On the Density of Nearly Regular Graphs with a Good Edge-Labelling

Abbas Mehrabian

amehrabi@uwaterloo.ca

University of Waterloo

Ryerson University
April 5th, 2012

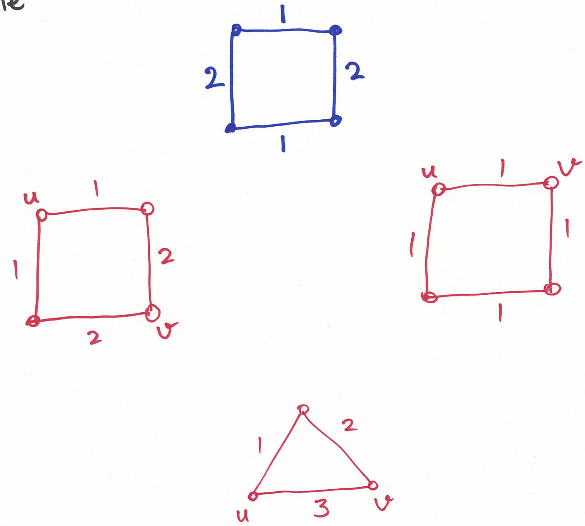
What is a *Good Edge-Labelling*?

Graphs are simple and undirected and have n vertices.

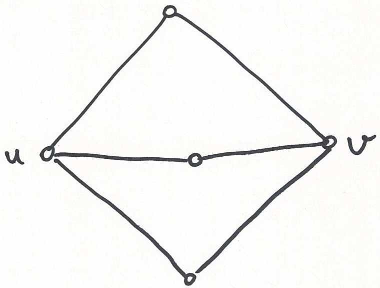
Definition (Bermond, Cosnard, and Pérennes 2009)

A good edge-labelling is a labelling of edges with integers such that for any ordered pair (u, v), there is at most one non-decreasing path from u to v.

Example



Example



$K_{2,3}$

What is a *good* graph?

Definition

A graph is good if it admits a good edge-labelling; otherwise it is bad.

Example

C_4 is good, K_3 and $K_{2,3}$ are bad.

Density of Good Graphs

Question

What is the maximum number of edges of a good graph?

Araújo, Cohen, Giroire, and Havet (2009) observed:

$$\Omega(n \log n) \leq \gamma(n) \leq O(n^{3/2}) \, .$$

Today we will see that a good regular graph has $n^{1+o(1)}$ edges.

Density of Good Graphs

Question

What is the maximum number of edges of a good graph?

Araújo, Cohen, Giroire, and Havet (2009) observed:

$$\Omega(n \log n) \leq \gamma(n) \leq O(n^{3/2}) .$$

Today we will see that a good regular graph has $n^{1+o(1)}$ edges.

Girth of Bad Graphs

Question [Bode, Farzad, and Theis 2011]

Is having a small girth the obstacle for being good?

NO! We will see there exist bad graphs with arbitrarily large girth.
I believe having too many (superlinear) edges is the obstacle.

Girth of Bad Graphs

Question [Bode, Farzad, and Theis 2011]

Is having a small girth the obstacle for being good?

NO! We will see there exist bad graphs with arbitrarily large girth.
I believe having too many (superlinear) edges is the obstacle.

The Main Result

Theorem (M 2012+)

For any integer t, there exists $\epsilon(t) > 0$ such that any d-regular graph with $\epsilon(t)d^t > n$ is bad.

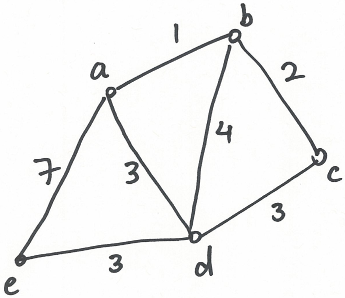
The Approach

Consider an arbitrary labelling of the graph,
and show there exist $> n^2$ non-decreasing paths.

Definition

A nice k-walk is a nondecreasing non-backtracking walk of length k.

Example



abcd

cba

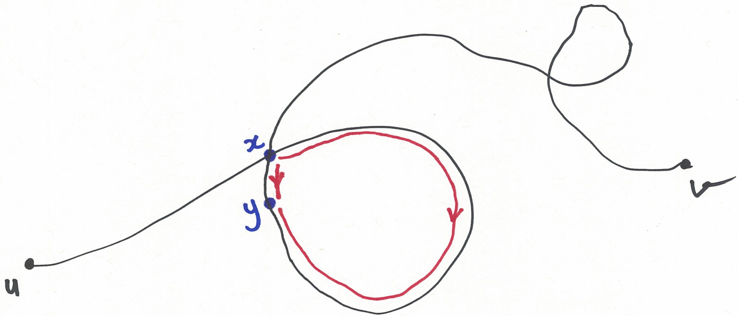
adea

bcdc

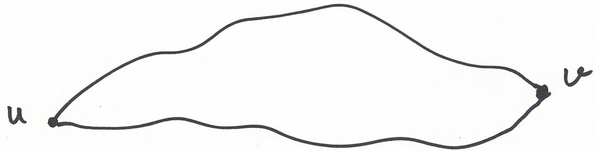
bd

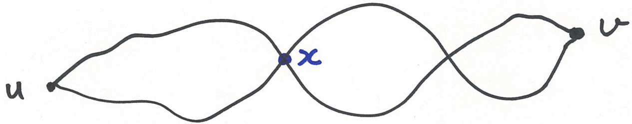
db

Claim. Self-intersecting nice walk means labelling is not good.



Claim. Two nice walks from u to v means labelling is not good.



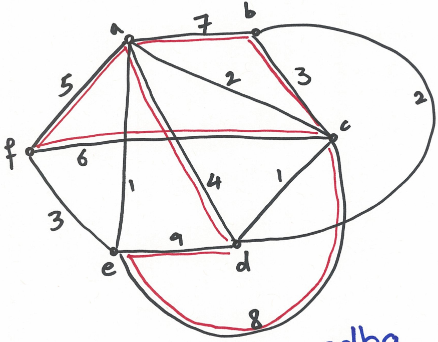


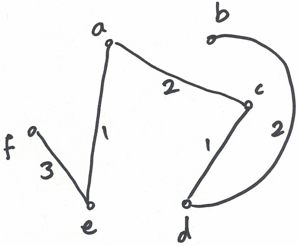
The Approach

Proposition

If for some k there exist $> n^2$ nice k-walks,
then the labelling is not good.

The Strategy





cdba
cdbc

cdb

25 nice 3-walks

5 nice 2-walks

The Main Lemma

Definition

Let $f_k(= n, \geq m, \leq \Delta)$ be the minimum number of nice k-walks of a graph with n vertices, at least m edges, and max degree at most Δ.

Example

- $f_1(= n, \geq m, \leq \Delta) = 2m$
- $f_2(= n, \geq \frac{nd}{2}, \leq d) \geq n\binom{d}{2}$

The Main Lemma

Definition

Let $f_k(= n, \geq m, \leq \Delta)$ be the minimum number of nice k-walks of a graph with n vertices, at least m edges, and max degree at most Δ.

Example

- $f_1(= n, \geq m, \leq \Delta) = 2m$
- $f_2(= n, \geq \frac{nd}{2}, \leq d) \geq n\binom{d}{2}$

The Main Lemma

Lemma

For any positive integer $a \leq \Delta/2$ we have

$$f_k(n, m, \Delta) \geq a\left[f_{k-1}(n, m - an, \Delta - a) - (n\Delta - 2m)a(\Delta - a)^{k-3}\right].$$

Proof.

If G has all degrees $\geq a$, then

$$\# \text{ of nice } k\text{−walks in } G \geq a\left[f_{k-1}(n, m - an, \Delta - a)\right].$$ □

The Main Lemma

Lemma

For any positive integer $a \leq \Delta/2$ we have

$$f_k(n, m, \Delta) \geq a\left[f_{k-1}(n, m-an, \Delta-a) - (n\Delta - 2m)a(\Delta-a)^{k-3}\right].$$

Proof.

If G has all degrees $\geq a$, then

$$\#\text{ of nice } k\text{−walks in } G \geq a\left[f_{k-1}(n, m-an, \Delta-a)\right]. \qquad \square$$

The Main Lemma

Lemma

For any positive integer $a \leq \Delta/2$ we have

$$f_k(n, m, \Delta) \geq a \left[f_{k-1}(n, m - an, \Delta - a) - (n\Delta - 2m) a (\Delta - a)^{k-3} \right].$$

Proof.

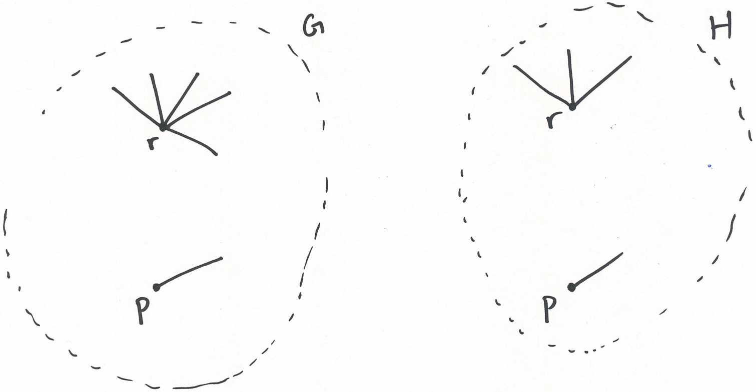
Say vertex v is poor if its degree $< a$, rich otherwise. Let b be the number of poor vertices. Then

$$b \times a + (n - b) \times \Delta \geq 2m ,$$

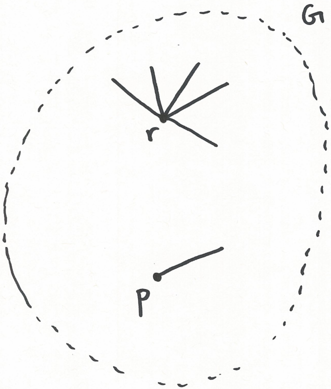
so

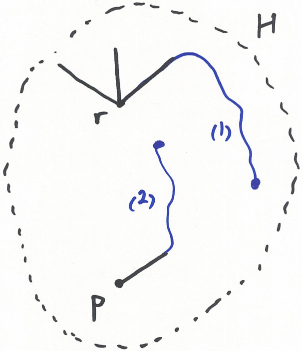
$$b \leq \frac{n\Delta - 2m}{\Delta - a} . \quad \square$$

Proof (Cont'd)



Proof (Cont'd)





$\geq f_{k-1}(n, \geq m - an, \leq \Delta - a)$ nice $(k-1)$-walks

$\leq b\,a\,(\Delta - a)^{k-2}$ $(k-1)$-walks of type (2)

The Main Lemma

Lemma

For any positive integer $a \leq \Delta/2$ we have

$$f_k(n, m, \Delta) \geq a\left[f_{k-1}(n, m-an, \Delta-a) - (n\Delta - 2m)a(\Delta-a)^{k-3}\right].$$

Proof.

$$f_k(n, m, \Delta) \geq a\left[f_{k-1}(n, m-an, \Delta-a) - ba(\Delta-a)^{k-2}\right].$$

Since

$$b \leq \frac{n\Delta - 2m}{\Delta - a},$$

we get

$$f_k(n, m, \Delta) \geq a\left[f_{k-1}(n, m-an, \Delta-a) - (n\Delta - 2m)a(\Delta-a)^{k-3}\right].$$

□

The Main Lemma

Lemma

For any positive integer $a \leq \Delta/2$ we have

$$f_k(n, m, \Delta) \geq a\left[f_{k-1}(n, m - an, \Delta - a) - (n\Delta - 2m)a(\Delta - a)^{k-3}\right].$$

Letting $a = q\Delta$ where $q \in (0, \frac{1}{2})$ and $p = 1 - q$ gives

$$f_k(n, m, \Delta) \geq q\Delta f_{k-1}(n, m - qn\Delta, p\Delta) - q^2 p^{k-3}\Delta^{k-1}(n\Delta - 2m).$$

The Main Lemma

Lemma

For any positive integer $a \leq \Delta/2$ we have

$$f_k(n, m, \Delta) \geq a\left[f_{k-1}(n, m - an, \Delta - a) - (n\Delta - 2m)a(\Delta - a)^{k-3}\right].$$

Letting $a = q\Delta$ where $q \in (0, \frac{1}{2})$ and $p = 1 - q$ gives

$$f_k(n, m, \Delta) \geq q\Delta f_{k-1}(n, m - qn\Delta, p\Delta) - q^2 p^{k-3} \Delta^{k-1} (n\Delta - 2m).$$

A Recursive Formula

Letting $a = q\Delta$ where $q \in (0, \frac{1}{2})$ and $p = 1 - q$ gives

$$f_k(n, m, \Delta) \geq q\Delta f_{k-1}(n, m - qn\Delta, p\Delta) - q^2 p^{k-3} \Delta^{k-1} (n\Delta - 2m) \,.$$

Using $f_1(n, m, \leq \Delta) \geq m$ and induction, this gives

$$f_k(n, m, \Delta) \geq a_k m \Delta^{k-1} - b_k n \Delta^k \,,$$

where

$$\begin{aligned} a_k &= qp^{k-2} a_{k-1} + 2q^2 p^{k-3} & a_1 &= 1 \,, \\ b_k &= q^2 p^{k-2} a_{k-1} + qp^{k-1} b_{k-1} + q^2 p^{k-3} & b_1 &= 0 \,. \end{aligned}$$

The ratio a_k / b_k is decreasing, and the rate is determined by q.

A Recursive Formula

Letting $a = q\Delta$ where $q \in (0, \frac{1}{2})$ and $p = 1 - q$ gives

$$f_k(n, m, \Delta) \geq q\Delta f_{k-1}(n, m - qn\Delta, p\Delta) - q^2 p^{k-3} \Delta^{k-1} (n\Delta - 2m) \, .$$

Using $f_1(n, m, \leq \Delta) \geq m$ and induction, this gives

$$f_k(n, m, \Delta) \geq a_k m \Delta^{k-1} - b_k n \Delta^k \; ,$$

where

$$\begin{aligned} a_k &= qp^{k-2} a_{k-1} + 2q^2 p^{k-3} & a_1 &= 1 \, , \\ b_k &= q^2 p^{k-2} a_{k-1} + qp^{k-1} b_{k-1} + q^2 p^{k-3} & b_1 &= 0 \, . \end{aligned}$$

The ratio a_k / b_k is decreasing, and the rate is determined by q.

A Recursive Formula

Letting $a = q\Delta$ where $q \in (0, \frac{1}{2})$ and $p = 1 - q$ gives

$$f_k(n, m, \Delta) \geq q\Delta f_{k-1}(n, m - qn\Delta, p\Delta) - q^2 p^{k-3} \Delta^{k-1}(n\Delta - 2m)\,.$$

Using $f_1(n, m, \leq \Delta) \geq m$ and induction, this gives

$$f_k(n, m, \Delta) \geq a_k m\Delta^{k-1} - b_k n\Delta^k\,,$$

where

$$\begin{aligned} a_k &= qp^{k-2} a_{k-1} + 2q^2 p^{k-3} & a_1 &= 1\,, \\ b_k &= q^2 p^{k-2} a_{k-1} + qp^{k-1} b_{k-1} + q^2 p^{k-3} & b_1 &= 0\,. \end{aligned}$$

The ratio a_k / b_k is decreasing, and the rate is determined by q.

A Technial Lemma

Lemma

Let $(a_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$ be sequences, defined as

$$\begin{aligned} a_k &= qp^{k-2}a_{k-1} + 2q^2p^{k-3} & a_1 &= 1\,, \\ b_k &= q^2p^{k-2}a_{k-1} + qp^{k-1}b_{k-1} + q^2p^{k-3} & b_1 &= 0\,. \end{aligned}$$

Then for any positive integer t, there is a $q(t) > 0$ such that for $0 < q \le q(t)$,

$$a_t > 2b_t\,.$$

The Main Theorem

Theorem

For any positive t, there exists a positive $\epsilon(t)$ such that any d-regular n-vertex graph G with $\epsilon(t)d^t > n$ is bad.

Proof.

Consider an arbitrary edge-labelling. Pick q small enough so that $a_t > 2b_t$. Then

$$\epsilon(t) := \frac{a_t}{2} - b_t > 0 .$$

Hence

$$f_t\left(n, \frac{nd}{2}, d\right) \geq a_t \left(\frac{nd}{2}\right) d^{t-1} - b_t nd^t$$
$$= nd^t \left(\frac{a_t}{2} - b_t\right) = nd^t \epsilon(t) > n^2 . \qquad \square$$

The Main Theorem

Theorem

For any positive t, there exists a positive $\epsilon(t)$ such that any d-regular n-vertex graph G with $\epsilon(t)d^t > n$ is bad.

Proof.

Consider an arbitrary edge-labelling. Pick q small enough so that $a_t > 2b_t$. Then

$$\epsilon(t) := \frac{a_t}{2} - b_t > 0\,.$$

Hence

$$\begin{aligned} f_t\left(n, \frac{nd}{2}, d\right) &\geq a_t\left(\frac{nd}{2}\right)d^{t-1} - b_t nd^t \\ &= nd^t\left(\frac{a_t}{2} - b_t\right) = nd^t\epsilon(t) > n^2\,. \end{aligned}$$

$\square$

The First Corollary

Corollary

Let $\gamma_r(n)$ be the maximum number of edges of a good regular graph. Then

$$\gamma_r(n) \leq n^{1+o(1)} .$$

Proof.

Consider a sequence of regular graphs with at least $n^{1+\frac{1}{k}}$ edges, k fixed. Then their degree is going to infinity with n.

Consider a d-regular graph in this sequence with $d > 1/\epsilon(k+1)$. Then for this graph

$$\epsilon(k+1)d^{k+1} > d^k = \left(\frac{2n^{1+\frac{1}{k}}}{n}\right)^k = 2^k n > n . \quad \square$$

The First Corollary

Corollary

Let $\gamma_r(n)$ be the maximum number of edges of a good regular graph. Then

$$\gamma_r(n) \leq n^{1+o(1)} .$$

Proof.

Consider a sequence of regular graphs with at least $n^{1+\frac{1}{k}}$ edges, k fixed. Then their degree is going to infinity with n.

Consider a d-regular graph in this sequence with $d > 1/\epsilon(k+1)$. Then for this graph

$$\epsilon(k+1)d^{k+1} > d^k = \left(\frac{2n^{1+\frac{1}{k}}}{n}\right)^k = 2^k n > n . \quad \square$$

The First Corollary

Corollary

Let $\gamma_r(n)$ be the maximum number of edges of a good regular graph. Then

$$\gamma_r(n) \leq n^{1+o(1)} .$$

Proof.

Consider a sequence of regular graphs with at least $n^{1+\frac{1}{k}}$ edges, k fixed. Then their degree is going to infinity with n.

Consider a d-regular graph in this sequence with $d > 1/\epsilon(k+1)$.

Then for this graph

$$\epsilon(k+1)d^{k+1} > d^k = \left(\frac{2n^{1+\frac{1}{k}}}{n}\right)^k = 2^k n > n . \quad \square$$

The Second Corollary

Corollary

For any g, there exists a bad graph with girth $\geq g$.

Proof.

Lazebnik, Ustimenko, and Woldar (1997) proved that for any odd prime power d, there exists a d-regular graph with girth g with $< 2d^{\frac{3}{4}g}$ vertices.

Let $d > 2/\epsilon(g)$. Then

$$\epsilon(g)d^g > 2d^{g-1} > 2d^{\frac{3}{4}g} > n\,. \quad \square$$

The Second Corollary

Corollary

For any g, there exists a bad graph with girth $\geq g$.

Proof.

Lazebnik, Ustimenko, and Woldar (1997) proved that for any odd prime power d, there exists a d-regular graph with girth g with $< 2d^{\frac{3}{4}g}$ vertices.

Let $d > 2/\epsilon(g)$. Then

$$\epsilon(g)d^g > 2d^{g-1} > 2d^{\frac{3}{4}g} > n\,. \qquad \square$$

The Second Corollary

Corollary

For any g, there exists a bad graph with girth $\geq g$.

Proof.

Lazebnik, Ustimenko, and Woldar (1997) proved that for any odd prime power d, there exists a d-regular graph with girth g with $< 2d^{\frac{3}{4}g}$ vertices.
Let $d > 2/\epsilon(g)$. Then

$$\epsilon(g)d^g > 2d^{g-1} > 2d^{\frac{3}{4}g} > n\,. \qquad \square$$

A Result in The Other Direction

Theorem (M 2012+)

Any graph with max degree Δ and girth $\geq 2k$ such that

$$4ek^2(\Delta - 1)^{k-1} < k!$$

is good.

Corollary

Any graph with max degree Δ and girth $\geq 40\Delta$ is good.

A Result in The Other Direction

Theorem

Any graph with max degree Δ and girth $\geq 2k$ such that

$$4ek^2(\Delta - 1)^{k-1} < k!$$

is good.

Proof.

1. The label of each edge: uniform from $[0, 1]$.
2. Labelling not good $\Rightarrow \exists$ nondecreasing path of length exactly k (k-path).
3. For any k-path P, $\mathbf{Pr}[P \text{ non}-\text{decreasing}] = \frac{2}{k!}$.
4. Any k-path intersects at most $2k^2(\Delta - 1)^{k-1}$ other k-paths.

□

A Result in The Other Direction

Theorem

Any graph with max degree Δ and girth $\geq 2k$ such that

$$4ek^2(\Delta-1)^{k-1} < k!$$

is good.

Proof.

1. The label of each edge: uniform from $[0,1]$.
2. Labelling not good $\Rightarrow \exists$ nondecreasing path of length exactly k (k-path).
3. For any k-path P, $\mathbf{Pr}[P \text{ non}-\text{decreasing}] = \frac{2}{k!}$.
4. Any k-path intersects at most $2k^2(\Delta-1)^{k-1}$ other k-paths.

A Result in The Other Direction

Theorem

Any graph with max degree Δ and girth $\geq 2k$ such that

$$4ek^2(\Delta - 1)^{k-1} < k!$$

is good.

Proof.

1. The label of each edge: uniform from $[0, 1]$.
2. Labelling not good $\Rightarrow \exists$ nondecreasing path of length exactly k (k-path).
3. For any k-path P, $\mathbf{Pr}[P \text{ non} - \text{decreasing}] = \frac{2}{k!}$.
4. Any k-path intersects at most $2k^2(\Delta - 1)^{k-1}$ other k-paths.

A Result in The Other Direction

Theorem

Any graph with max degree Δ and girth $\geq 2k$ such that

$$4ek^2(\Delta - 1)^{k-1} < k!$$

is good.

Proof.

1. The label of each edge: uniform from $[0, 1]$.
2. Labelling not good $\Rightarrow \exists$ nondecreasing path of length exactly k (k-path).
3. For any k-path P, $\mathbf{Pr}[P \text{ non} - \text{decreasing}] = \frac{2}{k!}$.
4. Any k-path intersects at most $2k^2(\Delta - 1)^{k-1}$ other k-paths.

A Result in The Other Direction

Theorem

Any graph with max degree Δ and girth $\geq 2k$ such that

$$4ek^2(\Delta - 1)^{k-1} < k!$$

is good.

Proof.

1. The label of each edge: uniform from $[0, 1]$.
2. Labelling not good $\Rightarrow \exists$ nondecreasing path of length exactly k (k-path).
3. For any k-path P, $\mathbf{Pr}[P \text{ non} - \text{decreasing}] = \frac{2}{k!}$.
4. Any k-path intersects at most $2k^2(\Delta - 1)^{k-1}$ other k-paths.

□

Maximum Density of Good Graphs

Recall that

$$\Omega(n \log n) \leq \gamma(n) \leq O(n^{3/2}) \, .$$

We showed for graphs with $\Delta = \overline{d}$ (i.e. regular graphs),

$$m = n^{1+o(1)} \, .$$

This result can be extended to graphs with $\Delta \leq c\overline{d}$, for fixed c.

Open Problem 1

What about general graphs?

Maximum Density of Good Graphs

Recall that

$$\Omega(n \log n) \leq \gamma(n) \leq O(n^{3/2}) .$$

We showed for graphs with $\Delta = \overline{d}$ (i.e. regular graphs),

$$m = n^{1+o(1)} .$$

This result can be extended to graphs with $\Delta \leq c\overline{d}$, for fixed c.

Open Problem 1

What about general graphs?

Finding Good Graphs

We showed any graph with max degree Δ and girth $\geq 40\Delta$ is good.

Open Problem 2

Improve the dependence on Δ !

Open Problem 3

Find other nice conditions that guarantee that a graph is good (note that the problem is NP-hard).

Finding Good Graphs

We showed any graph with max degree Δ and girth $\geq 40\Delta$ is good.

Open Problem 2

Improve the dependence on Δ !

Open Problem 3

Find other nice conditions that guarantee that a graph is good (note that the problem is NP-hard).

Finding Good Graphs

We showed any graph with max degree Δ and girth $\geq 40\Delta$ is good.

Open Problem 2

Improve the dependence on Δ !

Open Problem 3

Find other nice conditions that guarantee that a graph is good (note that the problem is NP-hard).

Thank you Slide

Thanks for your attention :-)