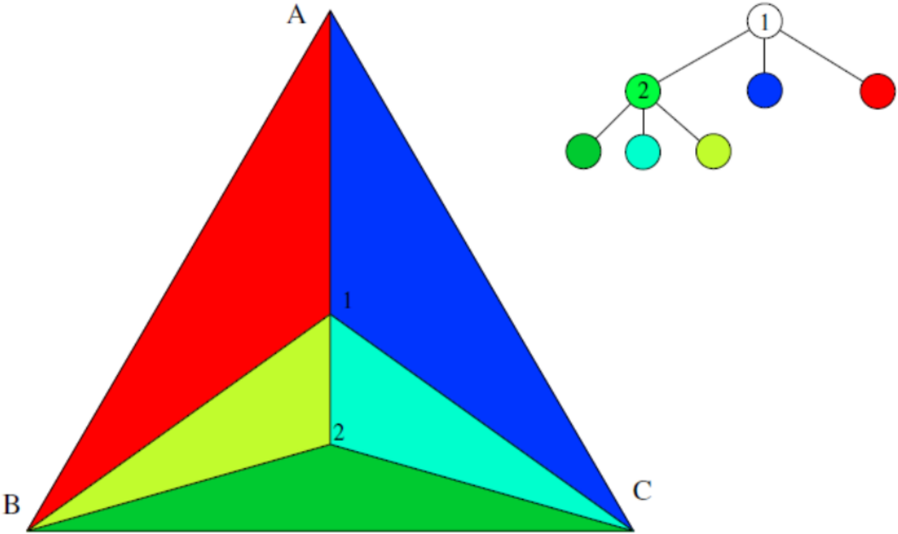
and

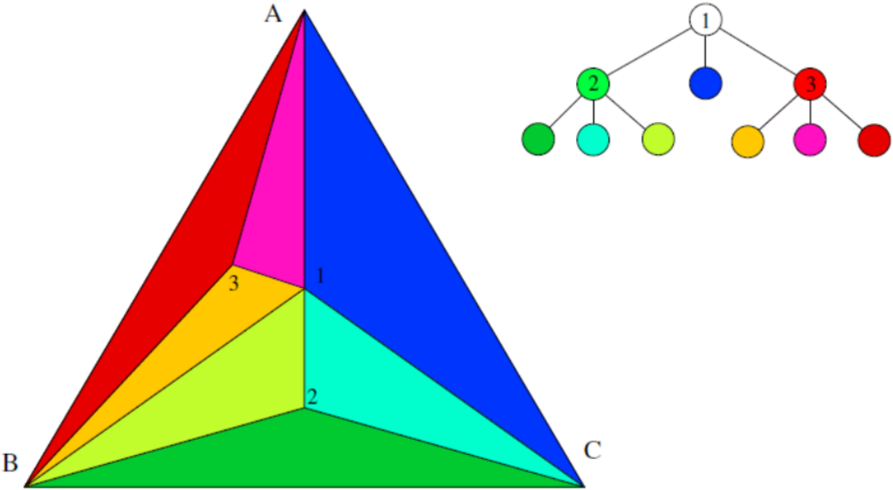
$$\mathbb{E}\left[\mathcal{L}_n\right] = \Omega\left(n^{0.88}\right)$$

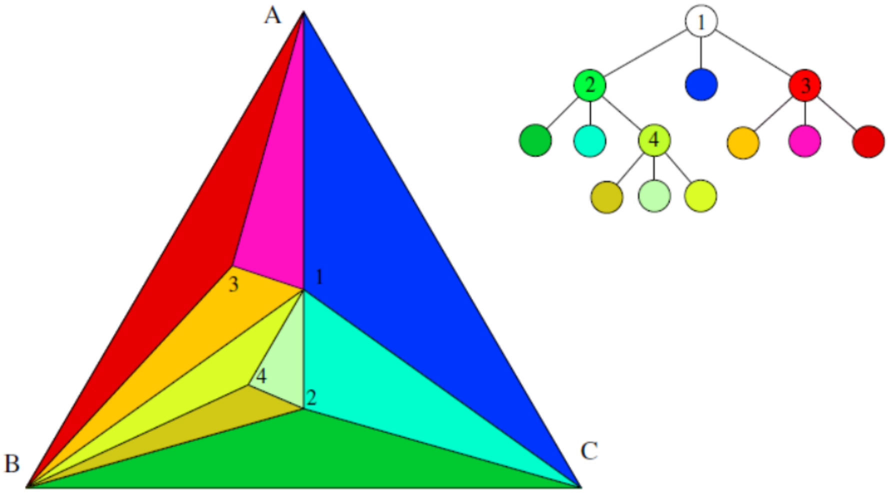
Δ-tree of a RAN

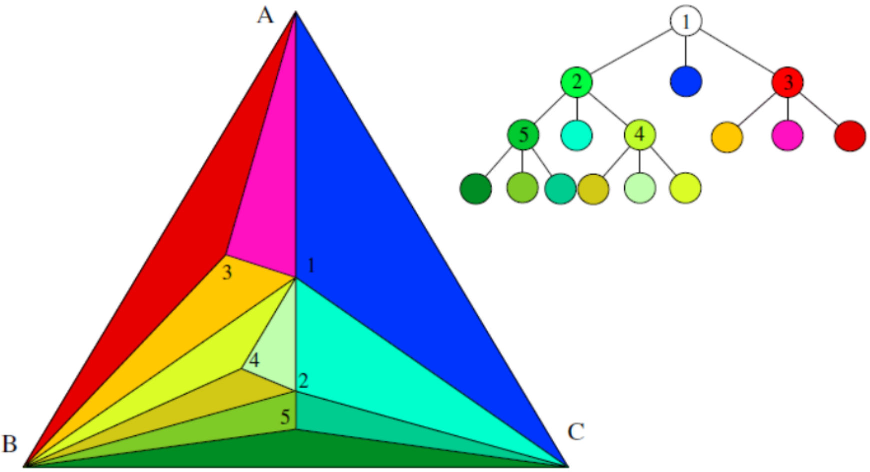

A
B
C


A
1
B
C
1


A
B
C
1
2
1
2


A
B
C
1
2
3
1
2
3


A
B
C
1
2
3
4
1
2
3
4

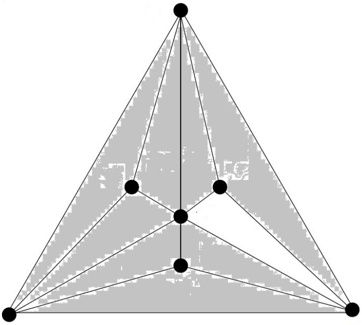


In each step, a random leaf gives birth to three children.
This is called a random (recursive) ternary tree.

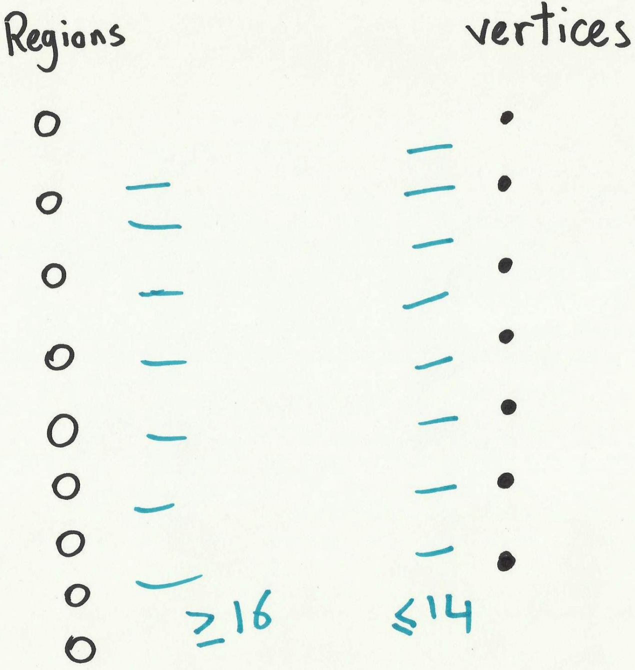
Proof outline for
length of the longest paths $< n^{0.99999996}$

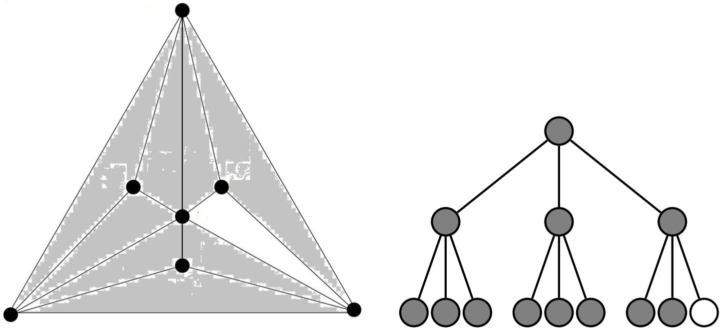
Upper Bound for Longest Path

The Main Idea

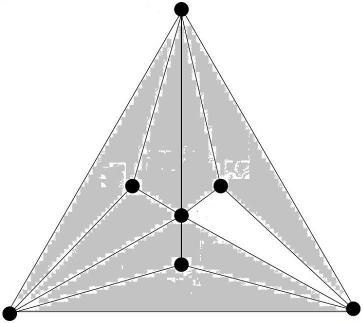


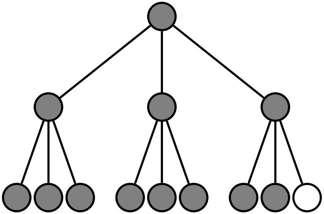
Claim: A path cannot contain internal vertices of all 9 faces.


Regions
vertices
≥16
≤14



If we colour those nodes of Δ-tree which a path goes inside, each coloured node can have at most 8 black grandchildren.





If we colour those nodes of Δ-tree which a path goes inside, each coloured node can have at most 8 black grandchildren.
New goal: bound the total number of coloured nodes in a random ternary tree.

Simplified goal

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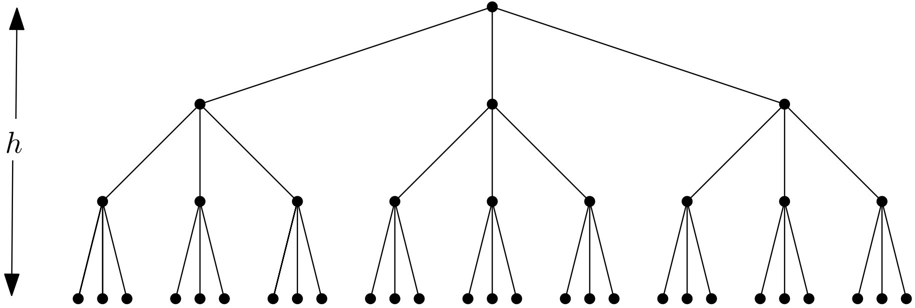

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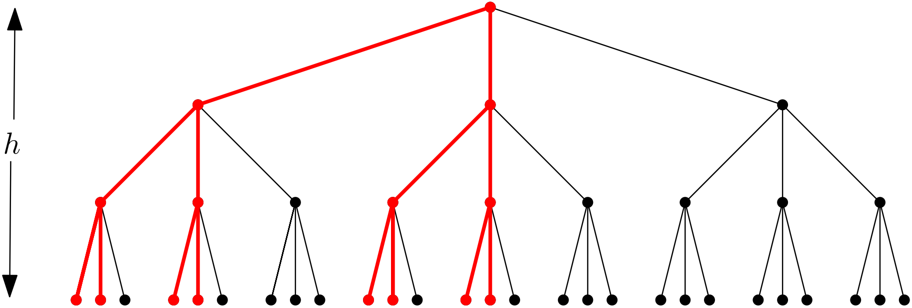

size of binary subtree $= 2^{h+1} - 1 \leq 2 \times (3^h)^{\log_3 2} < 2 \times n^{0.64}$

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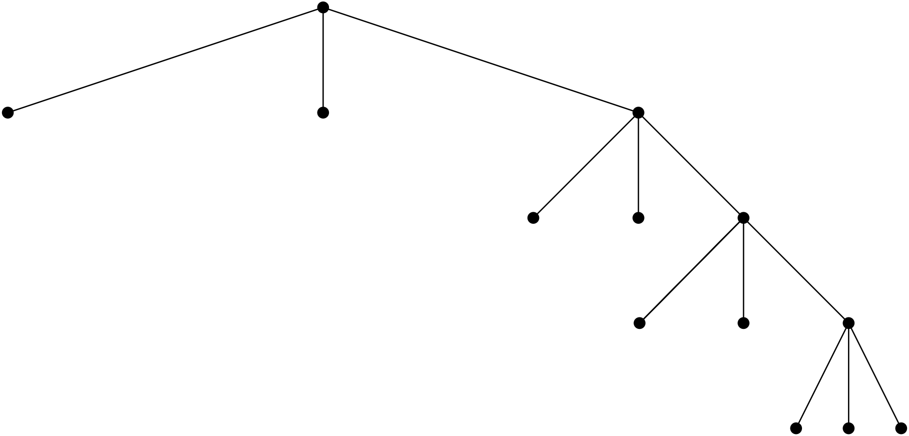

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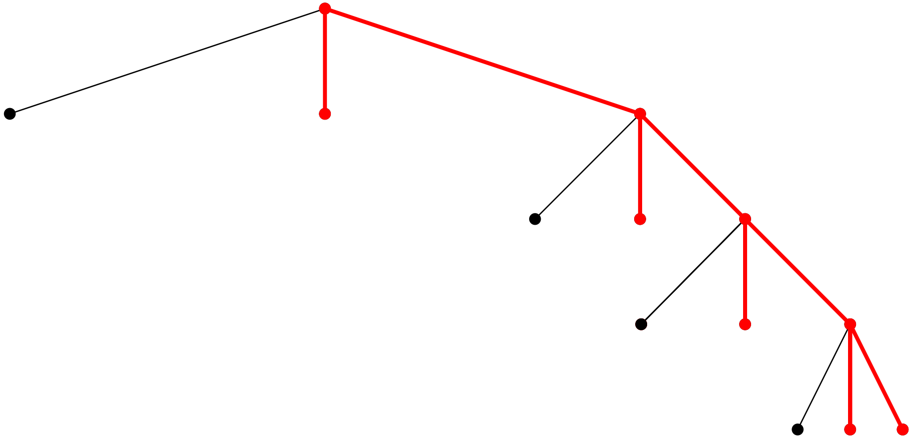
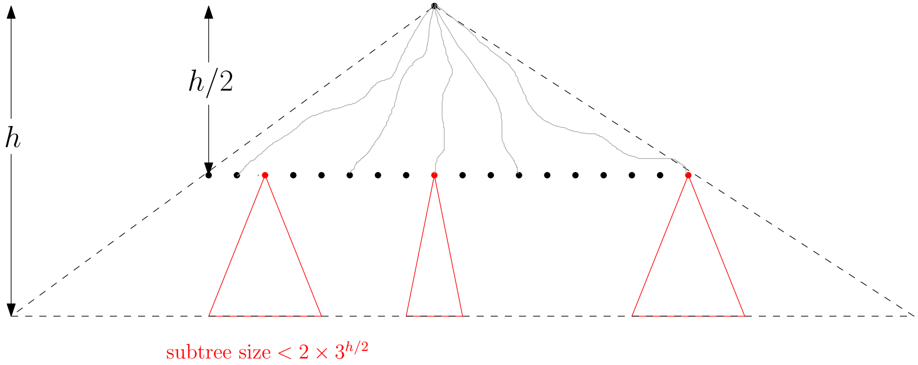


size of binary tree $= n - o(n)$

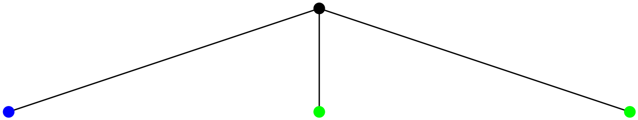
The strategy



size of any binary subtree

$$< 2 \times 3^{h/2} + 2^{h/2} \times 2 \times 3^{h/2} < O\left(3^{\log_3 2 \times h/2 + h/2}\right) < O\left(3^{0.82h}\right) < n^{0.83}$$

Analyzing subtree sizes



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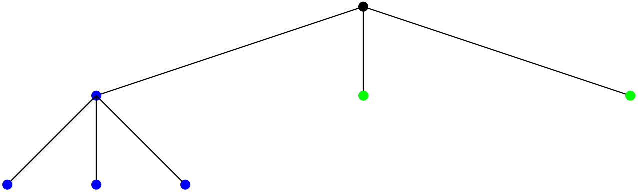

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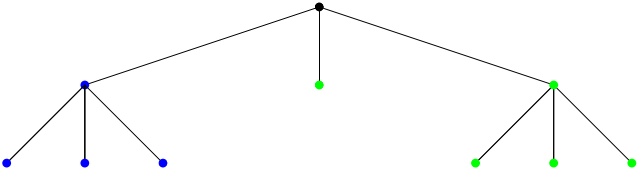

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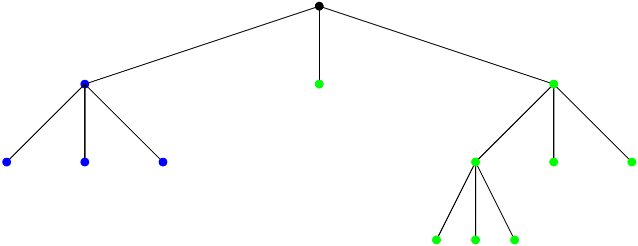

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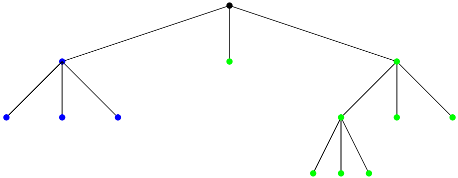

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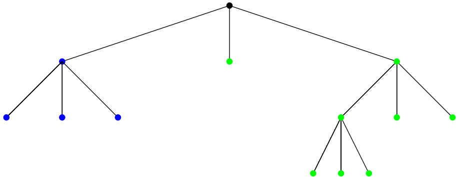

Growth rule: Start with one blue two green. In each step, choose a uniformly random leaf, and increase number of leaves of that colour by 2.

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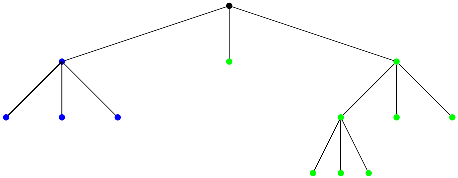

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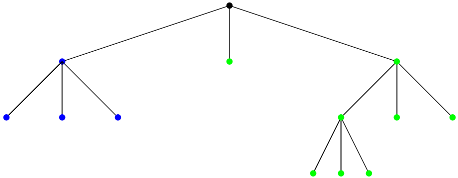

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$$\frac{\text{blue vertices}}{n-1} \xrightarrow[n\to\infty]{\text{in distribution}} \mathrm{Beta}(1/2, 1)$$

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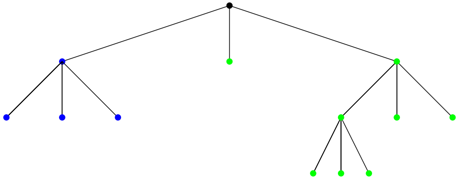

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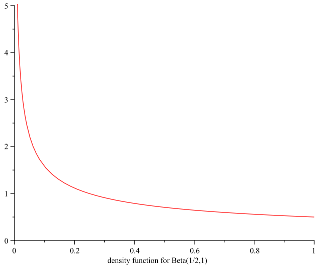

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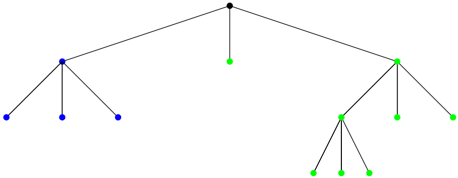

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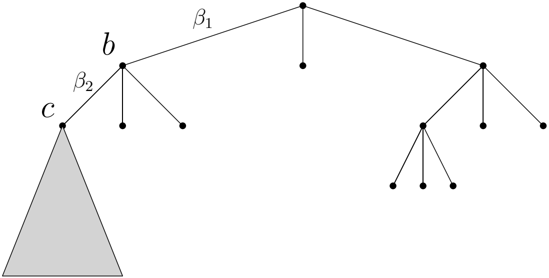
Draws from an E-P urn are exchangeable, so by de Finetti's theorem,

$$\text{blue vertices} \sim \text{Binomial}(n-1, \text{Beta}(1/2, 1))$$

analyzing subtree sizes



Suppose $\beta_1, \beta_2 \sim \mathrm{Beta}(1/2, 1)$ independent.

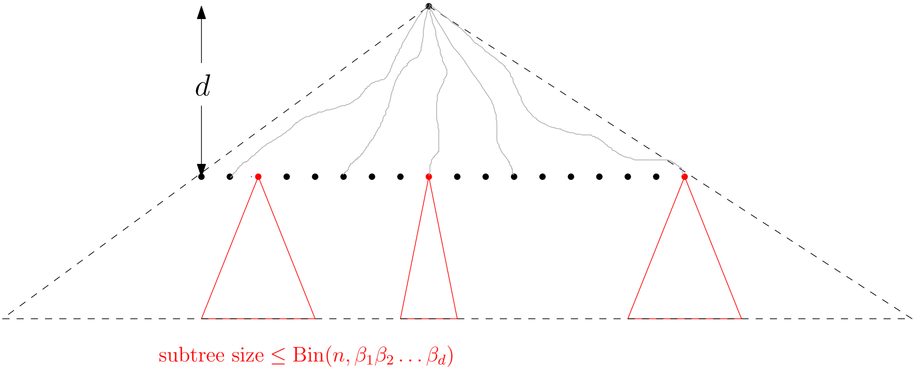
Size of subtree rooted at $b \sim \mathrm{Bin}(n-1, \beta_1)$

Size of subtree rooted at $c \sim \mathrm{Bin}(\text{size of } b, \beta_2)$

$$\sim \mathrm{Bin}(\mathrm{Bin}(n-1, \beta_1), \beta_2) \preccurlyeq \mathrm{Bin}(n, \beta_1\beta_2)$$

wrap up

goal:any binary subtree of random n-vertex ternary tree has size $\leq n^{0.9999}$



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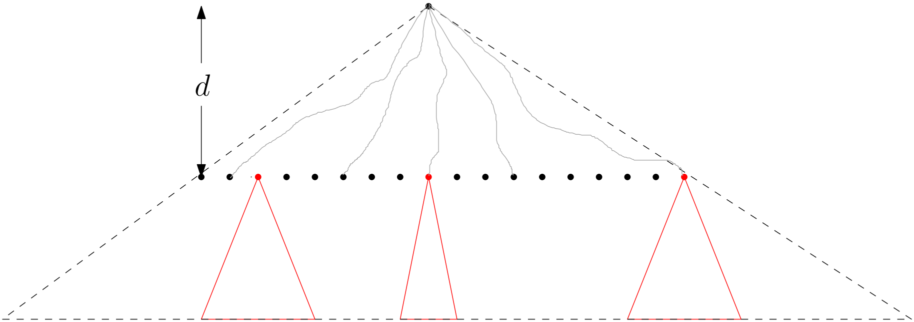

subtree size $\leq \mathrm{Bin}(n, \beta_1\beta_2 \ldots \beta_d)$

For suitable d, each red subtree size is sharply concentrated around $n/3^d$.

wrap up

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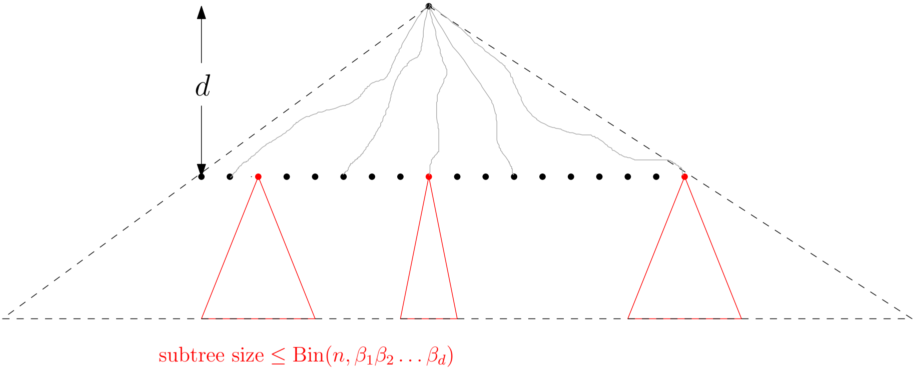

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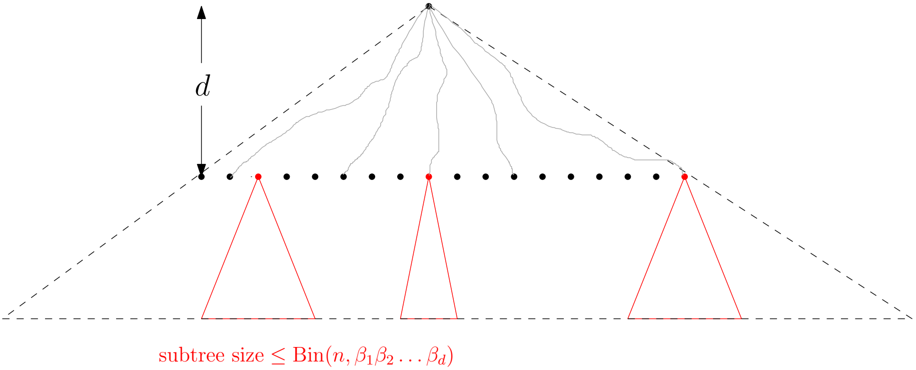
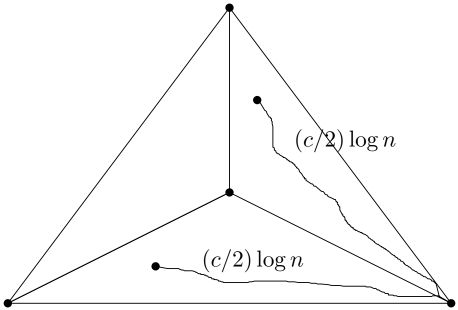


For suitable d, each red subtree size is sharply concentrated around $n/3^d$.
Apply union bound over 3^d nodes gives uniform bound for all red subtrees
Size of any binary subtree $\leq 2 \times 3^d + 2^d \times$ uniform bound $\leq n^{0.99986}$

Proof Outline for

$\frac{\text{diameter}}{\log n} \longrightarrow c \approx 1.668 \qquad$ in probability

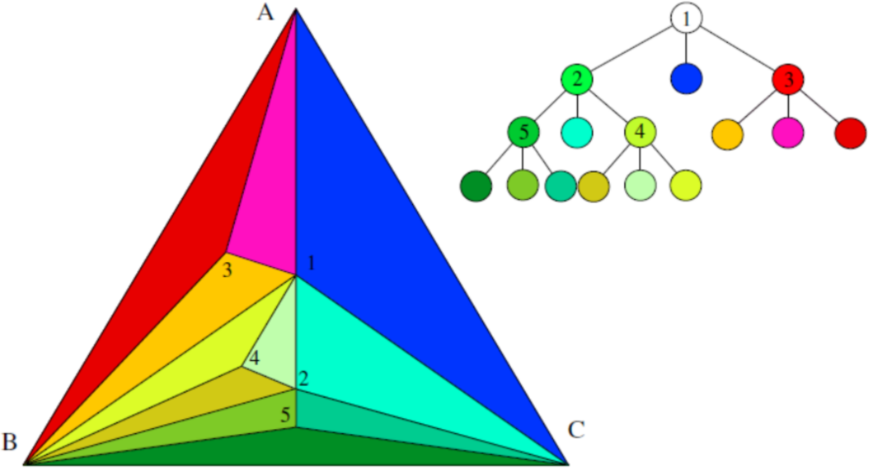
Radius



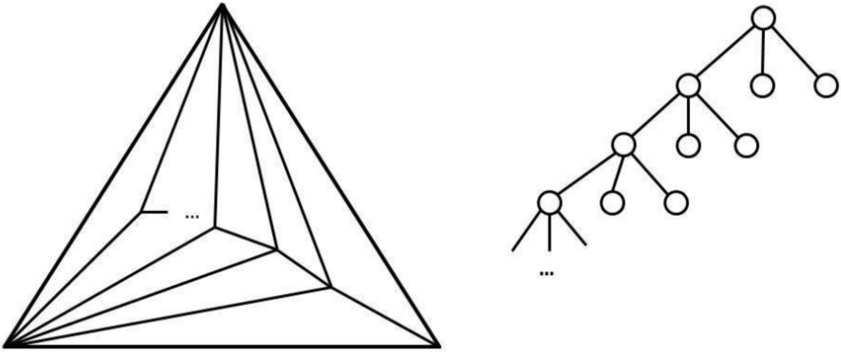
Radius : max distance between a vertex and the boundary

Lemma

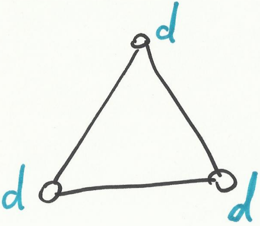
If radius $/\log n \to c/2$ *in probability,*
then diameter$/\log n \to c$ *in probability.*



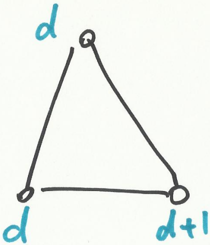
distance in graph $\leq$ distance in tree



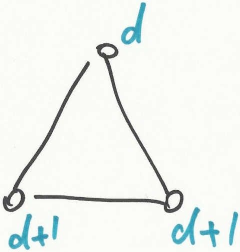
distance in graph $\ll$ distance in tree



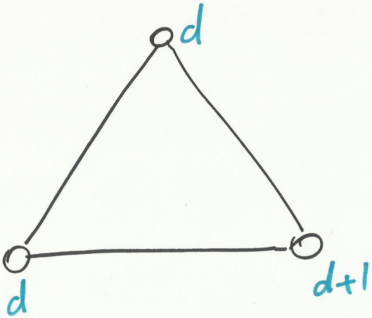
Type 1



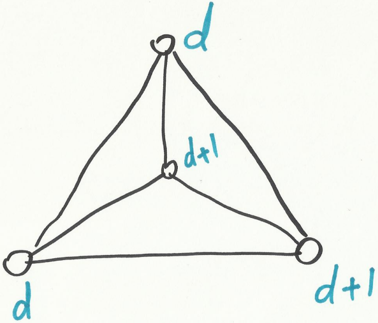
Type 2



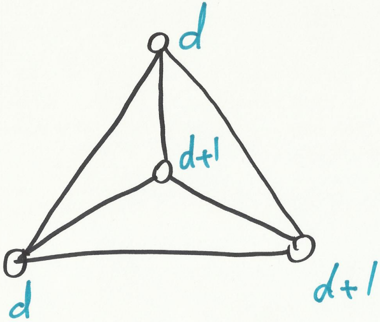
Type 3

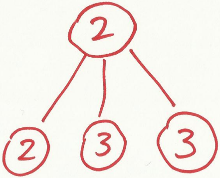


Type 2

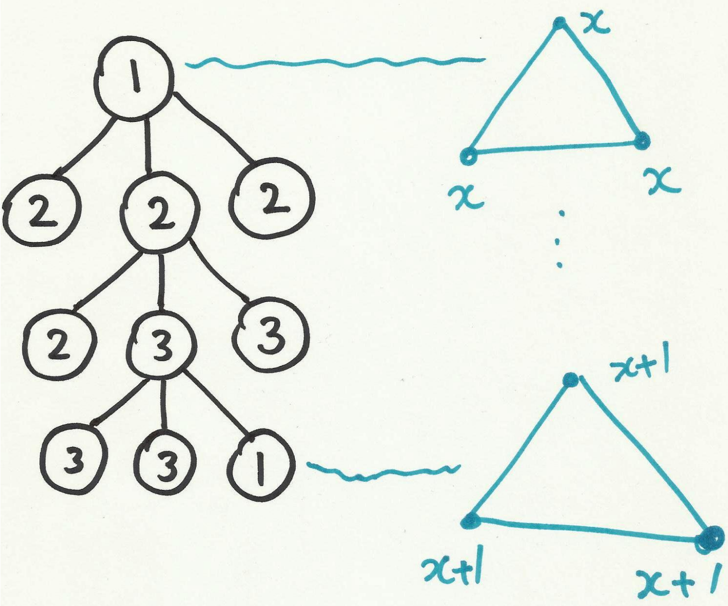


Type2

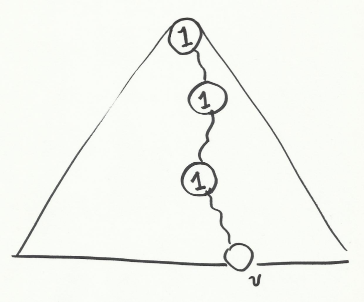




Type2

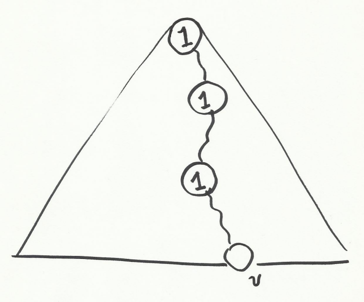

1
2
2
2
2
3
3
3
3
1
x
x
x
x+1
x+1
x+1

Crucial Observation



Distance of a vertex to the boundary (in graph) equals
number of type-1 nodes on path of the corresponding node to the root (in tree)

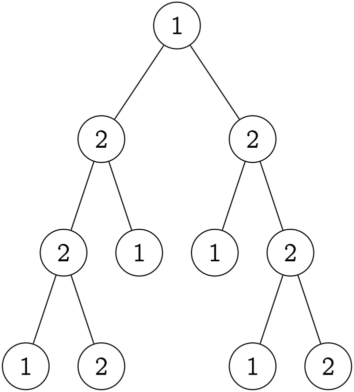
Crucial Observation



Distance of a vertex to the boundary (in graph) equals
number of type-1 nodes on path of the corresponding node to the root (in tree)
New goal: bound the largest number of type-1 nodes in any root-to-leaf path

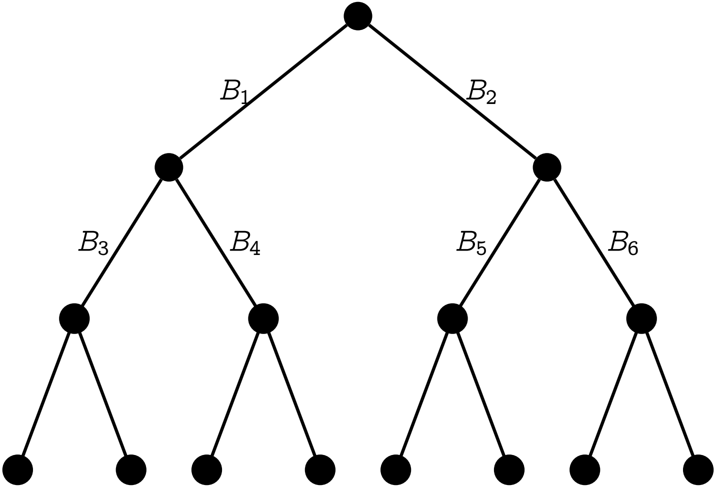
Simplified problem

Consider binary trees for simplicity:
a type-1 node has two type-2 children,
a type-2 node has on type-1 child and one type-2 child.
In every step a random leaf gives birth. After n steps,
what's the largest number of type-1 nodes in any root-to-leaf path?!



The theorem of Broutin and Devroye

Infinite binary tree:

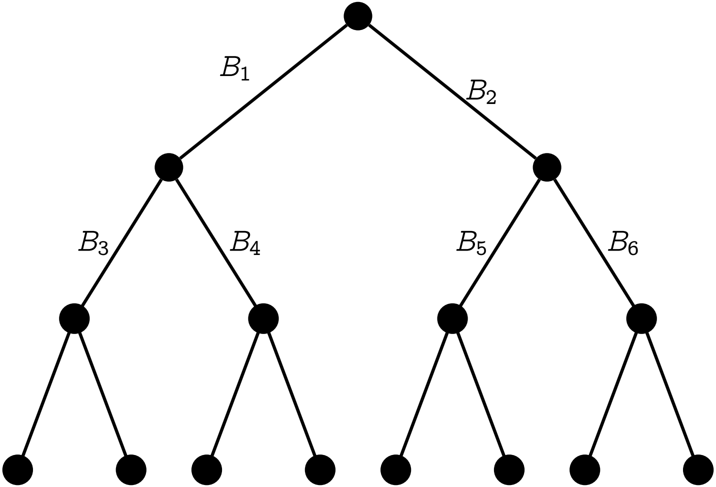


Theorem (Broutin and Devroye'06)

Assume:

- ✓ *All birth times have the same distribution.*
- ✓ *One-level offsprings of distinct vertices are mutually independent.*

Infinite binary tree:



$B_1 \sim B_2 \sim \cdots \sim B_6$ and $B_1 \perp B_3, B_4, B_5, B_6$ *etc.*

Theorem (Broutin and Devroye'06)

Assume:

- ✓ *All birth times have the same distribution.*
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Theorem (Broutin and Devroye'06)

Assume:

- ✓ *All birth times have the same distribution.*
- ✓ *One-level offsprings of distinct vertices are mutually independent.*

Then, *height* *of tree at time* t *is whp asymptotic to* ρt,
$\rho :=$ *unique solution to*

$$\sup_{\lambda \le 0}\{\lambda/\rho - \log(\mathbb{E}\left[\exp(\lambda E)\right])\} = \log 2 \,.$$

Back to random Apollonian networks...

Simplified problem

A type-1 node has two type-2 children,
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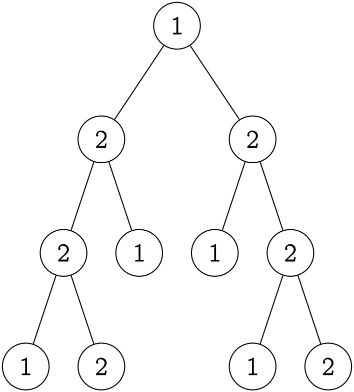
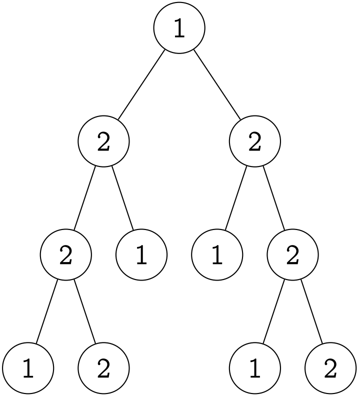
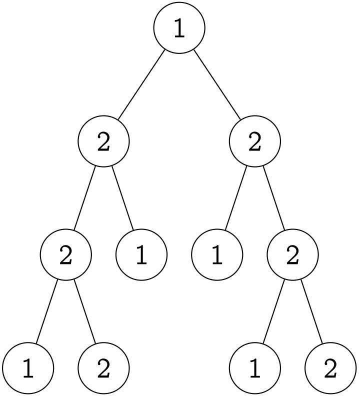
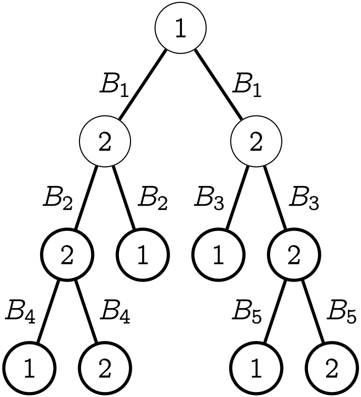
In every step a random leaf gives birth. After n steps,
what's the largest number of type-1 nodes in any root-to-leaf path?!



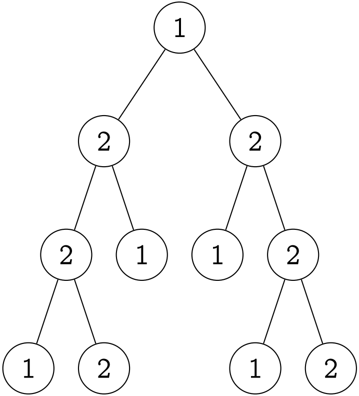


Use exponential birth times !





Use exponential birth times !


1
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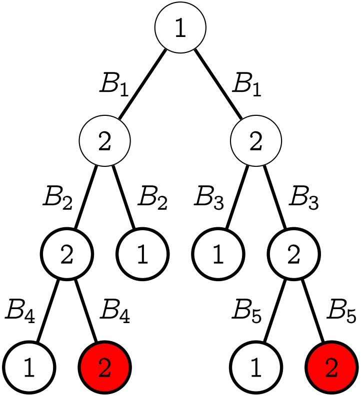
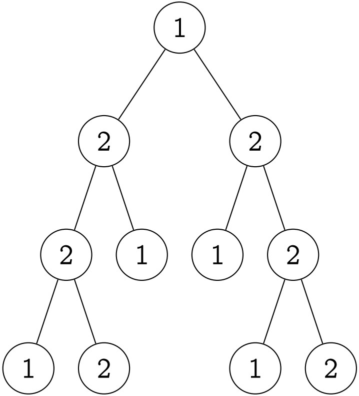

1
B_1
B_1
2
2
B_2
B_2
B_3
B_3
2
1
1
2
B_4
B_4
B_5
B_5
1
2
1
2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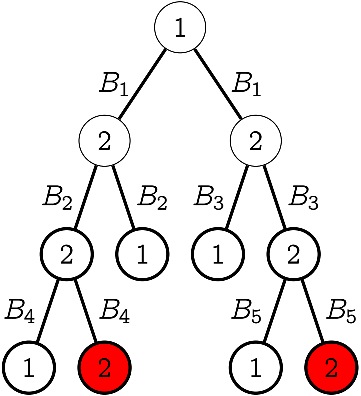
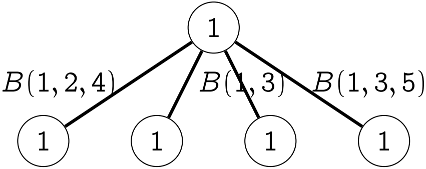
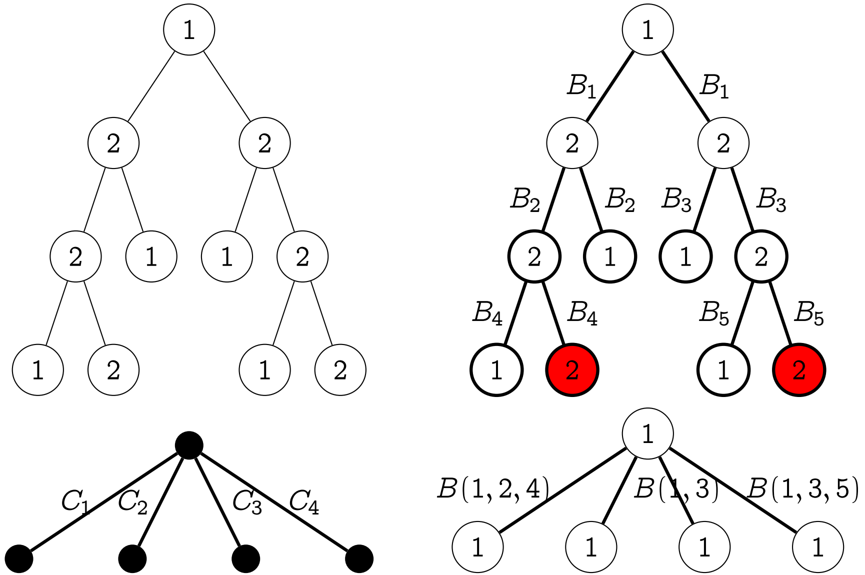

1
B_1
B_1
2
2
B_2
B_2
B_3
B_3
2
1
1
2
B_4
B_4
B_5
B_5
1
2
1
2


1
$B(1,2,4)$
$B(1,3)$
$B(1,3,5)$
1
1
1
1


1
2 2
2 1 1 2
1 2 1 2
B_1 B_1
B_2 B_2 B_3 B_3
B_4 B_4 B_5 B_5
C_1 C_2 C_3 C_4
$B(1,2,4)$ $B(1,3)$ $B(1,3,5)$

- ✓ For every fixed cut-off threshold k, we stochastically sandwich 1-height of our typed tree between heights of B&D-friendly trees.
- ✓ As $k \to \infty$, lower and upper bounds converge to $(c/2)\log n$.

Theorem (EFGMSWZ'13)

$$f(x) := \frac{12x^3}{1-2x} - \frac{6x^3}{1-x},$$

$y :=$ *unique solution to*

$$x(x-1)f'(x) = f(x)\log f(x), \quad x \in (0, 1/2),$$

$$c := (1 - y^{-1})/\log f(y) \approx 1.668$$

Then for every fixed $\varepsilon > 0$,

$$\mathbb{P}\left[(1-\varepsilon)c\log n \le \text{diameter of a RAN} \le (1+\varepsilon)c\log n\right] \to 1$$

Eggenberger-Pólya Urn

Theorem (Eggenberger and Pólya 1923)

Start: g green, r red balls.

In each step:

- ✓ *pick a random ball and return it to the urn;*
- ✓ *add s balls of the same colour.*

After n draws:

g_n: green balls

t_n: number of balls

For any $\alpha \in [0, 1]$

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{g_n}{t_n} < \alpha\right] = \frac{\Gamma((g+r)/s)}{\Gamma(g/s)\Gamma(r/s)} \int_0^\alpha x^{\frac{g}{s}-1}(1-x)^{\frac{r}{s}-1}\,\mathrm{d}x$$

$$= \mathbb{P}\left[Beta(g/s, r/s) < \alpha\right]$$

Broutin-Devroye's Theorem

Theorem (Broutin and Devroye 2006)

$E :=$ a positive random variable

$b :=$ a positive integer

$T_\infty :=$ an infinite b-ary tree.

Label the edges of T_∞ randomly,

1. *The label of every edge is distributed like E.*
2. *For vertices u and v, edges going down from u and v are independent.*

$H_t :=$ height of the subtree containing nodes whose sum of labels on their path to root $\leq t$.

Then $\frac{H_t}{t} \to \rho$ in probability

$\rho :=$ unique solution to

$$\sup\{\lambda/\rho - \log(\mathbb{E}\left[\exp(\lambda E)\right]) : \lambda \leq 0\} = \log b\ .$$