On a Generalization of Meyniel's Conjecture on the Cops and Robbers Game

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joint work with Noga Alon

Game Definition

Definition (The Game of Cops and Robber)

- Let G be a graph and s be a positive integer.
- There is a set of cops and a robber.
- In the beginning,
  - First, each cop chooses a starting vertex.
  - Then, the robber chooses a starting vertex.
- In each round,
  - First, each cop chooses to stay or go to an adjacent vertex.
  - Then, the robber chooses to stay, or move along a cop-free path of length $\leq s$.
- The cops capture the robber if, at some moment, a cop is at the same vertex with the robber.

Think of s as the speed of the robber.

Some Remarks/Assumptions About the Game

1. This is a perfect-information game: the players see each other.
2. More than one cops can be at the same vertex.
3. The robber cannot jump over a cop.
4. The moves are deterministic (no randomness).
5. The graph is simple and connected.

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Interested in: graphs that require lots of cops (not K_n or P_n!)

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Meyniel's Conjecture

Meyniel's Conjecture, 1987

$$f_1(n) = O(\sqrt{n})$$

$$k\sqrt{n} \leq f_1(n) \leq n2^{-(1-o(1))\sqrt{\log_2 n}} = n^{1-o(1)}$$

[Lu and Peng'09, Scott and Sudakov'10]

In general, let $\alpha = 1 + 1/s$. Then

$$kn^{s-3/s-2} \leq f_s(n) \leq n\alpha^{-(1-o(1))\sqrt{\log_\alpha n}}$$

[Frieze, Krivelevich, Loh'11]

Today we will prove $kn^{s/s+1} \leq f_s(n)$

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Controlling a Path

Definition

The cops control a vertex if there is a cop at that vertex or at an adjacent vertex.
The cops control a path if they control some vertex of it.

Proof of the Main Lemma

Lemma (The Main Lemma)

Let G be d-regular with girth $> 2s + 2$. Then $\Omega(d^s)$ cops are needed to capture the robber in G.

Proof.

Vertex r is safe if $\exists X \subseteq V$, $|X| = (d-1)^s/2$, such that $\forall x \in X$, $\exists (r, x)$-path of length s not controlled by the cops:

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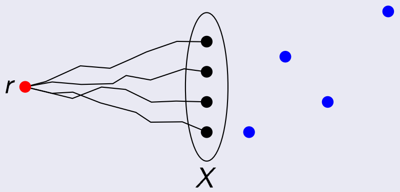

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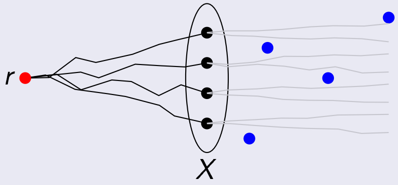

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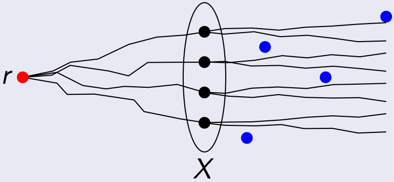

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Claim. After cops' move, each cop controls $\leq (s+1)d^s$ escaping paths.

This completes the proof, because

$$\text{number of cops} \geq \frac{|X|\frac{(d-1)^s}{2}}{(s+1)d^s} = \Omega(d^s)$$

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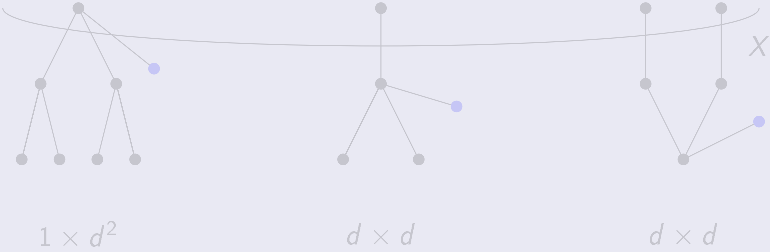

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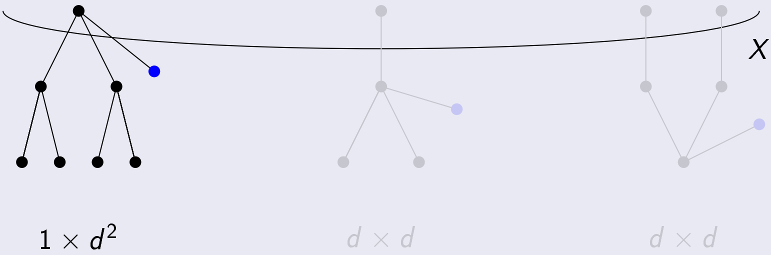

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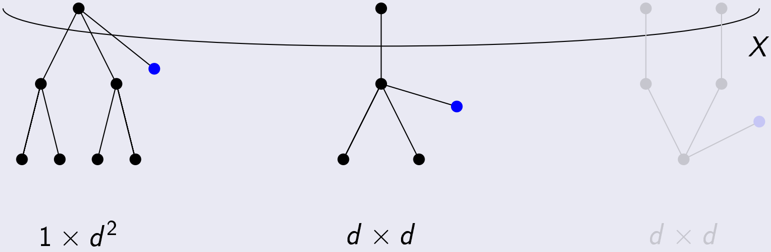

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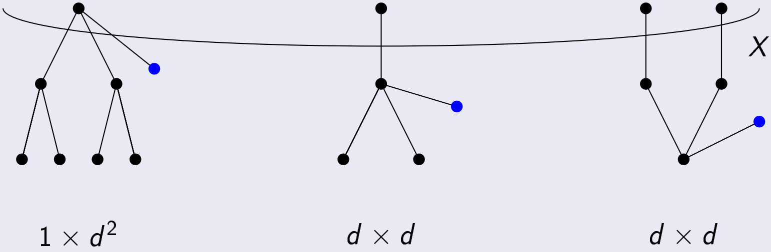

Insufficiency of the Main Lemma

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Let G be d-regular with girth $> 2s+2$. Then $\Omega(d^s)$ cops are needed to capture the robber in G.

Conjecture [Bollobas'78]

For all s and infinitely many n there is an $n^{\frac{1}{s+1}}$-regular graph with girth $> 2s+2$.

If true, the conjecture implies $f_s(G) = \Omega\left(n^{s/s+1}\right)$.

Only proved to be true for $s = 2, 4$!

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A Stronger Version of the Main Lemma

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Let G be d-regular bipartite graph with diameter larger than s, such that

1. *If u and v are vertices of distance $\leq s+1$, there are $O(1)$ distinct shortest (u,v)-paths.*
2. *For every vertex u and subset A of vertices having size $O(1)$, there exist $\Omega(d^s)$ vertices x of distance s from u, such that any shortest (u,x)-path avoids A.*

Then $\Omega(d^s)$ cops are needed to capture the robber in G.

Proof.

Similar to the proof of the Main Lemma. □

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Similar to the proof of the Main Lemma. $\square$

The Cayley Graph

Let $d := 2^r$,

$x_1, \ldots, x_d$: d elements of $GF(d)$ as 0,1-column vectors of length r,

$$H = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_d \\ x_1^3 & x_2^3 & \ldots & x_d^3 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2s+1} & x_2^{2s+1} & \ldots & x_d^{2s+1} \end{bmatrix}_{1+r(s+1)\times d}$$

Key property: Every $2s+2$ columns of H are independent.

G: the graph with vertex set $\mathbb{Z}_2^{1+r(s+1)}$,

v_1, v_2 adjacent if $v_1 - v_2$ is a column of H.

G is d-regular, has $2d^{s+1}$ vertices.

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$$x - y = a_1 + a_2 + \cdots + a_{s+1}$$

Another shortest path between x and y:

$$x - y = a_1' + a_2' + \cdots + a_{s+1}'$$

Then

$$a_1 + a_2 + \cdots + a_{s+1} + a_1' + a_2' + \cdots + a_{s+1}' = 0$$

So $\{a_1', a_2', \ldots, a_{s+1}'\}$ is a permutation of $\{a_1, a_2, \ldots, a_{s+1}\}$.
There are $(s+1)! = O(1)$ shortest (x, y)-paths.

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A Generalization of Meyniel's Conjecture

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For all s,

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Known Results
for a robber with speed n

Write $c_\infty(G)$ for the cop number of G if the robber has speed n.

- Computing $c_\infty(G)$ is NP-hard.
  [Fomin, Golovach, Kratochvíl'08]
- Computing $c_\infty(G)$ is in P if G is an interval graph.
  [Gavenciak'11]
- For every n, there exists G with $c_\infty(G) = \Theta(n)$.
  [Frieze, Krivelevich, Loh'11]

New Results
for a robber with speed n

Theorem (M'11+)

$$\frac{tw(G)+1}{\Delta+1} \leq c_\infty(G) \leq tw(G)+1$$

$$G \text{ planar} \Rightarrow c_\infty(G) = \Theta\left(tw(G)\right)$$

$$G \text{ interval} \Rightarrow c_\infty(G) = O(\sqrt{n})$$

$$\exists \text{ chordal } G \text{ s.t. } c_\infty(G) = \Omega\left(\frac{n}{\log n}\right)$$

$$np \geq 5\ln n \Rightarrow \frac{k_1}{p} \leq c_\infty(\mathcal{G}(n,p)) \leq \frac{k_2 \ln(np)}{p}$$