Cops and Robber Game with a Fast Robber on Expander Graphs and Random Graphs

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Game Definition

Definition (The Game of Cops and Robber)

- The game is played on a graph.
- There is a set of cops and a robber.
- In the beginning,
  - First, each cop chooses a starting vertex.
  - Then, the robber chooses a starting vertex.
- In each round,
  - First, each cop chooses to stay or go to an adjacent vertex.
  - Then, the robber chooses to stay, or move along a cop-free path.
- The cops capture the robber if, at some moment, a cop is at the same vertex with the robber.

Some Remarks

1. This is a perfect-information game: the players see each other.
2. More than one cops can be at the same vertex.
3. The robber cannot jump over a cop.
4. The moves are deterministic (no randomness).

Cop Number

Definition

The minimum number of cops that are needed to capture the (clever) robber is denoted by $c_\infty(G)$, and is called the cop number of G.

Example

- If G is the complete graph, then $c_\infty(G) = 1$.
- If G is a cycle with > 3 vertices, then $c_\infty(G) = 2$.
- If G is the $m \times m$ grid, then $c_\infty(G) = m$.

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Known Results

- Computing $c_\infty(G)$ is NP-hard.

  [Fomin, Golovach, Kratochvíl'08]

- For every n, there exists a connected graph G on n vertices with $c_\infty(G) = \Theta(n)$. [Frieze, Krivelevich, Loh'11]

Today:

- Bounds for $c_\infty(G)$ when G is an expander graph
- Results in bounds for the cop number of random graphs

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Notation

G	the graph of the game, simple and connected
n	the number of vertices of G
δ, Δ	the minimum, maximum degree of G
$\log$	the natural logarithm

The (Closed) Neighbourhood of a Subset

Definition

Let $S \subseteq V(G)$. The (closed) neighbourhood of S, written $\overline{N}(S)$, is the set of vertices that are in S or have a neighbour in S.

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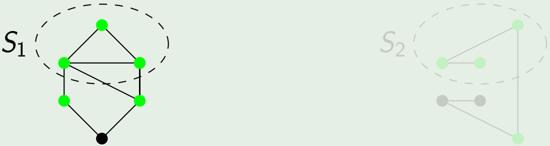

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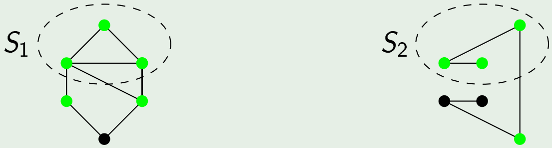

The Large Component Lemma

Lemma

Assume that for every subset S of vertices of size $\leq m$, $G - \overline{N}(S)$ has a connected component of size $> n/2$. Then $c_\infty(G) > m$.

Proof.

Let there be m cops. We give an escaping strategy for robber.

Invariant: Robber in largest component of $G - \overline{N}(S)$, $S =$ cops' position

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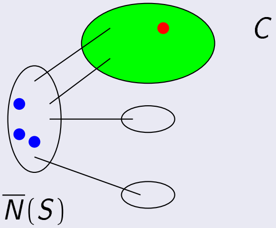

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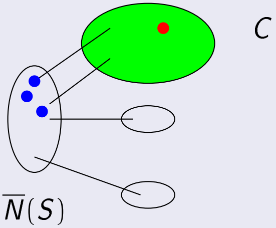

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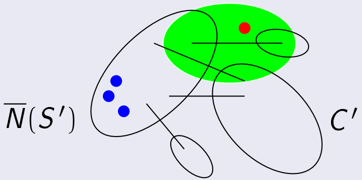

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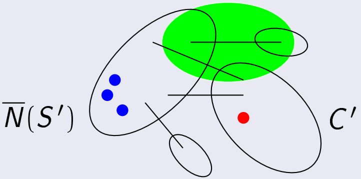

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In other words,

Lemma

Let $c = c_\infty(G)$. There exists a subset S of size $\leq c$ such that $G - \overline{N}(S)$ has no component of size $> n/2$.

Vertex Expansion

Definition

Let G be a graph. The vertex expansion of G, $\iota(G)$, is the following quantity:

$$\iota(G) = \min_{|S| \leq n/2} \frac{|\overline{N}(S) \setminus S|}{|S|}.$$

Example

$\iota(G_1) = 2/3$

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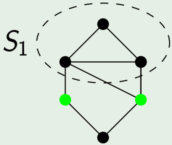

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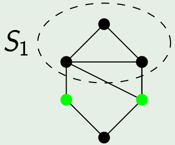

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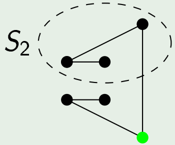

$\iota(G_2) = 1/3$

Lower Bound for Expander Graphs

Theorem

$$c_\infty(G) \geq \frac{\iota n}{4(\Delta + 1)}$$

Proof.

Let $c = c_\infty(G)$. There exists a subset S of size $\leq c$ such that $G - \overline{N}(S)$ has no component of size $> n/2$. Clearly $\overline{N}(S) \leq c(\Delta + 1)$. Let $C_1, \ldots, C_m$ be the components of $G - \overline{N}(S)$.

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If $|C_1| + \cdots + |C_m| < n/4$, then $\overline{N}(S) \geq 3n/4$, so

$$c(\Delta+1) \geq \overline{N}(S) \geq \frac{3n}{4} > \frac{\iota n}{4} \quad \square$$

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If $|C_1| + \cdots + |C_m| \geq n/4$, as each C_i has size $\leq n/2$, one can pick some of the C_i's such that their union U has $n/4 \leq |U| \leq n/2$.

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Set U has at least $\iota|U|$ neighbours outside, so

$$c(\Delta + 1) \geq |\overline{N}(S)| \geq \iota|U| \geq \iota n/4 \qquad \square$$

The Erdös-Rényi Random Graph

Definition

$\mathcal{G}(n,p)$ is a random graph on a set of vertices of size n, in which each edge appears in $\mathcal{G}(n,p)$ independently and with probability p. For a graph property A, we say $\mathcal{G}(n,p)$ asymptotically almost surely (a.a.s.) satisfies A, if we have

$$\lim_{n\to\infty} \mathbf{Pr}\left[\mathcal{G}(n,p(n)) \text{ satisfies } A\right] = 1$$

Vertex Expansion of Random Graphs

Theorem

Let $0 < b < 1$ be fixed, and t, k be constants such that

$$t > \frac{1 + \log 2}{1 - b} - \log(1 - b), \qquad k > \frac{2t}{1 - e^{-t}}.$$

If $np \geq k \log n$ then a.a.s. $\iota(\mathcal{G}(n,p)) \geq b$.

Proof.

Two pages of calculations and using Chernoff bounds ... □

Corollary

If $np \geq 4.2 \log n$, then a.a.s. $\iota(\mathcal{G}(n,p)) \geq 10^{-3}$.

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An Obvious Upper Bound For The Cop Number

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Set $X \subseteq V(G)$ is a dominating set if every vertex is either in X or adjacent to a vertex in X.
The domination number of graph G is the minimum size of a dominating set of G.

Proposition

The cop number $\leq$ the domination number.

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The cops start at a dominating set. They will capture the robber in their first move. $\square$

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Upper Bounds for Random Graphs

Corollary (from previous slides)

If $np \geq 4.2 \log n$, then a.a.s. $c_\infty(\mathcal{G}(n,p)) = \Omega\left(\frac{1}{p}\right)$.

Theorem (Alon, Spencer'92)

The domination number of any graph G is $O(n \log \delta/\delta)$.

Corollary

If $np \geq 4.2 \log n$, then a.a.s.

$$k_1\left(\frac{1}{p}\right) \leq c_\infty(\mathcal{G}(n,p)) \leq k_2\left(\frac{\log(np)}{p}\right)$$

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Tighter Bounds for Denser Random Graphs

Theorem (Bonato, Prałat, and Wang'07)

Consider the original game. If $np = n^{\alpha+o(1)}$, where $1/2 < \alpha < 1$, then a.a.s $\Omega(\log n/p)$ cops are needed.

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Use the result on the domination number for upper bounds, and the result for the "slow robber" version for lower bounds. □

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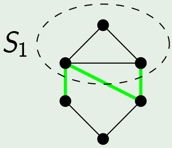

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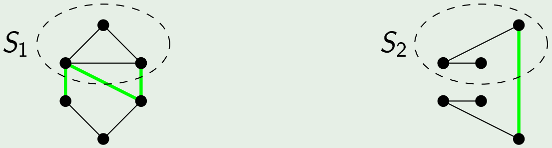

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Asymptotic Cop Number of Random Regular Graphs

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Fix $d \geq 3$. A.a.s. a random d-regular labelled graph G on n vertices has $c_\infty(G) = \Theta(n)$.

Proof.

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Open Problem

When $np \geq 4.2 \log n$, we proved that a.a.s.

$$\frac{k_1}{p} \leq c_\infty(\mathcal{G}(n,p)) \leq \frac{k_2 \log(np)}{p}$$

What is the correct value?

Thank You!

Any Questions?