On the Longest Paths in Random Apollonian Networks

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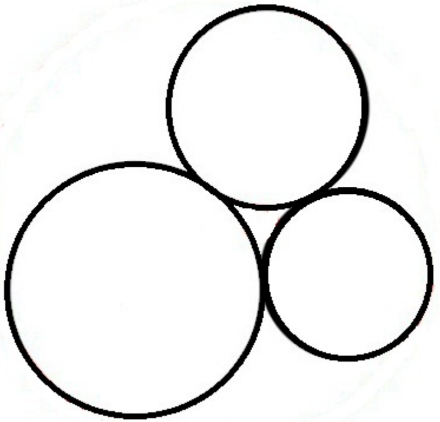
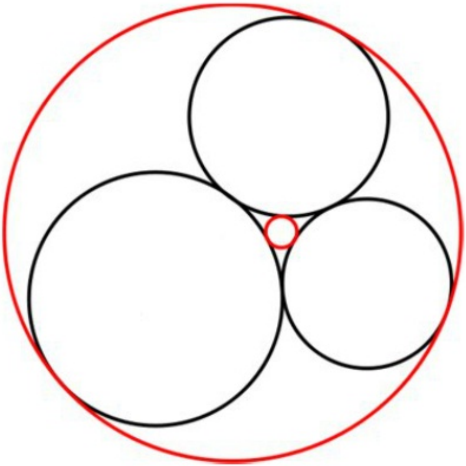
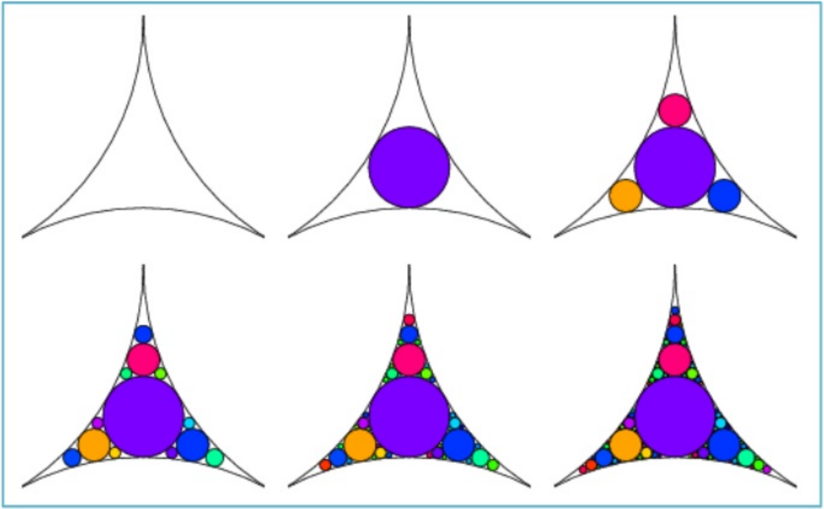
Pictures: Charalampos (Babis) Tsourakakis

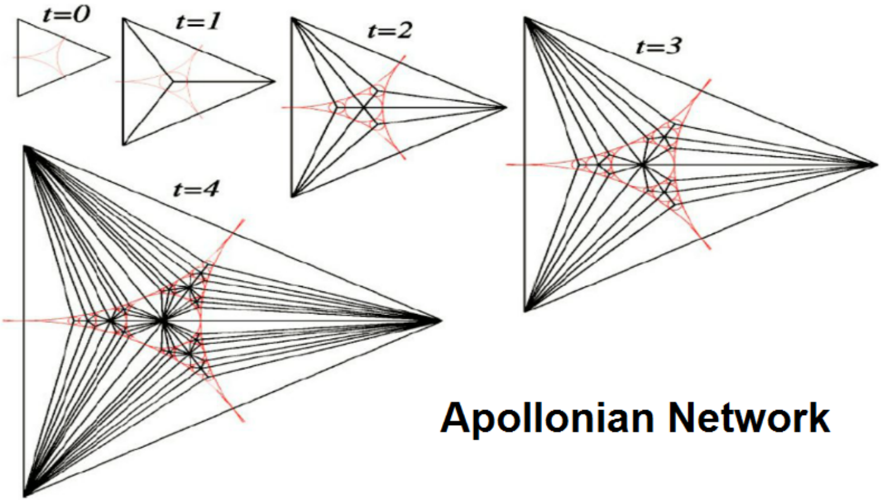
Paper: http://arxiv.org/abs/1303.5213



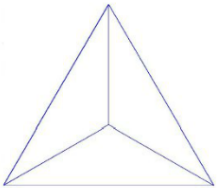




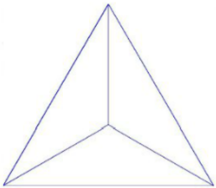
Apollonian Gasket



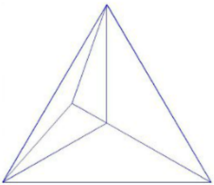
Apollonian Network



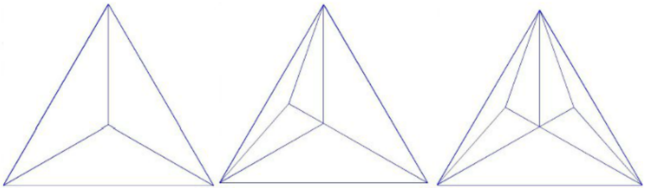
$t = 1$



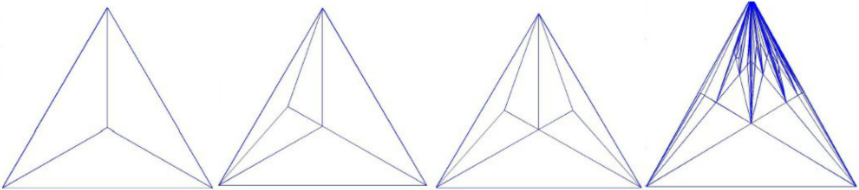
$t = 1$



$t = 2$



$t = 1$ $t = 2$ $t = 3$



$t = 1$ $t = 2$ $t = 3$ $t = 100$

Random Apollonian Network

After subdividing t times,

- a random triangulated plane graph
- $t+3$ vertices
- $3t+3$ edges
- $2t+1$ faces

called a Random Apollonian Network (RAN).
Zhou, Yan, Wang, Physical Review (2005)

Motivation

Modelling real-world networks:

- The Internet
- The Web graph
- (Online) Social networks
- Brain neurons
- Protein interactions

Properties of Real-World Networks

1. Power-law degree distribution:

$$\mathbb{P}\left[\mathrm{deg(a\ random\ vertex)} = k\right] = Ck^{-\beta}$$

2. Small-world phenomenon (six degrees of separation) :
There is a short path connecting every pair of vertices.

Motivation

Known models include:

- Erdös-Renyi random graphs
- preferential attachment models
- Kronecker graphs
- Cooper-Frieze model
- Random surfer graphs
- Fabrikant-Koutsoupias-Papadimitriou model

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RANs are an interesting model for generating random planar graphs.

Known Results

Degree sequence

$n :=$ number of vertices

$Z_{k,n} :=$ number of vertices with degree k.

Theorem (Frieze and Tsourakakis 2012)

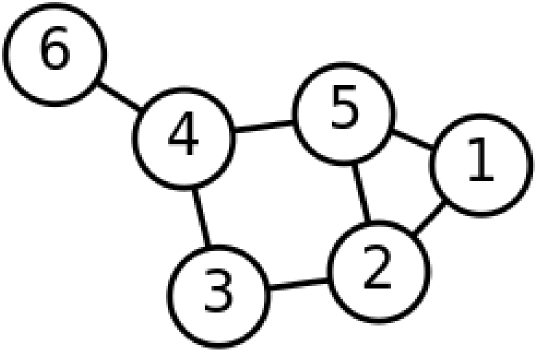
For every $k = k(n) \geq 3$*,*

$$\mathbb{E}\left[Z_{k,n}\right] = \Theta(nk^{-3}),$$

and,

$$\mathbb{P}\left[\left|Z_{k,n} - \mathbb{E}\left[Z_{k,n}\right]\right| > 10\sqrt{n \log n}\right] \to 0$$

The Diameter of a Graph



Diameter = 3

Known Results

The diameter

Theorem (Albenque and Marckert 2008)

Distance between two random vertices $\to 0.55 \log n$.

Theorem (Frieze and Tsourakakis 2012)

$$\mathbb{P}\left[\text{diameter} > 7.1 \log n\right] \to 0$$

Known Results

The diameter

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Distance between two random vertices $\longrightarrow 0.55 \log n$.

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$$\mathbb{P}\left[\text{diameter} > 7.1 \log n\right] \longrightarrow 0$$

Theorem (ELPANCJ'13+)

$$\frac{\text{diameter}}{\log n} \longrightarrow c \approx 1.668 \qquad \text{in probability}$$

Our Result on the Diameter

Theorem (ELPANCJ'13+)

$$f(x) := \frac{12x^3}{1-2x} - \frac{6x^3}{1-x},$$

$y :=$ unique solution to

$$x(x-1)f'(x) = f(x)\log f(x), \quad x \in (0, 1/2),$$

$$c := (1 - y^{-1}) / \log f(y) \approx 1.668$$

Then for every fixed $\varepsilon > 0$,

$$\mathbb{P}\left[(1-\varepsilon)c\log n \le \text{diameter} \le (1+\varepsilon)c\log n\right] \to 1$$

The Longest Path

The conjecture

Conjecture (Frieze and Tsourakakis 2012)

$$\mathbb{P}\left[\exists \text{ a path containing a positive fraction of vertices}\right] \longrightarrow 1$$

The Longest Path

The conjecture

Conjecture (Frieze and Tsourakakis 2012)

$$\mathbb{P}\left[\exists \text{ a path containing a positive fraction of vertices}\right] \longrightarrow 1$$

Not true!

The Longest Path

Our results

$m :=$ number of faces $= 2n - 5$

$L_m :=$ length of the longest path

Theorem (ELPANCJ'13+)

(A)

$$\exists \theta > 0 \ : \ \mathbb{P}\left[L_m < n/(\log n)^{\theta}\right] \to 1$$

The Longest Path

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$m :=$ number of faces $= 2n - 5$

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Theorem (ELPANCJ'13+)

(A)

$$\exists \theta > 0 \ : \ \mathbb{P}\left[L_m < n/(\log n)^{\theta}\right] \to 1$$

(B)

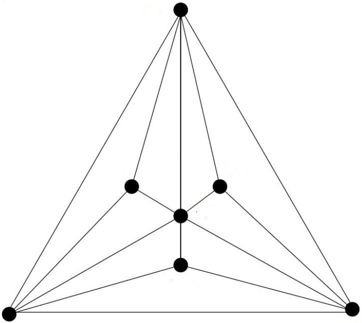
$$L_m > m^{0.63}$$

(C)

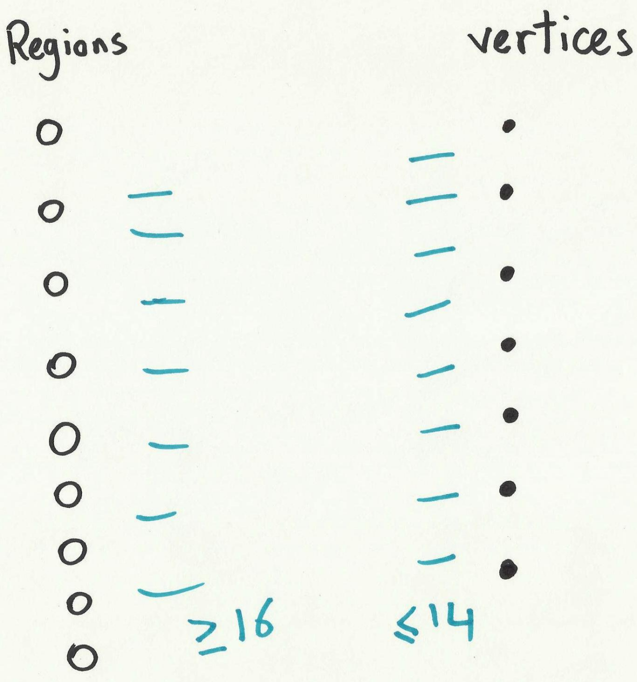
$$\mathbb{E}\left[L_m\right] = \Omega\left(m^{0.88}\right)$$

(A) Upper Bound for Longest Path

The Main Idea

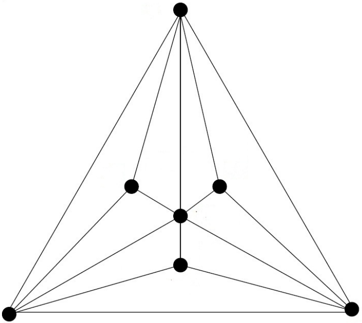


Claim: A simple path cannot contain internal vertices of all 9 regions.


Regions
vertices
≥16
≤14

(A) Upper Bound for Longest Path

The Main Idea



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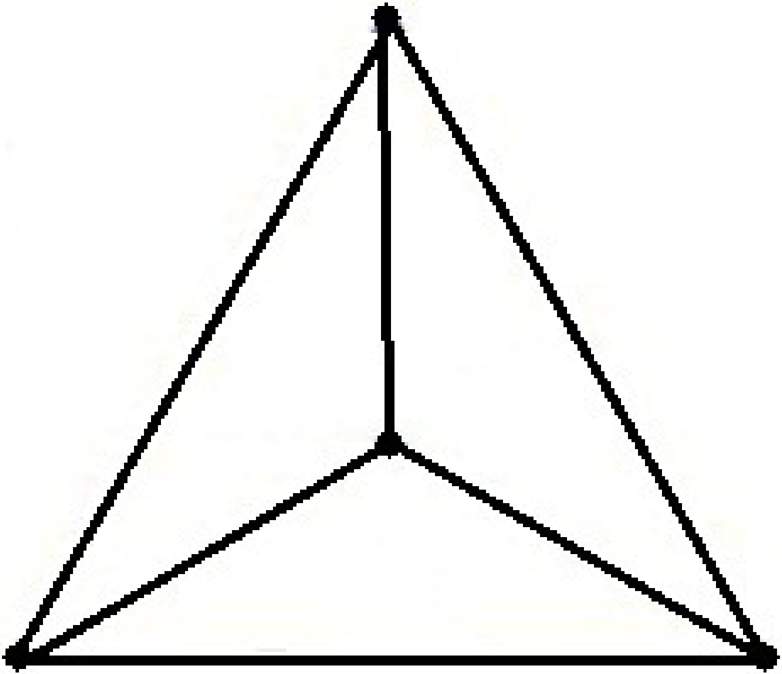
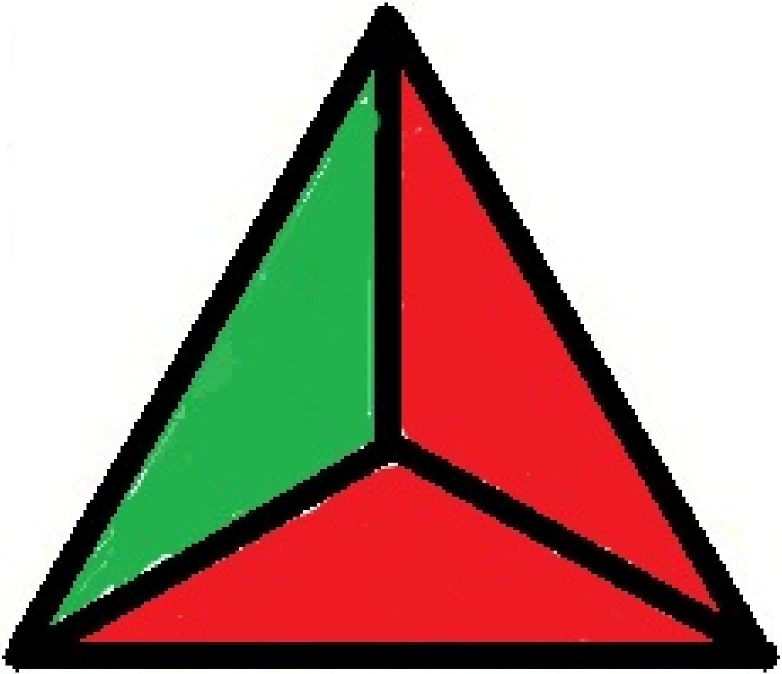




(A) Eggenberger-Pólya Urn

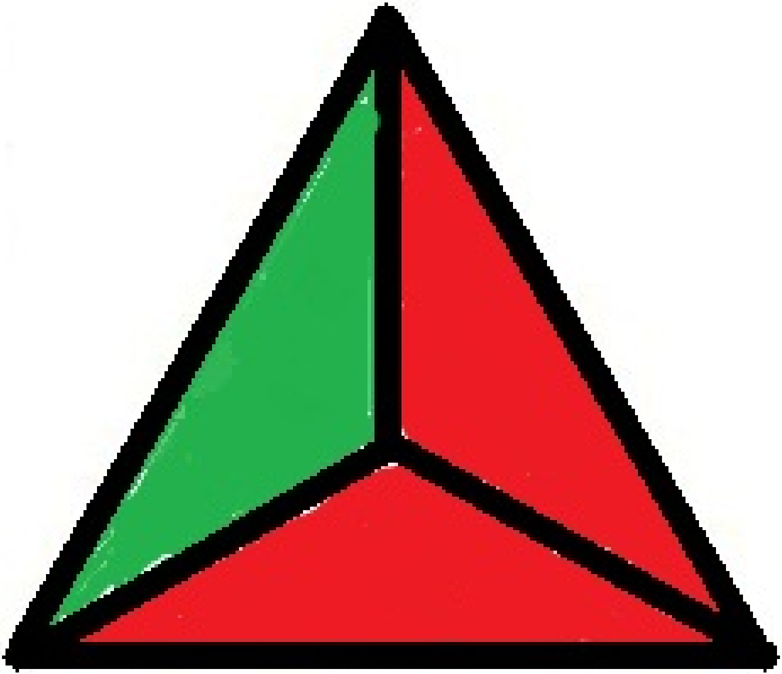
Theorem (Eggenberger and Pólya 1923)

Start: g green, r red balls.
In each step:

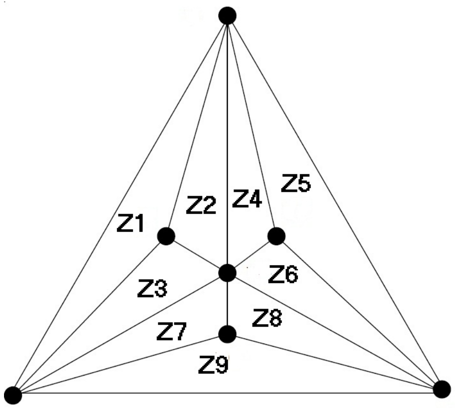
- *pick a random ball and return it to the urn;*
- *add s balls of the same colour.*

After n draws:
g_n: green balls
t_n: number of balls
For any $\alpha \in [0,1]$

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{g_n}{t_n} < \alpha\right] = \frac{\Gamma((g+r)/s)}{\Gamma(g/s)\Gamma(r/s)} \int_0^{\alpha} x^{\frac{g}{s}-1}(1-x)^{\frac{r}{s}-1}\,\mathrm{d}x$$
$$= \mathbb{P}\left[Beta(g/s, r/s) < \alpha\right]$$



(A) Upper Bound for Longest Path



Corollary

$$\mathbb{P}\left[\frac{\min\{Z_1,\cdots,Z_9\}}{n} < \epsilon\right] < 13\sqrt[4]{\epsilon}\,.$$

(A) Upper Bound for Longest Path

Fix a small ϵ. We lose

$$n\left[\varepsilon+(1-\varepsilon)\varepsilon+(1-\varepsilon)^2\varepsilon+\cdots+(1-\varepsilon)^k\varepsilon\right]=n\left[1-(1-\varepsilon)^{k+1}\right]$$

vertices in any simple path.

(A) Upper Bound for Longest Path

Fix a small ϵ. We lose

$$n\left[\varepsilon + (1-\varepsilon)\varepsilon + (1-\varepsilon)^2\varepsilon + \cdots + (1-\varepsilon)^k\varepsilon\right] = n\left[1-(1-\varepsilon)^{k+1}\right]$$

vertices in any simple path.

Theorem (ELPANCJ'13+)

(A) $\exists\theta > 0$ *such that*

$$\mathbb{P}\left[L_m < n/(\log n)^{\theta}\right] \to 1$$

Lower Bounds for Longest Path

$m :=$ number of faces $= 2n - 5$
$L_m :=$ length of the longest path
$\eta := \log 2 / \log 3 \approx 0.63$

Theorem (ELPANCJ'13+)

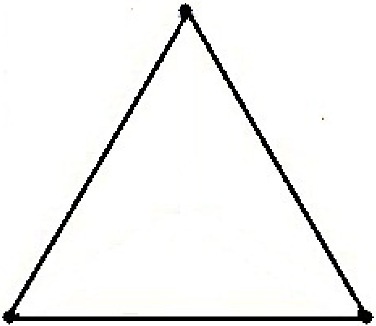
(B)

$$L_m > m^{\eta}$$

(C)

$$\mathbb{E}\left[L_m\right] = \Omega\left(m^{0.88}\right)$$

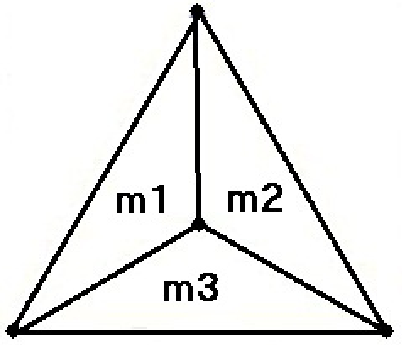
(B) Lower Bounds for Longest Path



Lemma

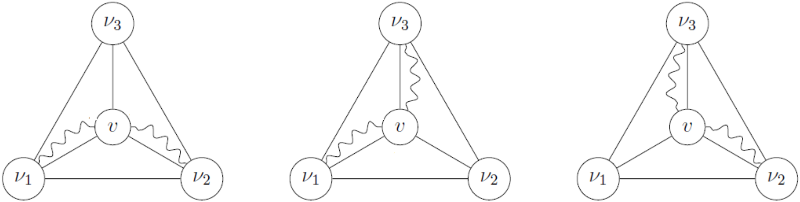
For any two boundary vertices,
$\exists$ a path of length $> m^{\eta}$ connecting them
not containing the third boundary vertex.

(B) Lower Bounds for Longest Path



Assume that $m_1 \geq m_2 \geq m_3$

(B) Lower Bounds for Longest Path



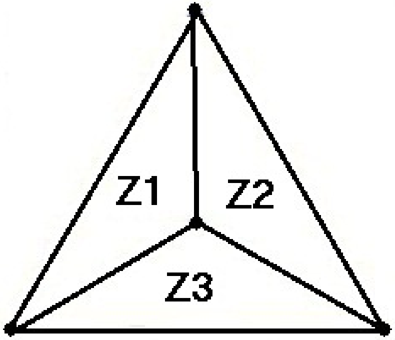
$$L_m \geq m_1^\eta + m_2^\eta \geq 2\left(\frac{m}{3}\right)^\eta = m^\eta$$

since $m_1 \geq m_2 \geq m_3$ and $m_1 + m_2 + m_3 = m$

(C) Lower Bounds for Longest Path

Finally, we sketch the proof of (C) !

(C) Lower Bounds for Longest Path



$$\mathbb{E}\left[L_m\right] \geq \mathbb{E}\left[Z_1^{0.88} + Z_2^{0.88} \middle| Z_1 \geq Z_2 \geq Z_3\right]$$

(C) Lower Bounds for Longest Path

$$\frac{\mathbb{E}\left[L_m\right]}{m^{0.88}} \geq \mathbb{E}\left[\left(\frac{Z_1}{m}\right)^{0.88} + \left(\frac{Z_2}{m}\right)^{0.88} \middle| Z_1 \geq Z_2 \geq Z_3\right]$$

$$= 6 \sum_{s \geq t \geq 1-s-t} \mathbb{P}\left[\left(\frac{Z_1}{m}, \frac{Z_2}{m}\right) = (s,t)\right]\left(s^{0.88} + t^{0.88}\right)$$

$$\to 6 \int_{s \geq t \geq 1-s-t} f(s,t)(s^{0.88} + t^{0.88}) \mathrm{d}s\mathrm{d}t \geq 1$$

Open Problems

The Longest Path

We showed $\exists\theta > 0$ such that

$$\mathbb{P}\left[L_m < m/(\log m)^{\theta}\right] \to 1$$

and

$$L_m > m^{0.63}$$

and

$$\mathbb{E}\left[L_m\right] = \Omega\left(m^{0.88}\right)$$

All these bounds can perhaps be improved.
Concentration of L_m around its expected value?

Open Problems

Cheeger constant

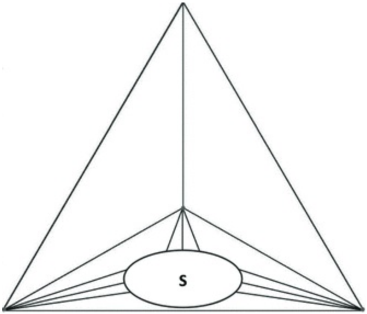
Definition (Cheeger constant)

$$\min\left\{\frac{|E(S, V \setminus S)|}{|S|} : |S| \leq \frac{n}{2}\right\}$$

Open Problems

Cheeger constant

Frieze and Tsourakakis: maximum degree is $O(\sqrt{n})$.



$$\text{Cheeger constant} \leq \frac{O(\sqrt{n})}{n/6} = O\left(\frac{1}{\sqrt{n}}\right)$$

Is this bound tight?

Thanks for your attention!