The Cops and Robber Game with a Fast Robber

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Outline

Game Definition

Definition (The Game of Cops and Robber)

- Let G be a graph and s be a positive integer.
- There is a set of cops and a robber.
- In the beginning,
  - First, each cop chooses a starting vertex.
  - Then, the robber chooses a starting vertex.
- In each round,
  - First, each cop chooses to stay or go to an adjacent vertex.
  - Then, the robber chooses to stay, or move along a cop-free path of length $\leq s$.
- The cops capture the robber if, at some moment, a cop is at the same vertex with the robber.

Think of s as the speed of the robber.

Some Remarks/Assumptions About the Game

1. This is a perfect-information game: the players see each other.
2. More than one cops can be at the same vertex.
3. The robber cannot jump over a cop.
4. The moves are deterministic (no randomness).
5. When describing a strategy for the cops, we assume the robber is clever; and vice versa.

Cop Number

Definition

The minimum number of cops that are needed to capture the (clever) robber is denoted by $c_s(G)$.

Example

- For every s, if G is the complete graph, then $c_s(G) = 1$.
- For every s, if G is a path, then $c_s(G) = 1$.
- For every s, if G is a cycle with > 3 vertices, then $c_s(G) = 2$.
- If G is the 4×8 grid, then $c_1(G) = 2$.
- If G is the 4×4 grid, then $c_3(G) > 2$.
- If G is the $m \times m$ grid, then $c_{2m}(G) = m$.

Some Known Results

for $s = 1$

- The game was defined independently by Quilliot'78 and Nowakowski and Winkler'83.
- If G is planar, then $c_1(G) \leq 3$. [Aigner and Fromme'84]
- If G is chordal, then $c_1(G) = 1$. [Quilliot'86]
- If G has no cycle with less than g vertices, then $c_1(G) > (\delta - 1)^{g/8}$. [Frankl'87]
- If G is a d-dimensional grid, then $c_1(G) = \lceil \frac{d+1}{2} \rceil$. [Neufeld and Nowakowski'98]
- If G has no cycle with more than m vertices, then $c_1(G) \leq \lceil m/2 \rceil$. [Joret, Kamiński, and Theis'10]

Known Results
for general s

- For every fixed s, computing $c_s(G)$ is NP-hard in general, but is in P when G is an interval graph. [Fomin, Golovach, Kratochvíl'08]
- When $s > 1$, there is no constant upper bound for $c_s(G)$ when G is a planar graphs. [Nisse and Suchan'08]

In my thesis, I further studied this game, especially the case $s > 1$.

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Notation

- G the graph of the game, which is simple and connected.
- n the number of vertices of G.
- s is the speed of the robber.

Meyniel's Conjecture

Meyniel's Conjecture, 1987

For every graph G on n vertices,

$$c_1(G) = O(\sqrt{n})$$

- The best known bound is

$$c_1(G) \leq n2^{-(1-o(1))\sqrt{\log_2 n}} = n^{1-o(1)}$$

[Lu and Peng'09, Scott and Sudakov'10]

- There exist graphs with $c_1(G) = \Omega(\sqrt{n})$.

For general s:

- For every fixed s, there exist graphs with $c_s(G) = \Omega\left(n^{s-3/s-2}\right)$. [Frieze, Krivelevich, Loh'11]
- We will show there exist graphs with $c_s(G) = \Omega\left(n^{s/s+1}\right)$.

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Controlling a Path

Definition

The cops control a vertex if there is a cop at that vertex or at an adjacent vertex.
The cops control a path if they control some vertex of it.

Maximum Cop Number of Connected Graphs

When $s \in \{2, 4\}$

Lemma

Let G be d-regular with girth $> 2s + 2$. Then $c_s(G) = \Omega(d^s)$.

Proof.

Vertex r is safe if $\exists X \subseteq V$, $|X| = (d-1)^s/2$, such that $\forall x \in X$, $\exists (r, x)$-path of length s not controlled by the cops:

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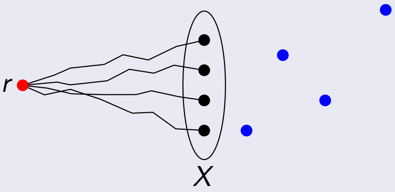

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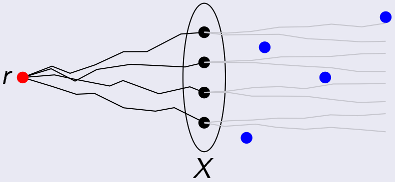

After their move, each cop controls $\leq (s+2)d^s$ escaping paths. $\square$

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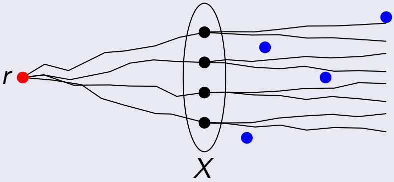

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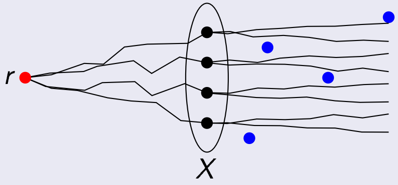

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Lemma

Let G be d-regular with girth $> 2s + 2$. Then $c_s(G) = \Omega(d^s)$.

Theorem

Let $s \in \{2, 4\}$ be fixed. For every n, there exists a graph on n vertices with $c_s(G) = \Omega\left(n^{s/s+1}\right)$.

Proof.

- $s = 2$ There exist d-regular graphs on $\leq 2d^3$ vertices with girth 7. [Lazebnik, Ustimenko, Woldar'97]
- $s = 4$ There exist d-regular graphs on $\leq 2d^5$ vertices with girth 12. [Araujo, González, Montellano-Ballesteros, Serra'07]

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Maximum Cop Number of Connected Graphs

For general s

Lemma

Let G be d-regular bipartite graph with diameter larger than s, such that

1. *If u and v are vertices of distance $\leq s+1$, there are $O(1)$ distinct shortest (u, v)-paths.*
2. *For every vertex u and subset A of vertices, there exist $\Omega(d^s)$ vertices x of distance s from u, where any shortest (u, x)-path avoids A.*

Then $c_s(G) = \Omega(d^s)$.

Maximum Cop Number of Connected Graphs

For general s

Theorem

Let s be fixed. For every n, there exists a graph on n vertices with $c_s(G) = \Omega\left(n^{s/s+1}\right)$.

Proof.

For d large enough, there exist Cayley graphs on $O(d^{s+1})$ vertices satisfying the conditions of the lemma. □

Remark: This result was proved jointly with Noga Alon.

Maximum Cop Number of Connected Graphs

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A Generalization of Meyniel's Conjecture

Theorem

Let s be fixed. For every n, there exists a graph on n vertices with $c_s(G) = \Omega\left(n^{s/s+1}\right)$.

Conjecture

For every graph G on n vertices, $c_s(G) = O\left(n^{s/s+1}\right)$.

[Meyniel's Conjecture, 1987]

For every graph G on n vertices, $c_1(G) = O(\sqrt{n})$.

Infinitely Fast Robber

Definition

Let G be a connected graph on n vertices. Then

$$c_\infty(G) = c_n(G)$$

And we call $c_\infty(G)$ the cop number of G.
If the robber has speed n,
we say the robber is infinitely fast.

Known Results
for infinitely fast robber

- Computing $c_\infty(G)$ is NP-hard.
  [Fomin, Golovach, Kratochvíl'08]
- For every n, there exists a graph on n vertices with $c_\infty(G) = \Theta(n)$. [Frieze, Krivelevich, Loh'11]

For the rest of the talk, we will assume the robber is infinitely fast.

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An Obvious Bound For The Cop Number

Definition

Set $X \subseteq V(G)$ is a dominating set if every vertex is either in X or adjacent to a vertex in X.
The domination number of graph G is the minimum size of a dominating set of G.

Proposition

The cop number $\leq$ the domination number.

Proof.

The cops start at a dominating set. They will capture the robber in the first move. □

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The Treewidth of a Graph

Definition

Let G be a graph, T be a tree, and $\{W_t : t \in V(T)\}$ be a family of subsets of $V(G)$, called the bags.
The pair (T, W) is a tree decomposition of G if it satisfies:

(i) $\cup_{t \in V(T)} W_t = V(G)$.

(ii) Forall $uv \in E(G)$, there is a bag containing both u and v.

(iii) Forall $v \in V(G)$ the set of bags containing v induces a subtree of T

The width of (T, W) is the maximum size of a bag, minus 1.
The treewidth of G, written $tw(G)$, is the minimum number w such that G has a tree decomposition having width w.

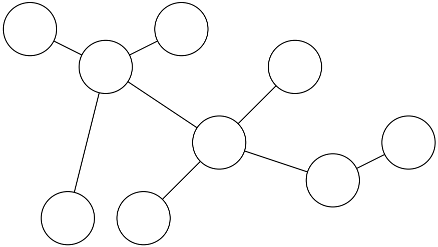
Treewidth quantifies how much is G similar to a tree.
for example, a tree has treewidth 1.

The Relation Between Cop Number and Treewidth

Proposition

For any G,

$$c_\infty(G) \leq tw(G) + 1$$



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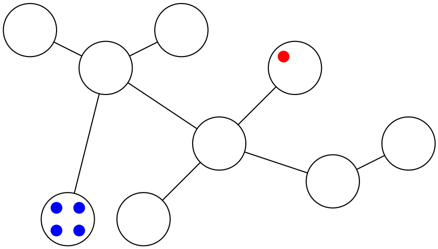

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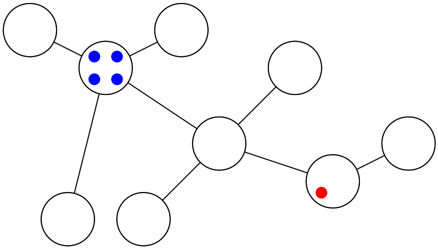

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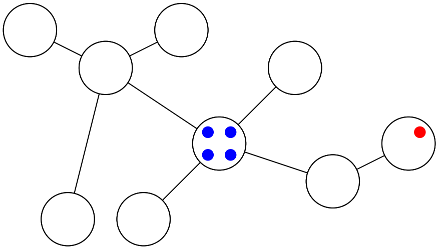

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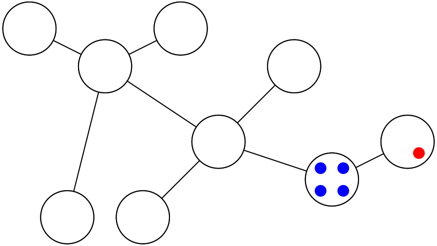

Helicopter Cops and Robber Game

Definition (Helicopter Cops and Robber Game)

- There is a set of cops and a robber.
- It is a continuous-time game.
- At any moment, the robber is at a vertex.
- At any moment, each cop is either
  - standing at a vertex, or
  - in a helicopter.
- The cops want to land via a helicopter on the robber's vertex.
- The robber can see the helicopter approaching its landing spot, and may run along a cop-free path to a new vertex.

In a complete graph, n cops are needed.

In a path, 2 cops are needed.

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A Lower Bound for Cop Number

Using helicopter cops and robber Game

Theorem (Seymour and Thomas'93)

Exactly $tw(G) + 1$ cops are needed to capture the robber in the Helicopter Cops and Robber game.

Theorem

For every graph G with maximum degree Δ,

$$tw(G) + 1 \leq c_\infty(G)(\Delta + 1)$$

Punchline: If the robber is able to predict the movement of the cops, the number of required cops is at most multiplied by $\Delta + 1$.

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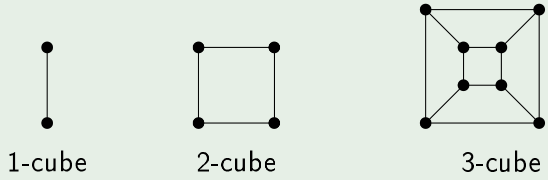
The m-Dimensional Hypercube

Definition

The *m-dimensional hypercube*, or the *m-cube*, has vertex set $\{0,1\}^m$ with two vertices being adjacent if they differ in exactly one coordinate.

Example



The Cop Number of The m-dimensional Hypercube

Theorem (from previous slides)

For every graph G with treewidth tw and maximum degree Δ,

$$\frac{tw+1}{\Delta+1} \leq c_\infty(G)$$

Corollary

If G is the m-dimensional hypercube with $n = 2^m$ vertices, then

$$\frac{\eta_1 n}{m\sqrt{m}} \leq c_\infty(G) \leq \frac{\eta_2 n}{m}$$

Proof.

$tw(G) = \Theta(n/\sqrt{m})$ [Sunil Chandran, Kavitha'06], $\Delta = m$, and domination number $= O(n/m)$. □

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Interval Graphs

Definition

Consider a set of closed intervals on the real line. Put a vertex for each interval, and join two vertices if their corresponding intervals intersect. The result is an interval graph.

Known Results

for interval graphs

- For every fixed finite s, computing $c_s(G)$ is NP-hard when G is a general graph.
- If G is an interval graph, then

$$c_s(G) \leq 5s - 1$$

- Leads to a polynomial algorithm for fixed finite s.

[Fomin, Golovach, Kratochvíl'08]

- For $s = \infty$, the complexity for interval graphs is open.
- We give a 3-approximation algorithm.

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A Path Decomposition of a Graph

Definition

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(i) $\cup_{1\leq i\leq m} W_i = V(G)$.

(ii) For every $uv \in E(G)$, there is a bag containing both u and v.

(iii) For every $v \in V(G)$, v is contained in a consecutive set of bags.

Fact: Every interval graph G has a path decomposition, in which every bag induces a clique in G.

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3-Approximation For Interval Graphs

Definition

A subgraph H of G is k-wide if

(i) H is k-vertex-connected, and

(ii) No $k-1$ vertices of G dominate H.

Let M be the maximum number s. t. G has an M-wide subgraph.

Lemma

$$M \leq c_\infty(G) \leq 3M$$

Proof (lower bound).

Assume H be M-wide, and there are $< M$ cops in the game. The robber stays in H all the time. In every round, there is an uncontrolled vertex in H, and M disjoint paths to go there. □

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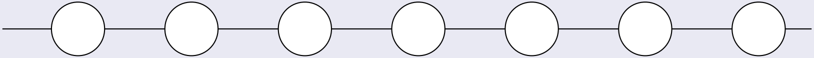

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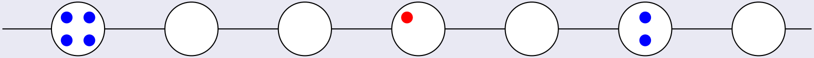

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3-Approximation For Interval Graphs

Lemma

If M is the maximum number s. t. G has an M-wide subgraph,

$$M \leq c_\infty(G) \leq 3M$$

Proof (upper bound).

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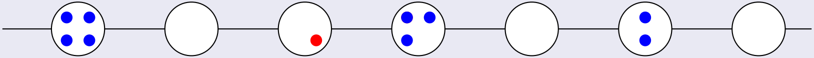

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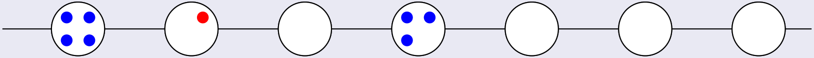

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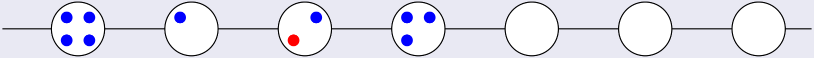

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It is possible to calculate M in polynomial time.

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The number of interval subgraphs is $O(n^2)$.
Each of them is an interval graph, and its dominating number and connectivity can be found in polynomial time. □

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Let M be the maximum number s. t. G has an M-wide subgraph.

$$M \leq c_\infty(G) \leq 3M$$

and it is possible to calculate M in polynomial time. Thus,

Theorem

There is a polynomial time 3-approximation algorithm for finding the cop number of an interval graph.

Maximum Cop Number of Interval Graphs

Theorem

Let G be an interval graph. No subgraph of G is $(\sqrt{5n}+3)$-wide. Hence

$$c_\infty(G) = O(\sqrt{n})$$

The theorem is asymptotically tight:

The above graph is $\left(\frac{\sqrt{n}}{3}\right)$-wide.

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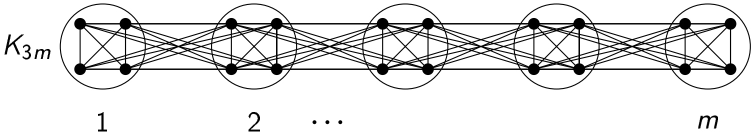

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Chordal Graphs

Definition

Graph G is chordal if it does not have an induced cycle with more than 3 vertices.

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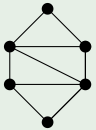

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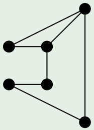

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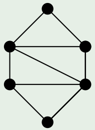

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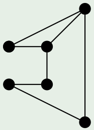

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As we saw, the cop number of an interval graph is $O(\sqrt{n})$.
What about the cop number of a chordal graph?
Next we show there are chordal graphs with cop number $\Omega\left(\frac{n}{\log n}\right)$.

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Accessible Sets

Definition

A subset $X \subseteq V(G)$ is called accessible if

- $c_\infty(G) \geq |X|$, and
- if there are $|X| - 1$ cops in the game, then there exists a strategy for the robber, in which the robber has access to X in every round. That is, in every round, the cops are not separating the robber from X.

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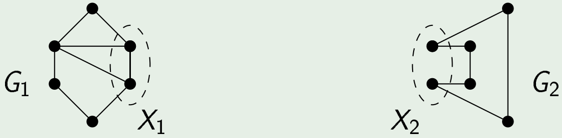

Construction of Chordal Graphs With Large Cop Number

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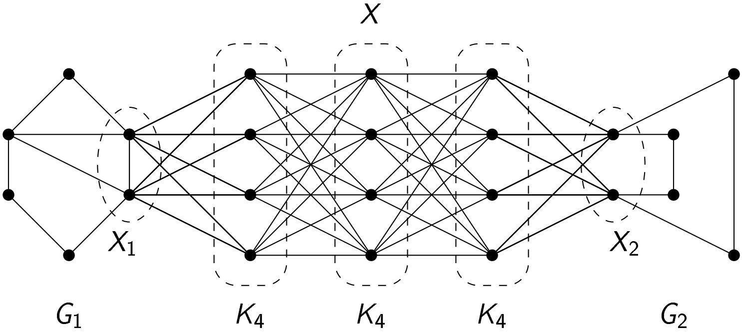

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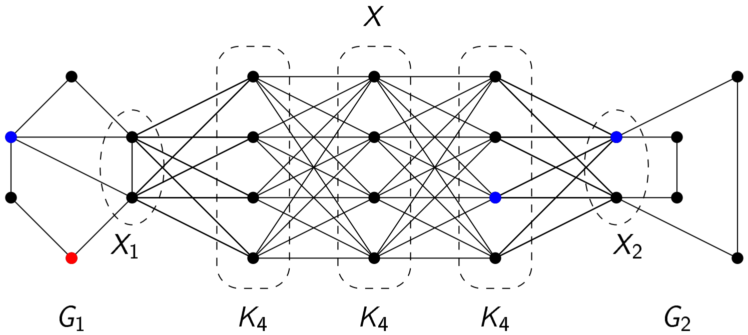

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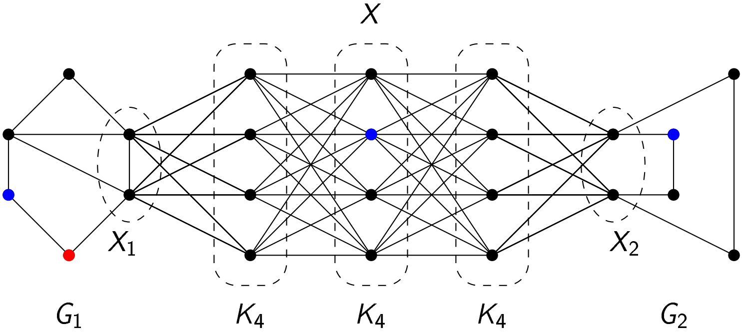

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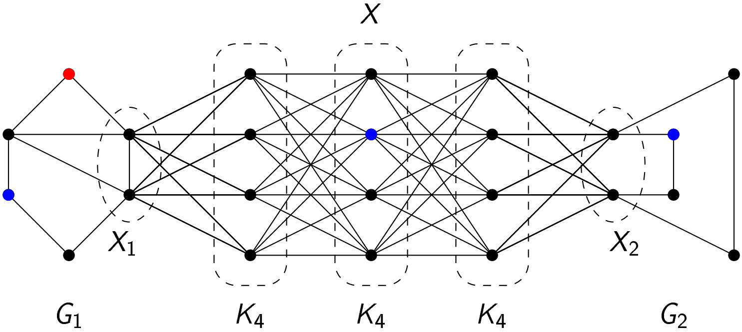

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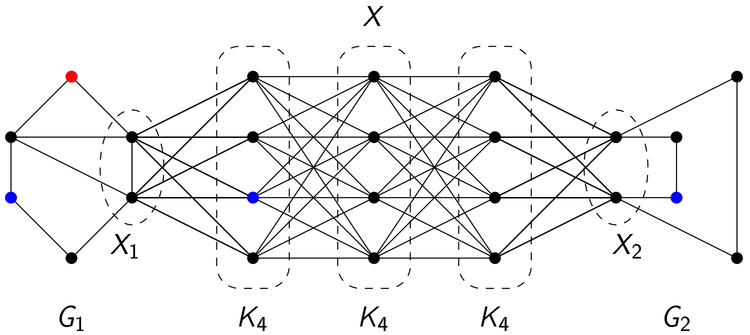

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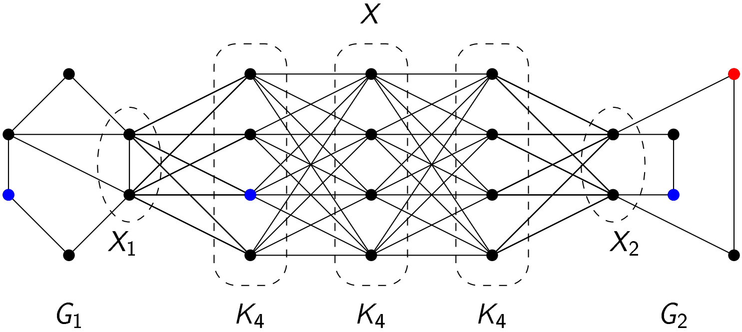

The Maximum Cop Number of Chordal Graphs

Theorem

There exist chordal graphs with cop number $\Omega\left(\frac{n}{\log n}\right)$.

Proof.

Let $g(m)$ be the minimum size of a graph with an accessible subset of m vertices. One can build a graph with an accessible subset of $2m$ vertices by using two graphs with accessible subsets of m vertices, and 3 copies of K_{2m}. Hence we have

$$g(2m) \leq 2g(m) + 3 \times 2m$$

so $g(m) = O(m \log m)$. Thus there are chordal graphs on $O(m \log m)$ vertices with cop number $\geq m$. $\square$

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Definition

Let $S \subseteq V(G)$. The (closed) neighbourhood of S, written $\overline{N}(S)$, is the set of vertices that are in S or have a neighbour in S.

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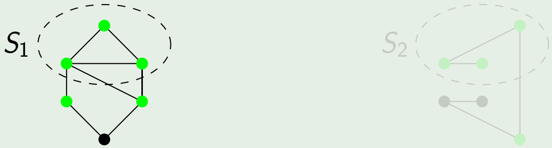

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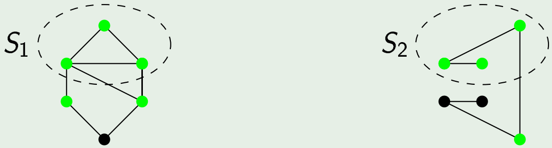

The Large Component Lemma

Lemma

Assume that for every subset S of vertices of size $\leq m$, $G - \overline{N}(S)$ has a connected component of size $> n/2$. Then $c_\infty(G) > m$.

Proof.

Let there be m cops. We give an escaping strategy for robber:

Invariant: Robber in largest component of $G - \overline{N}(S)$, $S =$ cops' position

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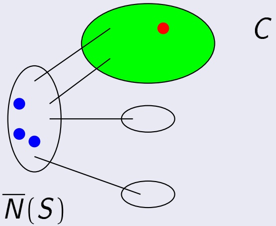

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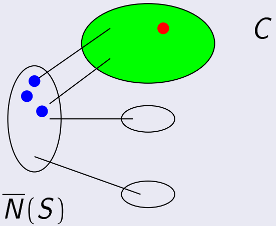

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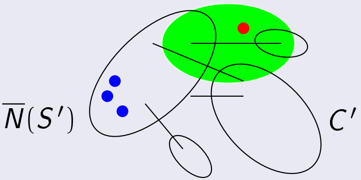

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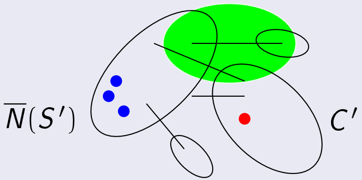

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Corollary

Let $c = c_\infty(G)$. There exists a subset S of size $\leq c$ such that $G - \overline{N}(S)$ has no component of size $> n/2$.

Vertex Expansion

Definition

Let G be a graph. The vertex expansion of G, $\iota(G)$, is the following quantity:

$$\iota(G) = \min_{|S| \leq n/2} \frac{|\overline{N}(S) \setminus S|}{|S|}.$$

Example

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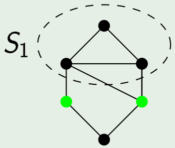

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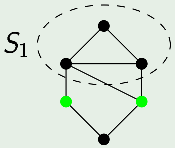

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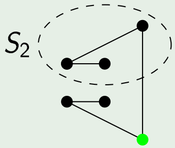

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Lower Bound for Expander Graphs

Theorem

$$c_\infty(G) \geq \frac{\iota n}{4(\Delta+1)}$$

Proof.

Let $c = c_\infty(G)$. There exists a subset S of size $\leq c$ such that $G - \overline{N}(S)$ has no component of size $> n/2$. Clearly $\overline{N}(S) \leq c(\Delta+1)$. Let $C_1, \ldots, C_m$ be the components of $G - \overline{N}(S)$.

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If $|C_1| + \cdots + |C_m| < n/4$, then $\overline{N}(S) \geq 3n/4$, so

$$c(\Delta + 1) \geq \frac{3n}{4} > \frac{\iota n}{4} \qquad \square$$

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If $|C_1| + \cdots + |C_m| \geq n/4$, as each C_i has size $\leq n/2$, one can pick some of the C_i's such that their union U has $n/4 \leq |U| \leq n/2$. Set U has at least $\iota|U|$ neighbours outside, so

$$c(\Delta+1) \geq |\overline{N}(S)| \geq \iota|U| \geq \iota n/4 \quad \square$$

The Erdös-Rényi Random Graph

Definition

Let n be a positive integer and p be a real number in $[0, 1]$. The Erdös-Rényi random graph $\mathcal{G}(n, p)$ is a random labelled graph on n vertices such that each edge appears in $\mathcal{G}(n, p)$ independently and with probability p. For a function $p : \mathbb{N} \longrightarrow [0, 1]$ and a graph property A, we say $\mathcal{G}(n, p)$ asymptotically almost surely (a.a.s.) satisfies A, if we have

$$\lim_{n \longrightarrow \infty} \mathbf{Pr}\left[\mathcal{G}(n, p(n)) \text{ satisfies } A\right] = 1$$

Lower Bounds for Random Graphs

Theorem (from previous slides)

$$c_\infty(G) \geq \iota \frac{n}{4(\Delta+1)}$$

Theorem

If $np \geq 20 \ln n$, *then a.a.s.* $\iota(\mathcal{G}(n,p)) \geq 10^{-3}$.

Corollary

If $np \geq 20 \ln n$, *then a.a.s.* $c_\infty(\mathcal{G}(n,p)) = \Omega\left(\frac{1}{p}\right)$.

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Theorem

If $np > 2 \ln n$, then a.a.s. the domination number of $\mathcal{G}(n,p)$ is $O\left(\frac{\log(np)}{p}\right)$. *[Alon, Spencer'92]*

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Tighter Bounds for Denser Random Graphs

Theorem

- *If* $np = n^{\alpha+o(1)}$, *where* $1/2 < \alpha < 1$, *then a.a.s*

$$c_\infty(\mathcal{G}(n,p)) = \Theta\left(\frac{\log n}{p}\right)$$

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Proof.

Use bounds on domination number for upper bounds,
and results for the "slow robber" version (proved by Bonato, and Prałat, and Wang'07) for the lower bounds. □

Tighter Bounds for Denser Random Graphs

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Edge Expansion

Definition

For $S \subseteq V(G)$, let ∂S denote the set of edges with exactly one endpoint in S. Then the edge expansion of G, written $\iota_e(G)$, is defined as:

$$\iota_e(G) = \min_{|S| \leq n/2} \frac{|\partial S|}{|S|}$$

Example

$\iota_e(G_1) = 1$

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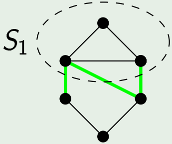

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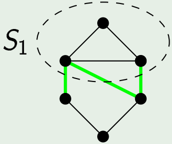

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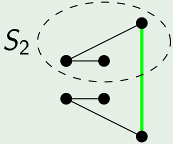

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Asymptotic Cop Number of Random Regular Graphs

Theorem

$$c_\infty(G) \geq \frac{\iota_e n}{2\Delta^2}$$

Corollary

Fix $d \geq 3$. With probability $\to 1$ as $n \to \infty$, a random d-regular labelled graph G on n vertices has $c_\infty(G) = \Theta(n)$.

Proof.

$$\text{A.a.s. } \iota_e(G) \geq d/2 - \sqrt{d \ln 2} \qquad \text{[Bollobás'88], so}$$

$$c_\infty(G) \geq \frac{d - 2\sqrt{d \ln 2}}{4d^2} n$$

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Results Not Mentioned Here

1. Characterization of graphs with cop number one, $O(n^2)$ algorithm
2. Results on Cartesian products of graphs
3. The same-speed variation

Open Problems

1. If the robber has finite speed s, we proved that there exist graphs with
$$c_s(G) = \Omega\left(n^{s/s+1}\right)$$
We conjecture that this bound is tight, that is,
$$c_s(G) = O\left(n^{s/s+1}\right)$$
This seems to be difficult: even if $s = 1$, the best known upper bound is
$$c_1(G) \leq n^{1-o(1)}$$

Open Problems

2. We proved that for every G,

$$\frac{tw(G)+1}{\Delta+1} \leq c_\infty(G) \leq tw(G)+1$$

Using this we showed that if G is the m-dimensional hypercube with $n = 2^m$ vertices, then

$$\frac{\eta_1 n}{m\sqrt{m}} \leq c_\infty(G) \leq \frac{\eta_2 n}{m}$$

Can we eliminate this $\sqrt{m}$ factor?

3. We proved that finding the cop number of an interval graph is 3-approximable. Is this problem polynomial-time solvable? Does a better approximation algorithm exist?

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Open Problems

4. We proved that there are chordal graphs with $c_\infty(G) = \Omega(n/\log n)$. Is this bound tight? Do there exist chordal graphs with $c_\infty(G) = \Theta(n)$?

5. When $np \geq 20 \ln n$, we proved that a.a.s.

$$\frac{k_1}{p} \leq c_\infty(\mathcal{G}(n,p)) \leq \frac{k_2 \log(np)}{p}$$

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Thank You!

Any Questions?