The diameters of two random graph models

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Random Structures and Algorithms
Poznań

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- ✓ Nick Wormald (Waterloo and Monash)
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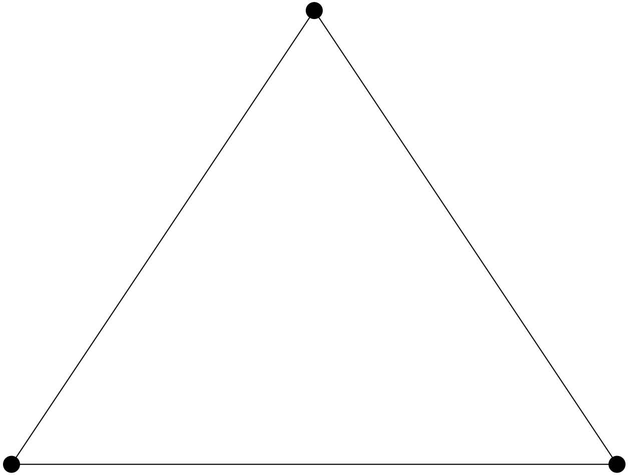
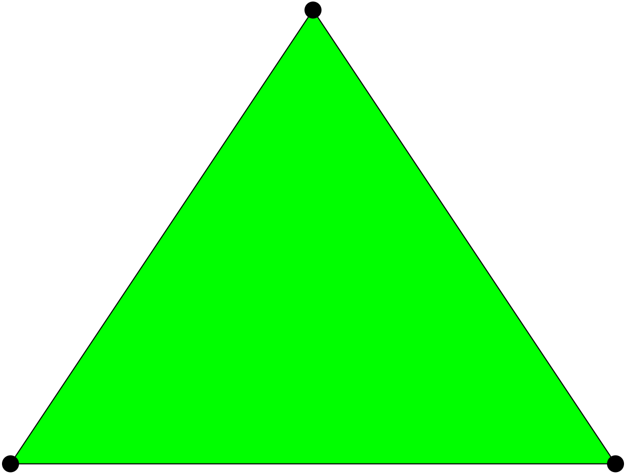
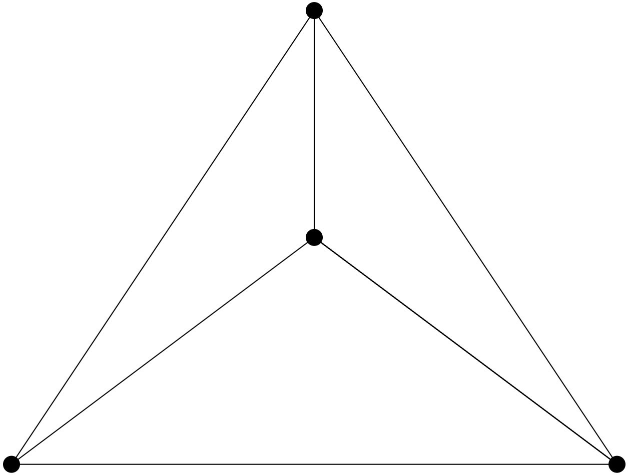
Random apollonian networks



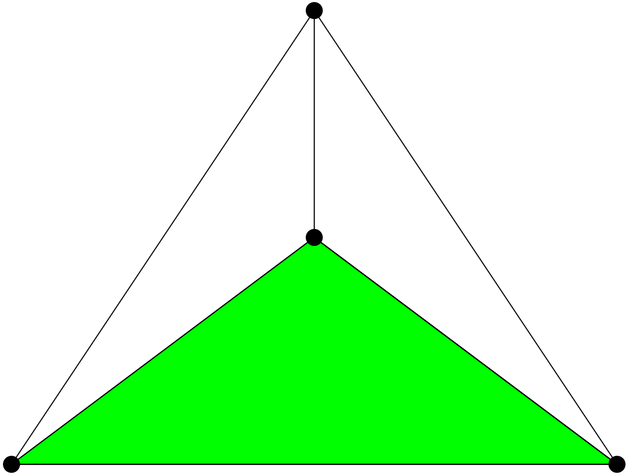
$t = 0$



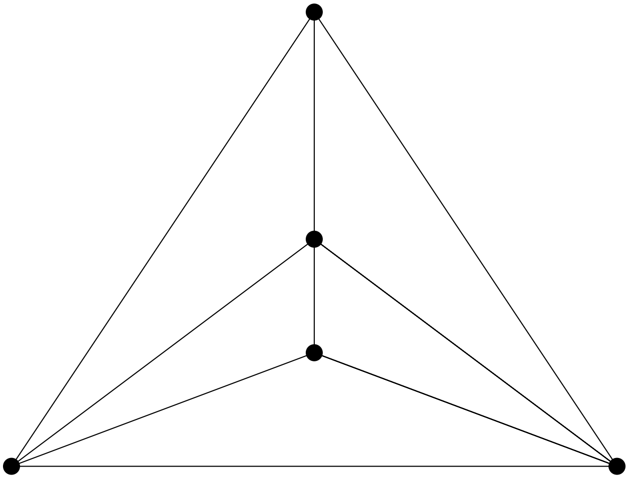
$t = 0$



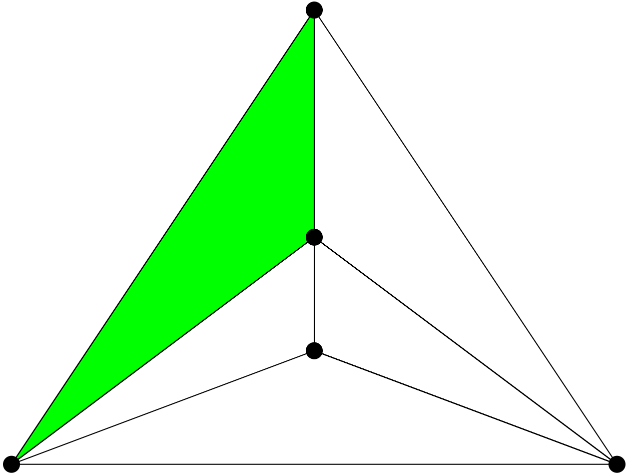
$t = 1$



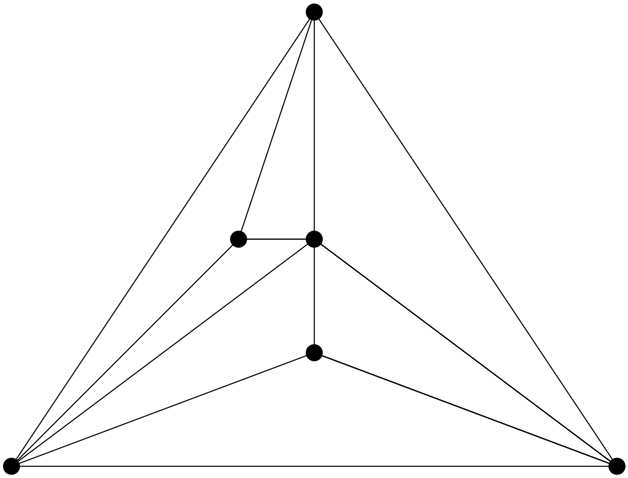
$t = 1$



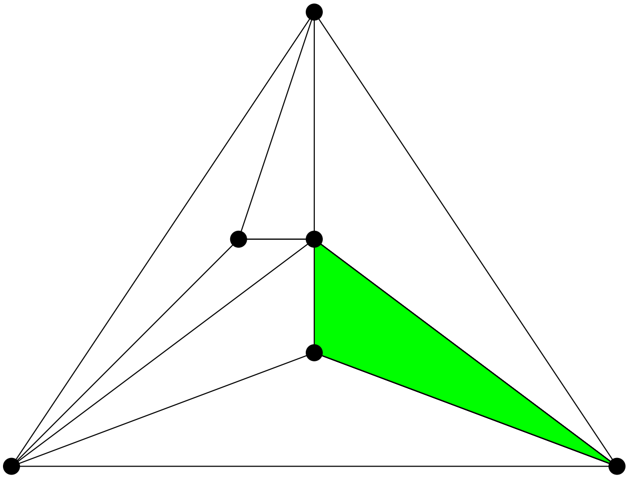
$t = 2$



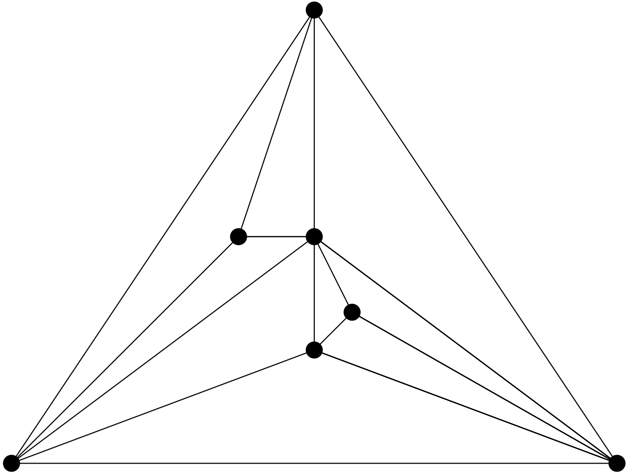
$$t = 2$$



$t = 3$



$t = 3$



$t = 4$

After t steps,

- ✓ a random triangulated plane graph
- ✓ $n = t + 3$ vertices
- ✓ $3t + 3$ edges
- ✓ $2t + 1$ faces

called a Random Apollonian Network (RAN).

Zhou, Yan, Wang'05: generating real-world planar graphs.

Known results

Theorem (Albenque and Marckert'08)

Distance between two random vertices $\to 0.55 \log n$.

Theorem (Frieze and Tsourakakis'12)

$$\mathbb{P}\left[\text{diameter} > 7.1 \log n\right] \to 0$$

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Theorem (EFGMSWZ'13+)

$$\frac{\text{diameter}}{\log n} \to c \approx 1.668 \qquad \text{in probability}$$

Our result

Theorem (EFGMSWZ'13+)

$$f(x) := \frac{12x^3}{1-2x} - \frac{6x^3}{1-x},$$

$y :=$ unique solution to

$$x(x-1)f'(x) = f(x)\log f(x), \quad x \in (0, 1/2),$$

$$c := (1 - y^{-1})/\log f(y) \approx 1.668$$

Then for every fixed $\varepsilon > 0$,

$$\mathbb{P}\left[(1-\varepsilon)c\log n \le \text{diameter} \le (1+\varepsilon)c\log n\right] \to 1$$

Parallel to this work

- ✓ Cooper and Frieze: similar result on the diameter, extension to higher dimensions.
- ✓ Kolossváry: average distance (central limit theorem) for all dimensions.

The longest path

$L_n :=$ length of the longest path

Theorem (EFGMSWZ'13+)

(A)

$$\forall \varepsilon > 0 \ : \ \mathbb{P}\left[L_n > \varepsilon n\right] \to 0$$

(B)

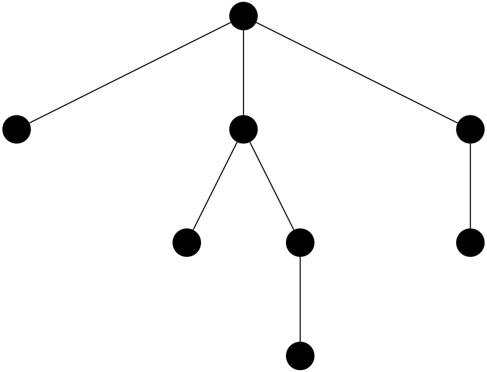
$$L_n > n^{0.63}$$

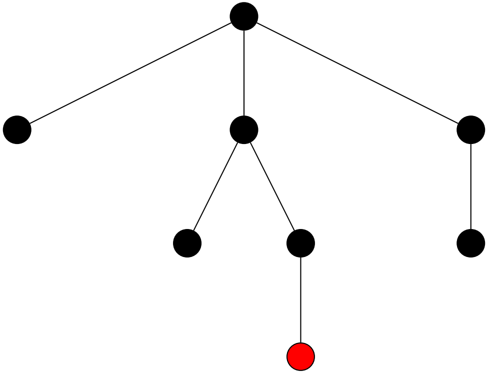
(C)

$$\mathbb{E}\left[L_n\right] = \Omega\left(n^{0.88}\right)$$

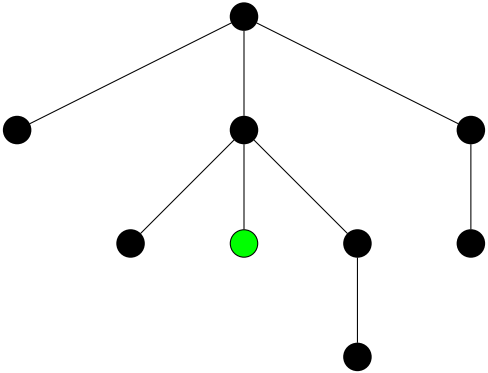
Random-surfer trees

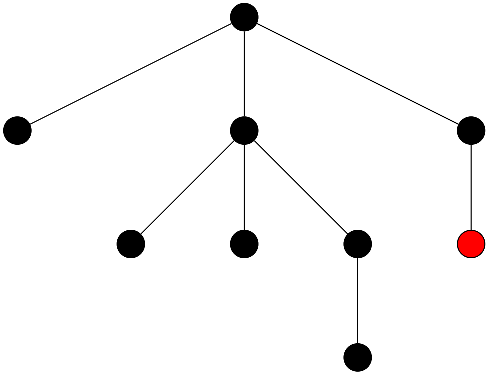
- ✓ Parameters: n and p
- ✓ Consider a pool of independent Geo(p) random variables.
- ✓ Build a random tree: start with a single vertex, add a new vertex in each step.





next geom.r.v. $= 2$





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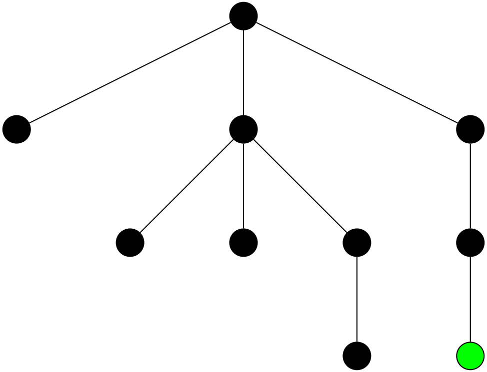
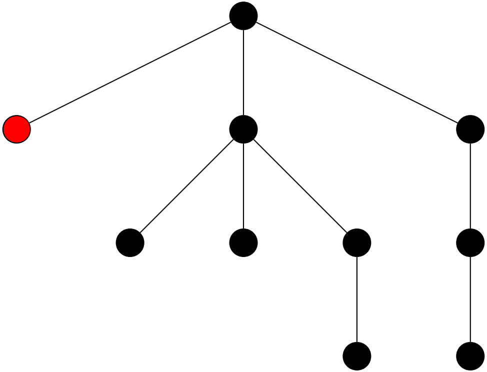
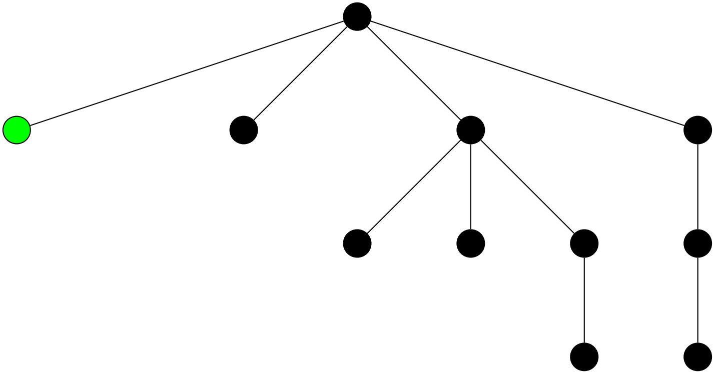
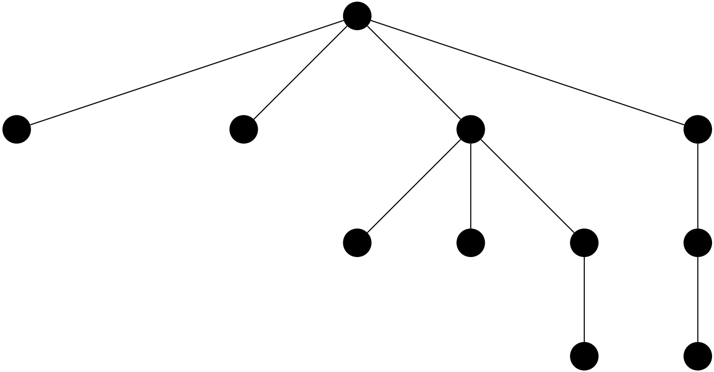


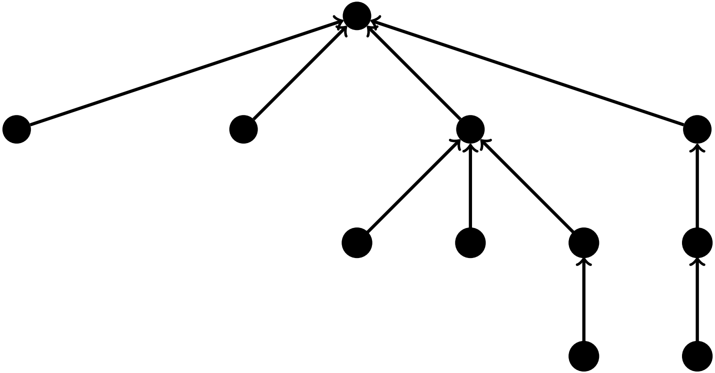


next geom.r.v. $= 3$





Blum, Chan, Rwebangira'06: Random-surfer web-graph model



Pandurangan, Raghavan, Upfal'02: the pagerank-based selection model

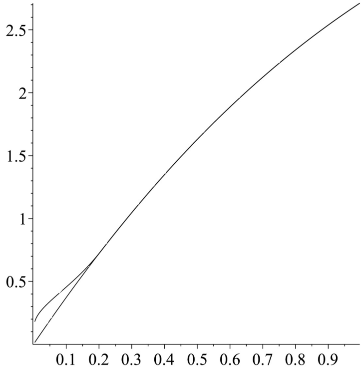
Known results

- ✓ Chebolu and Melsted'08 observed the models are equivalent.
- ✓ Nothing known about height/diameter: previously focused on degree sequence

Our result

Theorem (M, Wormald'13+)

Given p and $\varepsilon > 0$, a.a.s. the height is between $(L(p) - \varepsilon)\log n$ and $(U(p) + \varepsilon)\log n$, and the diameter is between twice these values.



Let $p_0 \approx 0.206$ be the unique solution in $(0, 1/2)$ to

$$\log\left(\frac{1-p}{p}\right) = \frac{1-p}{1-2p}\ .$$

Let s be the solution in $(0, 1)$ to

$$s\log\left(\frac{(1-p)(2-s)}{1-s}\right) = 1\ .$$

Then,

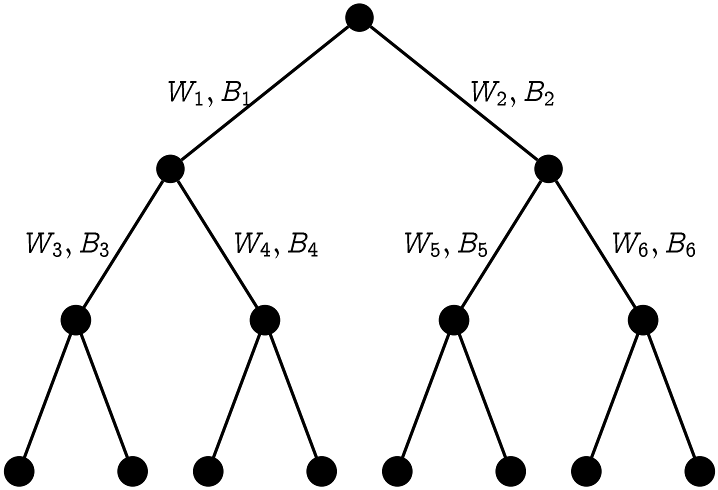
$$L(p) = \exp(1/s)s(2-s)p\ ,$$

and

$$U(p) = \begin{cases} L(p) & \text{if } p_0 \le p < 1 \\ \left(\log\left(\frac{1-p}{p}\right)\right)^{-1} & \text{if } 0 < p < p_0\ . \end{cases}$$

The theorem of Broutin and Devroye

Infinite binary tree:

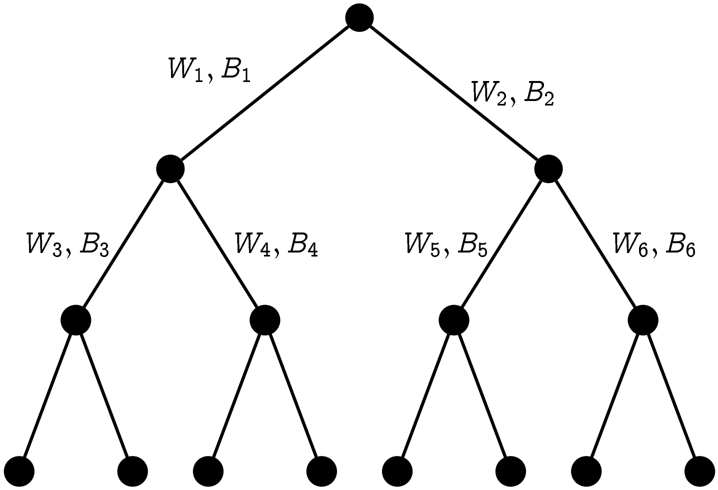


Theorem (Broutin and Devroye'06)

Assume:

- ✓ *All weights (birth times) have the same distribution.*
- ✓ *Weights are independent of birth times.*
- ✓ *One-level offsprings of distinct vertices are mutually independent.*

Infinite binary tree:



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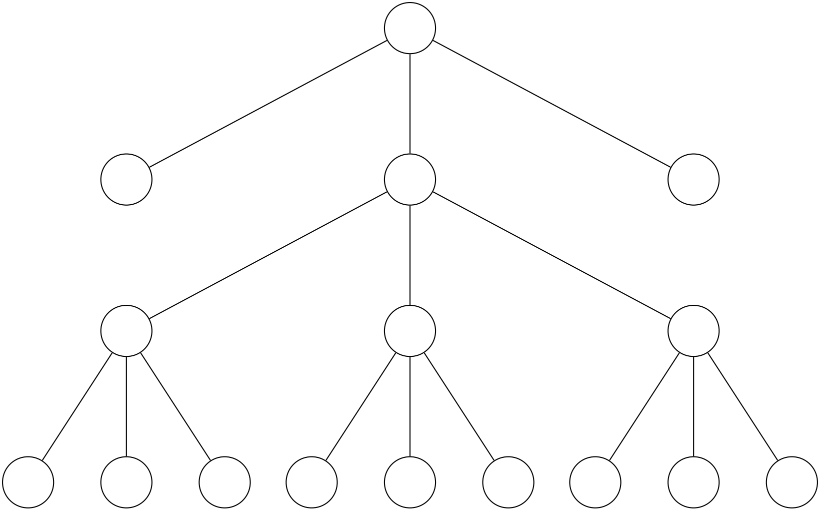
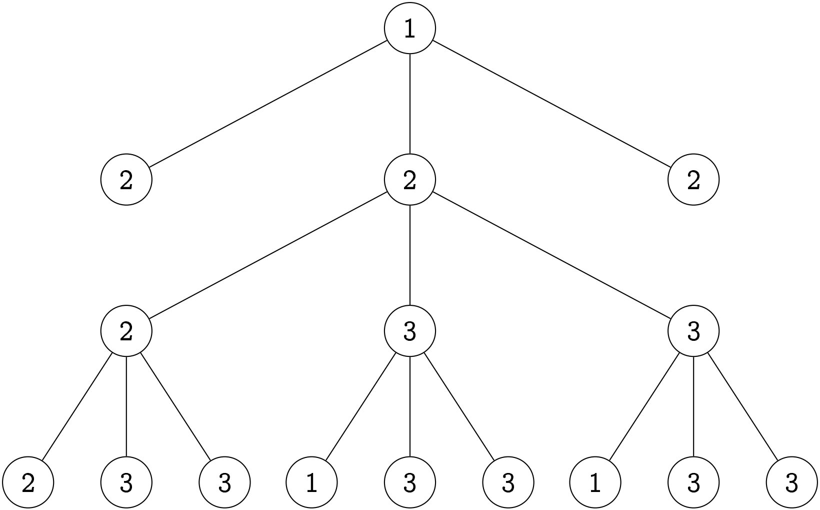
Then, height of tree at time t is a.a.s. asymptotic to ct,

$$c = \sup\left\{\frac{\alpha}{\rho} : \Lambda_W^*(\alpha) + \Lambda_B^*(\rho) = \log 2\right\}$$

Back to random apollonian networks...

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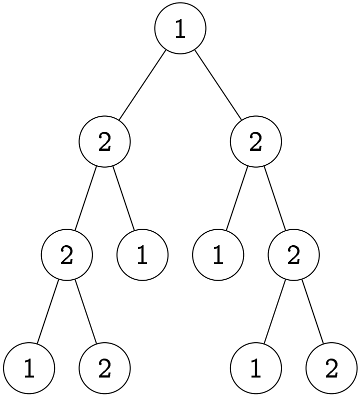
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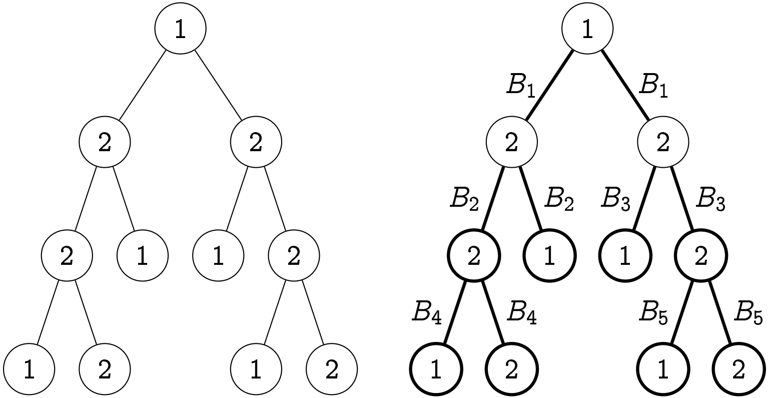
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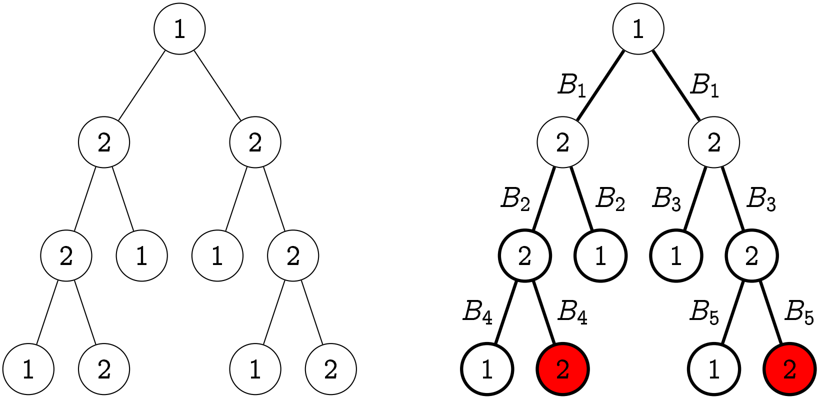
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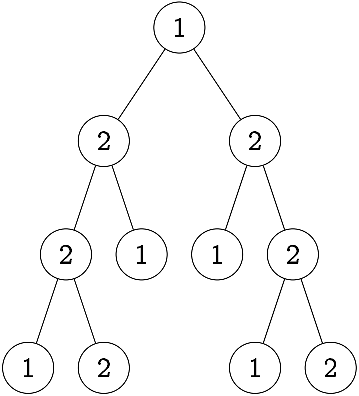

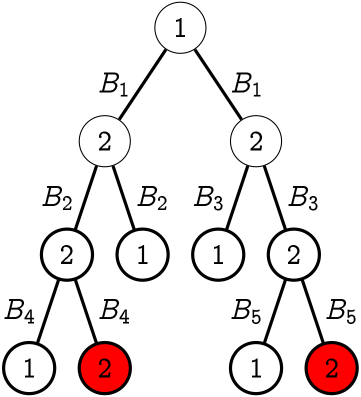
B_1
B_1
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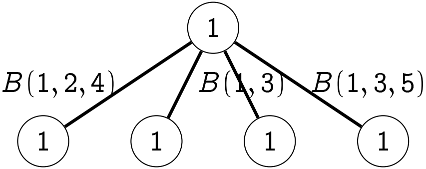
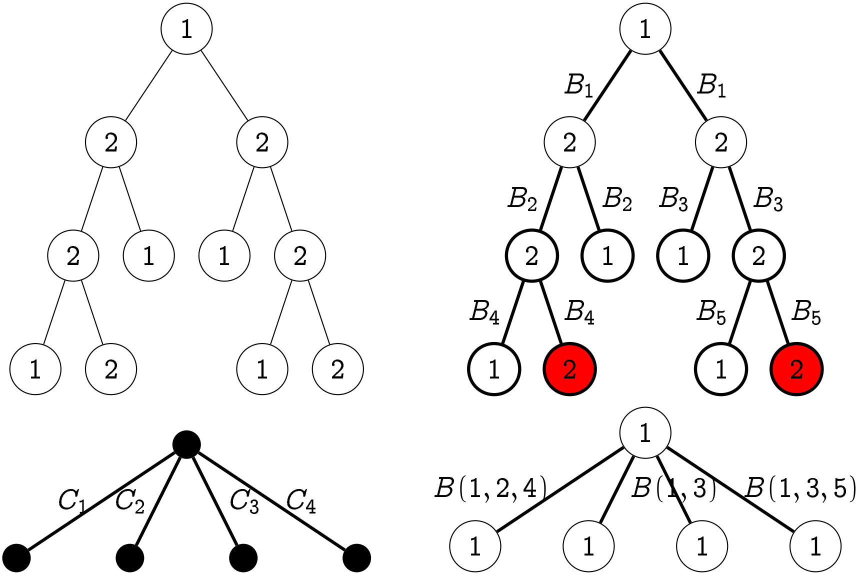
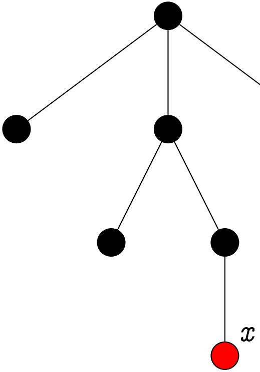
B_2
B_3
B_3
B_4
B_4
B_5
B_5


$B(1,2,4)$
$B(1,3)$
$B(1,3,5)$

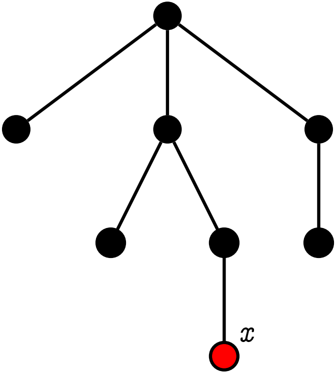

1
2 2
2 1 1 2
1 2 1 2
B_1 B_1
B_2 B_2 B_3 B_3
B_4 B_4 B_5 B_5
C_1 C_2 C_3 C_4
$B(1,2,4)$ $B(1,3)$ $B(1,3,5)$
1
1 1 1 1

- ✓ For every fixed cut-off threshold k, we stochastically sandwich 1-height of our typed tree between heights of B&D-friendly trees.
- ✓ As $k \to \infty$, lower and upper bounds converge to $(c/2)\log n$.

Back to random-surfer trees...



next geom.r.v. = 2



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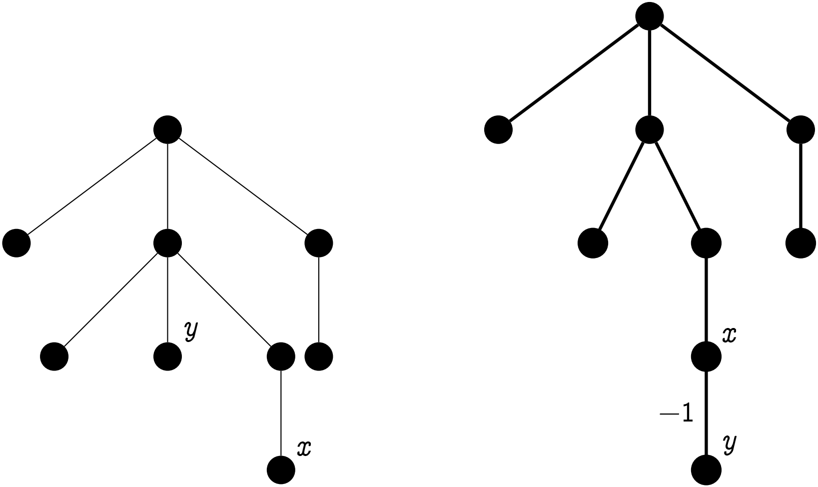
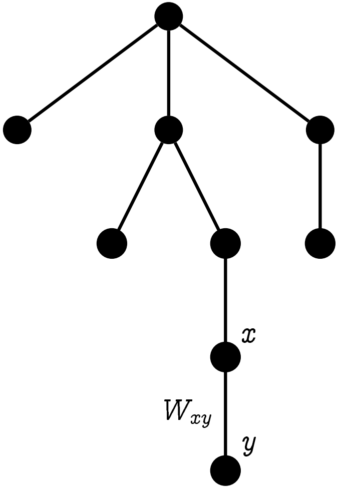
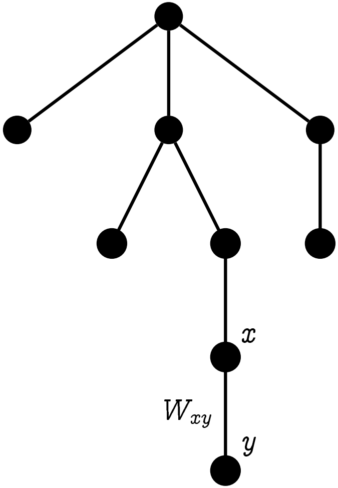
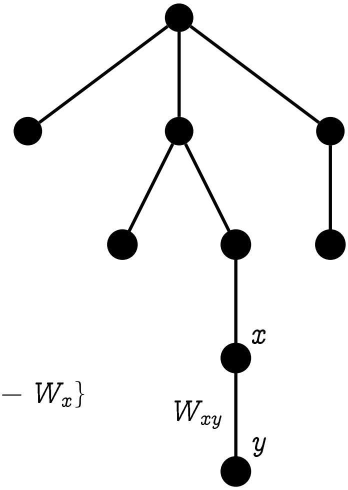

y
x
x
−1
y


x
W_{xy}
y



$$W_{xy} = 1 - \text{Geo}(p)$$



$$W_{xy} = \max\{1 - \mathrm{Geo}(p), 1 - W_x\}$$

Theorem (follows from proof of B&D's upper bound)

Assume:

- ✓ *Weights are independent of birth times.*
- ✓ *Sum of weights (birth times) along every vertical path starting from root is sharply concentrated (has exponential decay).*

Then, height of tree at time t is a.a.s. $\leq (U + o(1))t$, U depends on decay rates.

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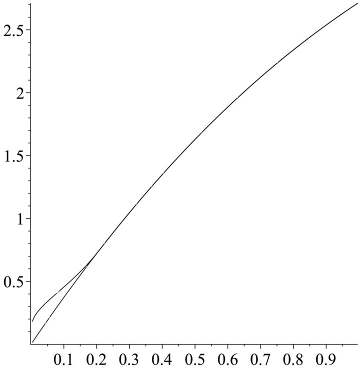
Diameter is similar

The result on the height

Theorem (M, Wormald'13+)

Given p and $\varepsilon > 0$, a.a.s. the height is between $(L(p) - \varepsilon) \log n$ and $(U(p) + \varepsilon) \log n$, and the diameter is between twice these values.



Final remarks

- ✓ Flexibility of B&D: playing with birth times/weights
- ✓ The gap for $p < p_0$ means what?
- ✓ Close the gap !
- ✓ The graph case
- ✓ A challenge!

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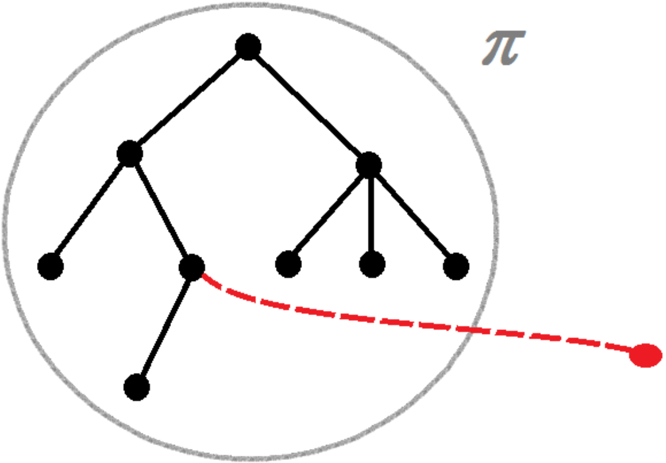
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