Cops and Robber Game with a Fast Robber

by

Abbas Mehrabian

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Abstract

Graph searching problems are described as games played on graphs, between a set of searchers and a fugitive. Variants of the game restrict the abilities of the searchers and the fugitive and the corresponding search number (the least number of searchers that have a winning strategy) is related to several well-known parameters in graph theory. One popular variant is called the Cops and Robber game, where the searchers (cops) and the fugitive (robber) move in rounds, and in each round they move to an adjacent vertex. This game, defined in late 1970's, has been studied intensively. The most famous open problem is Meyniel's conjecture, which states that the cop number (the minimum number of cops that can always capture the robber) of a connected graph on n vertices is $O(\sqrt{n})$.

We consider a version of the Cops and Robber game, where the robber is faster than the cops, but is not allowed to jump over the cops. This version was first studied in 2008. We show that when the robber has speed s, the cop number of a connected n-vertex graph can be as large as $\Omega(n^{s/s+1})$. This improves the $\Omega(n^{\frac{s-3}{s-2}})$ lower bound of Frieze, Krivelevich, and Loh (Variations on Cops and Robbers, J. Graph Theory, to appear). We also conjecture a general upper bound $O(n^{s/s+1})$ for the cop number, generalizing Meyniel's conjecture.

Then we focus on the version where the robber is infinitely fast, but is again not allowed to jump over the cops. We give a mathematical characterization for graphs with cop number one. For a graph with treewidth tw and maximum degree Δ, we prove the cop number is between $\frac{tw+1}{\Delta+1}$ and $tw+1$. Using this we show that the cop number of the m-dimensional hypercube is between $\frac{c_1 n}{m\sqrt{m}}$ and $\frac{c_2 n}{m}$ for some constants c_1 and c_2. If G is a connected interval graph on n vertices, then we give a polynomial time 3-approximation algorithm for finding the cop number of G, and prove that the cop number is $O(\sqrt{n})$. We prove that given n, there exists a connected chordal graph on n vertices with cop number $\Omega(n/\log n)$. We show a lower bound for the cop numbers of expander graphs, and use this to prove that a random $G \in \mathcal{G}(n,p)$ that is not very sparse, asymptotically almost surely has cop number between $\frac{d_1}{p}$ and $\frac{d_2 \log(np)}{p}$ for suitable constants d_1 and d_2. Moreover, we prove that a fixed-degree regular random graph with n vertices asymptotically almost surely has cop number $\Theta(n)$.

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Chapter 1

Introduction

1.1 Game definition

The game of Cops and Robber is a perfect information game, in which the players are a set of cops and a robber. Let G be a graph and s be a positive integer. Initially, the cops are placed at vertices of their choice in G (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops' placement, positions herself at one of the vertices of G. Then the cops and the robber move in alternate rounds, with the cops moving first, where every cop may move in each round and players are permitted to remain stationary in their turn if they wish. In every round, a cop can move to an adjacent vertex, and the robber can take *any path of length at most* s from her current position, but she is not allowed to pass through a vertex occupied by a cop. The cops win and the game ends if eventually a cop moves to the vertex currently occupied by the robber; otherwise, i.e., if the robber can elude the cops forever, the robber wins. The parameter s is called, naturally, the *speed* of the robber. One interesting case is when s is equal to the number of vertices of G, in which the robber can move along *any path of arbitrary length* (with no cop present at its internal vertices) in her move. In this case we may abuse notation and write $s = \infty$. The parameter of interest is the *cop number of* G, which is defined as the minimum number of cops needed to ensure that the cops can win.

1.2 Notation

In this thesis all logarithms are in base $e \approx 2.718$ unless specified otherwise, the set of all positive integers is denoted by $\mathbb{N}$. Let G be a finite graph. n will always denote the number of vertices of G. We will write $\delta = \delta(G)$ and $\Delta = \Delta(G)$ for minimum and maximum degrees of G, respectively. For a subset A of vertices, the *neighbourhood* of A, written $N(A)$, is the set of vertices that have a neighbour in A, and the *closed neighbourhood* of A, written $\overline{N}(A)$, is the union $A \cup N(A)$. If $A = \{v\}$ then we may write $N(v)$ and $\overline{N}(v)$ instead of $N(A)$ and $\overline{N}(A)$, respectively. A *dominating set* is a subset A of vertices with $V(G) = \overline{N}(A)$, and the *domination number* of G, written $\gamma(G)$, is the minimum cardinality of a dominating set of G. The subgraph induced by A is written $G[A]$, and the subgraph induced by $V(G) - A$ is written $G - A$. We say A is a *cut-set* if $G - A$ has more connected components than G. For vertices u and v, the *distance* between u and v in G, written $d(u, v)$, is the length of the shortest (u, v)-path in G.

Let G be the graph in which the game is played. We will assume that the graph G is simple, because deleting multiple edges or loops does not affect the set of possible moves of the players. Usually we consider only connected graphs, since the cop number of a disconnected graph obviously equals the sum of the cop numbers for each connected component. Note that if we are only interested in studying the cop number, then we may assume without loss of generality that the cops choose vertices of our choice in the beginning, since they can simply move to the vertices of their choice later. If the speed of the robber is s, where $s \in \mathbb{N} \cup \{\infty\}$, then we denote the cop number of G by $c_s(G)$. We let $f_s(n)$ be the maximum of $c_s(G)$ among all connected graphs G with n vertices.

1.3 Previous work

1.3.1 The unit-speed variant

The variant of the game with $s = 1$, i.e. when the cops and the robber have speed one, has been studied intensively. The game was defined (for one cop) by Winkler and Nowakowski [43] and Quilliot [46]. For surveys of results on $c_1(G)$ and related search parameters, see [4, 30].

The famous open question in this area is Meyniel's conjecture, published by Frankl [25], which states that $f_1(n) = O(\sqrt{n})$. This is asymptotically tight, i.e. it is known that $f_1(n) = \Omega(\sqrt{n})$ (see [45] for instance). The best upper bound found so far is

$$f_1(n) \leq n 2^{-(1-o(1))\sqrt{\log_2 n}},$$

see [27, 36, 49] for various proofs.

Goldstein and Reingold [29] studied the complexity of computing $c_1(G)$ for an input graph G, and showed that if the starting configuration is given as part of the input, then deciding whether the cops can capture the robber is $\mathcal{EXP}$-complete (this is intuitively the class of problems that require exponential time to solve, see Chapter 2 of [6] for the precise definition). They also proved that if the game is played in a directed graph, such that the players should respect the direction of the arcs, then finding the number of cops necessary to capture the robber is $\mathcal{EXP}$-complete, even if the input graph is restricted to be strongly connected.

Joret, Kamiński, and Theis [35] studied $c_1(G)$ when G does not have certain (induced) subgraphs. For any $m > 3$, they proved that if G does not have the path on m vertices, or the cycle on m vertices, as an induced subgraph, then $c_1(G) \leq m - 2$. They also proved that if G has no cycle of length larger than m as a subgraph, then $c_1(G) \leq \lceil m/2 \rceil$. Andreae [5] studied $c_1(G)$ when G has forbidden minors. He proved that if u is a vertex of a graph H such that $H - u$ has no isolated vertex and G does not have H as a minor, then $c_1(G) \leq |E(H - u)|$. As a consequence, he proved that if K_m is not a minor of G then $c_1(G) \leq \binom{m-1}{2}$.

The following proposition, which was proved by Aigner and Fromme [1], is the basis for many upper bounds for the cop number (in the unit-speed setting).

Proposition 1.1. *Let u and v be two vertices of a graph G, and P be a shortest (u, v)-path in G. A single cop can play in such a way that, after a finite number of rounds, he will prevent the robber from entering P. That is, if the robber enters a vertex of P, the cop will capture her immediately.*

For lower bounds, the only nontrivial general bound is the following, which has been proved by Frankl [25]: for every $m \in \mathbb{N}$, if G has no cycle with less than $8m - 3$ vertices, then $c_1(G) > (\delta(G) - 1)^m$.

Winkler and Nowakowski [43], and independently Quilliot [46], characterized graphs G with $c_1(G) = 1$.

Proposition 1.2. *A graph G has cop number one, if and only if, there is an ordering $(v_1, v_2, \ldots, v_n)$ of its vertices, such that for every $1 \leq i \leq n - 1$, there exists an index j, $j > i$, with $\overline{N}(v_i) \subseteq \overline{N}(v_j)$.*

Graphs that have such an ordering are called *dismantlable* graphs. Later, Quilliot [47] proved that all chordal graphs are dismantlable (a chordal graph is a graph that has no induced cycle with more than three vertices).

Aigner and Fromme [1] studied $c_1(G)$ when G is a planar graph, and proved that $c_1(G) \leq 3$ in this case. Later, Schroeder [48] proved that if G has genus g then $c_1(G) \leq \lfloor 3g/2 \rfloor + 3$, and conjectured that in fact $c_1(G) \leq g + 3$. Clarke [19] proved that if G is outerplanar (can be drawn on the plane in such a way that all vertices are incident to the outer face), then $c_1(G) \leq 2$.

Frankl [26] studied $c_1(G)$ when G is a Cayley graph, and proved that if Γ is a commutative group and S is a generating subset of Γ with $S = S^{-1}$, then the cop number of the

Cayley graph of Γ with respect to S is at most $\lceil(|S|+1)/2\rceil$. (The Cayley graph of a group Γ with respect to a subset $S \subseteq \Gamma$, is a graph G with vertex set Γ and with $a, b \in \Gamma$ being adjacent if $a^{-1}b \in S$.) The *m-dimensional hypercube*, denoted by $\mathcal{H}_m$, is the graph with vertex set $\{0,1\}^m$ and two vertices being adjacent if they differ in exactly one component. It can be shown using the above bound for Cayley graphs that $c_1(\mathcal{H}_m) = \lceil(m+1)/2\rceil$. Frankl [25] also proved that if Γ is a group, and S is a generating subset of Γ such that for all $g \in \Gamma$ and $s \in S$, we have $gsg^{-1} \in S$, then the cop number of the Cayley graph of Γ with respect to S is at most $|S|$.

Tošić [51] considered the game played in the Cartesian product of graphs (for the definition of Cartesian product see Chapter 9). He proved that if G is the Cartesian product of G_1 and G_2, then $c_1(G) \le c_1(G_1) + c_1(G_2)$. Neufeld and Nowakowski [41] further studied $c_1(G)$ when G is the product of several graphs. If G is the Cartesian product of k cycles with at least 4 vertices and m trees with at least 2 vertices, then $c_1(G) = k + \lceil(m+1)/2\rceil$. This shows that the cop number of any m-dimensional grid is $\lceil(m+1)/2\rceil$. They also proved that the cop number of the Cartesian product of m complete graphs, each having at least 3 vertices, is equal to m. They also considered categorial and strong products of graphs (we do not define them here). They showed that the cop number of the categorial product of m complete graphs, each having at least 3 vertices, is at most $\lfloor m/2\rfloor + 2$; and that the cop number of the strong product of m cycles, each having at least 5 vertices, is at most $m+1$.

Several authors have studied $c_1(G)$ when G is an Erdös-Rényi random graph with parameters n and p (see Chapter 8 for the definition). Bollobás, Kun, and Leader [11] showed that if $np \ge 2.1 \log n$, then asymptotically almost surely (a.a.s.),

$$\frac{1}{(np)^2} n^{\frac{1}{2}\frac{\log\log(np)-9}{\log\log(np)}} \le c_1(G) \le 160000\sqrt{n}\log n,$$

which shows that the Meyniel's conjecture is true for such random graphs, up to a $\log n$ factor, and that sparse random graphs have cop number $n^{1/2-o(1)}$. Let us parameterize $np = n^{\alpha+o(1)}$. For $1/2 < \alpha \le 1$, Bonato, Prałat, and Wang [14] proved that a.a.s.

$$c_1(G) = \Theta(\log n/p) = n^{1-\alpha+o(1)}.$$

For $\alpha = 1/2$, the same authors proved that a.a.s. $c_1(G) = n^{1/2+o(1)}$. For $0 < \alpha < 1/2$, Łuczak and Prałat [37] have determined the asymptotic value of $c_1(G)$ up to a $\log^{O(1)} n$ factor, but their result is too complicated to be presented here.

Hahn, Laviolette, Sauer, and Woodrow [31] considered the game of Cops and Robber played in an infinite graph G. They proved that there exist infinite chordal graphs G with $c_1(G) > 1$. For more results in infinite graphs, see [13].

1.3.2 The fast robber variant

The generalized variant, $s > 1$, was first studied by Fomin, Golovach, Kratochvíl, Nisse, and Suchan [22, 42]. They merged those two papers and wrote a journal version [23], in which the following results appear.

- For every fixed $s \in \mathbb{N} \cup \{\infty\}$, calculating $c_s(G)$ is $\mathcal{NP}$-hard.
- For fixed $s \in (\mathbb{N} - \{1\}) \cup \{\infty\}$, the problem remains $\mathcal{NP}$-hard even if the input graph is a split graph (in contrast to the $s = 1$ case, where the cop number of a split graph is always one). A *split graph* is a graph whose vertex set can be partitioned into a clique and an independent set.
- For fixed $s \in \mathbb{N}$ there exists a polynomial time algorithm for finding $c_s(G)$ when G is an interval graph.
- For $s \in \{1, 2\}$, there exists a polynomial time algorithm for finding $c_s(G)$ when G has bounded cliquewidth. Cliquewidth is a graph parameter that measures in a certain sense the complexity of a graph, but we do not define it here.
- When G is planar, $c_2(G)$ can be as large as $\Omega(\sqrt{\log n})$ (in contrast to the $s = 1$ case, where the cop number is at most 3, see [1]).
- If a graph G has the $m \times m$ grid as an induced subgraph, then $c_2(G) = \Omega(\sqrt{\log m})$. However, there exist graphs G that have the $m \times m$ grid as a minor and yet $c_2(G) \leq 2$.

Although computing $c_s(G)$ is $\mathcal{NP}$-hard for every fixed s, it is known that for every fixed $c \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{\infty\}$, deciding if $c_s(G) \leq c$ can be done in polynomial time (see [32] for an algorithm for the $s = 1$ case, which can easily be generalized to greater s).

The game with $s > 1$ was further studied by Frieze, Krivelevich and Loh [27], where the authors' approach is based on expansion. Let $s \in \mathbb{N}$ be fixed and $\alpha = 1 + \frac{1}{s}$. It is shown in [27] that there exists a constant $k > 0$ such that for all n,

$$kn^{(s-3)/(s-2)} \leq f_s(n) \leq n\alpha^{-(1-o(1))\sqrt{\log_\alpha n}}.$$

Moreover, it is shown there that $f_\infty(n) = \Omega(n)$.

1.4 Motivation

One interesting fact about the classical (unit-speed) Cops and Robber game is that, many scholars have studied the game, and yet it is not really well understood: for example, it is not known how large the value of c_1 for a graph on n vertices can be. It was conjectured in 1987 that $O(\sqrt{n})$ cops suffice, but no upper bound better than $n^{1-o(1)}$ has been proved so far. As an another example, no nontrivial approximation algorithm for finding the c_1 of a graph has been developed.

One might try to change the rules of the game a little in order to get a more approachable problem, and/or to understand what property of the original game causes the difficulty. One way to do this, is limiting the visibility of the cops (see [18] for instance). If the visibility of both players are limited, it might be appropriate for the players to deploy randomized strategies, see [34] for some results on this. Another approach is to change the definition of capturing. For example, the version in which the cops capture the robber if some cop gets close enough to the robber has been studied in [12]. One can also make the edges unidirectional and allow the players to move only in a certain direction along each edge (see [27]).

The approach chosen in this thesis is to allow the robber move faster than the cops: we study the $s > 1$ variant, and especially the $s = \infty$ case. One nice fact about this variation is the following: if we let the robber move infinitely fast, and also change the rules to make this a real-time game, we get the so-called Helicopter Cops and Robber game (defined in [50], see Chapter 4 for the definition), for which it is known that the number of cops needed equals the treewidth of the graph (which is a fairly well-understood parameter) plus one (see [50]). Thus one may hope to get good bounds for c_∞ in terms of treewidth by relating the Cops and Robber game with infinitely fast robber with the Helicopter Cops and Robber game, and this is what we do in Chapter 4. However, this analogy should not deceive one; c_∞ can be arbitrarily smaller than treewidth: any graph with small domination number and large treewidth (say, a complete graph) is such an example. Therefore, this thesis can also be regarded as an attempt to find connections between the classical Cops and Robber game and the Helicopter Cops and Robber game by studying an in-between game.

The parameter c_1 seems to be related to many graph parameters, and this is another interesting fact about it. Connections are known with the shortest and longest cycles, excluded minors, genus, domination number, diameter and vertex expansion (see Section 1.3.1 for other examples). By studying $c_s(G)$ for $s > 1$, we try to understand which of these connections depend heavily on the fact that the robber has speed one. On one hand, diameter becomes irrelevant as soon as the robber gets faster. This is probably because most bounds in terms of diameter are based on Proposition 1.1, which does not hold if the robber is fast. On the other hand, the domination number is an upper bound for c_s for all values of s, since if the cops start by occupying a dominating set, they will capture the robber in the first round. In Chapter 3 we will observe that the parameter c_s is related to the number of geodesic paths of length $s+1$ between any two pair of vertices. Treewidth is closely connected with c_∞, and Chapter 4 is completely devoted to this connection. Tree and path decompositions arise naturally and are important when studying c_∞, and the idea of several proofs in Chapters 5 and 6 is based on them (although they do not appear explicitly in the mentioned chapters). Expansion properties of a graph also seem to be closely connected with c_∞ as shown in Chapter 7, and this fact is used to prove lower bounds for the c_∞ of random graphs in Chapter 8.

1.5 Summary of new results

The main results proved in this thesis are given below. To make this summary short, we omit the definition of some terms. The definition of each term can be found in the corresponding chapter. Let G be a connected graph on n vertices and recall that $c_s(G)$ is the cop number of G when the robber has speed s.

Chapter 2: We give a characterization of graphs G with $c_\infty(G) = 1$, and provide an $O(n^2)$ algorithm for deciding if G has this property.

Chapter 3: Fix $s \in \mathbb{N}$. We prove that for every n there exists G with

$$c_s(G) = \Omega\left(n^{s/(s+1)}\right).$$

This result appears in [2]. See [39] for a simpler proof when $s = 2, 4$. Frieze et al. [27] had proved that for every n there exists G with $c_s(G) = \Omega\left(n^{(s-3)/(s-2)}\right)$, and had asked whether there exist graphs G with $c_2(G) = \omega(\sqrt{n})$. This result improves their bound and gives a positive answer to their question, as we provide graphs with $c_2(G) = \Omega(n^{2/3})$. The best known general upper bound [27] is not better than $c_s(G) \le n^{1-o(1)}$.

Chapter 4: Let $tw(G)$ and $\Delta(G)$ denote the treewidth and maximum degree of G, respectively. We prove that for every G,

$$\frac{tw(G)+1}{\Delta(G)+1} \le c_\infty(G) \le tw(G)+1,$$

and provide examples for which these bounds are tight.

Chapter 5: If G is an interval graph, then we prove that $c_\infty(G) = O(\sqrt{n})$ and provide examples for which this bound is tight. We also give a polynomial time 3-approximation algorithm for finding $c_\infty(G)$.

Chapter 6: We prove that for every n there exists a chordal graph G with $c_\infty(G) = \Omega(n/\log n)$.

Chapter 7: Let $\iota_e(G)$ and $\iota_v(G)$ denote the edge isoperimetric and vertex isoperimetric numbers of G, respectively. We prove that for every G,

$$c_\infty(G) \geq \max\left\{\frac{\iota_e n}{2\Delta^2}, \frac{\iota_v n}{4\Delta}\right\}.$$

Chapter 8: Let $\lim_{n\to\infty} np - 20\log n = \infty$. We prove that asymptotically almost surely a random graph $G \in \mathcal{G}(n,p)$ has

$$c_\infty(G) = \Omega\left(\frac{n}{\Delta}\right) = \Omega\left(1/p\right).$$

If also $p = 1 - \Omega(1)$, then we prove that asymptotically almost surely G has

$$c_\infty(G) = O(\log(np)/p).$$

Chapter 9: Let P_n denote a path with n vertices. We prove that if G is the Cartesian product of $P_{k_1}, P_{k_2}, \ldots, P_{k_m}$, where $k_1 = \max\{k_i : 1 \leq i \leq m\}$, then

$$\frac{n}{4k_1 m^2} \leq c_\infty(G) \leq \frac{n}{k_1}.$$

Moreover, if each k_i is equal to 2 (i.e. if G is the m-dimensional hypercube), then there exist constants κ_1, κ_2 with

$$\frac{\kappa_1 n}{m\sqrt{m}} \leq c_\infty(G) \leq \frac{\kappa_2 n}{m}.$$

Chapter 2

Characterization of Graphs with Cop Number One

Graphs G with $c_1(G) = 1$ have been characterized in the earliest works on the Cops and Robber game [43, 46]. In this chapter we characterize graphs G with $c_\infty(G) = 1$, and give an $O(n^2)$ algorithm for their detection.

Definition (block, block tree)**.** Let G be a connected graph. By a *block* of G, we mean either a maximal 2-connected subgraph of G, or an edge of G that is not contained in any 2-connected subgraph. We may associate with G a bipartite graph $B(G)$ with bipartition $(\mathcal{B}, S)$, where $\mathcal{B}$ is the set of blocks of G and S is the set of cut vertices of G, a block B and a cut vertex v being adjacent in $B(G)$ if and only if B contains v. The graph $B(G)$ is a tree, called the *block tree* of G (see for example [15], page 121).

Lemma 2.1. *If $c_\infty(G) = 1$ then every block of G has domination number one.*

Proof. Suppose for the sake of contradiction that $c_\infty(G) = 1$ and B is a block of G with domination number larger than one. So B is a 2-connected subgraph. Assume that there is a single cop in the game. We claim that the robber can play in such a way that, at the end of each round, if the cop is at a vertex v, then the robber is at a vertex $r \in V(B) \setminus \overline{N}(v)$. This shows that she can elude the cop forever, which contradicts the assumption $c_\infty(G) = 1$.

Assume that the cop starts at $v_0 \in V(G)$. Since B has domination number larger than one, there exists $r_0 \in V(B) \setminus \overline{N}(v_0)$. The robber starts at r_0. For every positive i, suppose that in round i, the cop moves to v_i. Since B has domination number larger than one, there exists $r_i \in V(B) \setminus \overline{N}(v_i)$. As B is 2-connected, there are two disjoint (r_{i-1}, r_i)-paths in G, so there exists an (r_{i-1}, r_i)-path in G that does not contain v_i. The robber has infinite speed and moves along that path to r_i, and the proof is complete. ■

Definition (directed hole, hallway)**.** Let u be a cut vertex of G, and B be a block of G containing u. If $\{u\}$ is not a dominating set for B, then the pair (B, u) is called a *directed hole*. Let B, B' be two distinct blocks of G, and $Bu_1 \dots u_kB'$ be the unique (B, B')-path in $B(G)$. If both (B, u_1) and (B', u_k) are directed holes, then the pair $\{B, B'\}$ is called a *hallway*.

Note that if a block B is not 2-connected, then it consists of a single edge, and each of its vertices makes a dominating set. Hence, if $\{B, B'\}$ is a hallway, then both B and B' are maximal 2-connected subgraphs. We will prove that a graph G has $c_\infty(G) = 1$ if and only if each of its blocks has domination number one, and it does not have a hallway.

Lemma 2.2. *If $c_\infty(G) = 1$, then G does not have a hallway.*

Proof. Suppose for the sake of contradiction that $c_\infty(G) = 1$ and $\{B, B'\}$ is a hallway. By the discussion before the lemma, B and B' are maximal 2-connected subgraphs. Let $Bu_1 \dots u_kB'$ be the unique (B, B')-path in $B(G)$. Assume that there is a single cop in the game. Since (B, u_1) is a directed hole, there exists $b \in V(B) \setminus \overline{N}(u_1)$. Similarly, since (B', u_k) is a directed hole, there exists $b' \in V(B') \setminus \overline{N}(u_k)$. Note that the distance between b and u_1 in G is at least 2, and the distance between u_k and b' in G is at least 2, so the distance between b and b' in G is at least 4. We claim that the robber can play in such a way that, at the end of each round, if the cop is at a vertex v, then she is at a vertex $r \in \{b, b'\} \setminus \overline{N}(v)$. This shows that she can elude the cop forever, which contradicts the assumption $c_\infty(G) = 1$.

Assume that the cop starts at $v_0 \in V(G)$. As the distance between b and b' in G is at least 4, there exists $r_0 \in \{b, b'\} \setminus \overline{N}(v_0)$ and the robber starts at r_0. For every positive i, suppose that in round i, the cop moves to v_i. At the end of round $i-1$, the robber was either at b or at b', and by symmetry we may assume that she was at b. If $b \notin \overline{N}(v_i)$, then the robber remains at b. Otherwise $b \in \overline{N}(v_i)$ so $v_i \neq u$ since $b \notin \overline{N}(u_1)$, and $b' \notin \overline{N}(v_i)$ since the distance between b and b' in G is at least 4. There exists two disjoint (b, u_1)-paths, thus at least one of them is cop-free. There is also a cop-free (u_1, u_k)-path and a cop-free (u_k, b')-path so the robber can move to b' in her turn. ■

The two above lemmas prove the "only if" part of the result we are going to prove. For the other direction, we need another definition and a lemma.

Definition (end block)**.** Let G be a connected graph such that $B(G)$ has more than one vertex. The blocks of G which correspond to leaves of $B(G)$ are referred to as its *end blocks*.

Lemma 2.3. *Let B be an end block of graph G, and u be the unique cut vertex of G contained in B. Assume that $\{u\}$ is a dominating set for B. Let H be the graph obtained by contracting the subgraph B into vertex u. Then we have $c_\infty(H) \geq c_\infty(G)$.*

Proof. We need to show that for every positive c, if c cops can capture the robber in H, then c cops can capture the robber in G. Assume that c cops have a capturing strategy in H. They may use the following strategy in G: whenever the robber is at a vertex $r \in V(H)$, they move according to their strategy in H, and when the robber moves to a vertex in $r \in V(G) \setminus V(H)$, they just "imagine" that the robber is at u, and again move according to their strategy in H. Since the cops' strategy in H is winning, they eventually will either capture the robber in H, or capture the "imagined" robber at u. In the former case, the robber is captured in G as well. In the latter case, there would be a cop at u and the robber would be in $V(G) \setminus V(H)$. Now, that cop can capture the robber in the next move, as $\{u\}$ is a dominating set for B, and $V(G) \setminus V(H) \subseteq V(B)$. ■

We are now ready to prove the main result of this chapter.

Theorem 2.4. *A connected graph G has $c_\infty(G) = 1$ if and only if each of its blocks has domination number one, and it does not have a hallway.*

Proof. If $c_\infty(G) = 1$ then by Lemma 2.1 each of the blocks of G has domination number one, and by Lemma 2.2, G does not have a hallway.

Conversely, let G be a connected graph such that each of its blocks has domination number one, and it does not have a hallway. We perform the following operation on G: let B be an arbitrary end block of G, and u be the unique cut vertex of G contained in B. If $\{u\}$ is a dominating set for B, then we contract the subgraph B into vertex u. We repeat this operation until no such end block exists. Let H be the resulting graph. Note that each of the blocks of H is also a block of G.

Claim. The graph H has a single block.

Proof of Claim. If H has more than one block, then since $B(H)$ is a tree, it has at least two leaves. Let B and B' be two end blocks of H, u and u' be the unique cut vertices of H with $u \in V(B)$ and $u' \in V(B')$. Since we cannot perform the above operation on H, we know that $\{u\}$ is not a dominating set for B, and $\{u'\}$ is not a dominating set for B'. But then $\{B, B'\}$ would be a hallway in G, contradiction! □

Each block of H is also a block of G, hence H has domination number one, thus $c_\infty(H) = 1$. Lemma 2.3 shows that $c_\infty(G) \leq c_\infty(H)$, and the proof is complete. ■

We gave a mathematical characterization for graphs G with $c_\infty(G) = 1$. Using this we give a simple algorithm for detecting such graphs.

Corollary 2.5. *Let G be a connected graph on n vertices. There exists an $O(n^2)$ algorithm to decide whether $c_\infty(G) = 1$.*

Proof. The block tree of G can be built in time $O(|E(G)|)$ using depth-first search (see for example [15], page 142). If block B has m vertices, then it is possible to find in time $O(m^2)$ all vertices $u \in V(B)$ such that $\{u\}$ is a dominating set for B (using exhaustive search). Hence in time $O(n^2)$ one can determine if all blocks of G have domination number one, and also find all directed holes (B, u). Using a simultaneous depth-first search on $B(G)$ starting from all the directed holes, it is possible to decide if there is a hallway in G in time $O(|E(B(G))|) = O(n)$. Hence the total running time of the algorithm is $O(n^2)$. ∎

Chapter 3

A Lower Bound for the Maximum Cop Number of Connected Graphs

As mentioned in the introduction, the best upper bound for $f_s(n)$, when $s \in \mathbb{N}$, is the following. Let $\alpha = 1 + 1/s$. Frieze et al. [27] proved that

$$f_s(n) = n\alpha^{-(1-o(1))\sqrt{\log_\alpha n}}.$$

Their approach is based on vertex expansion of graphs. For the $s = 1$ case, two different proofs for the same bound are known, which are based on the idea of protecting shortest paths (see [36, 49]). The most important open question in the area of Cops and Robber game is Meyniel's conjecture, published by Frankl [25], which states that $f_1(n) = O(\sqrt{n})$.

When $s = \infty$, the authors of [27] proved by considering an appropriate random graph that

$$f_\infty(n) = \Omega(n).$$

As $f_\infty(n) \leq n$, this is asymptotically tight.

For lower bounds, it is well-known that

$$f_1(n) = \Omega(\sqrt{n}),$$

see [45] for an argument using incident graphs of projective planes. The case $s > 1$ was first considered in [27], where the authors proved using random graphs that

$$f_s(n) = \Omega\left(n^{(s-3)/(s-2)}\right).$$

Note that this bound is interesting only for $s \geq 5$. In this chapter we improve their result by showing that for all $s \in \mathbb{N}$,

$$f_s(n) = \Omega\left(n^{s/s+1}\right),$$

and thus generalize the lower bound for the $s = 1$ case. The material of this chapter also appears in [2]. For the cases $s = 2, 4$ a simpler proof can be found in [39].

Definition ($N_k(u)$, $N_k^A(u)$, diameter)**.** Let k be a positive integer. For a vertex u of a graph G, $N_k(u)$ denotes the set of vertices whose distance from u is exactly k. If A is a subset of vertices, then $N_k^A(u)$ denotes the set of vertices v such that

- The distance between u and v is k, and
- for every shortest (u, v)-path $uu_1u_2 \dots u_{k-1}v$, we have $u_1 \notin A$.

The *diameter* of G is the maximum distance between any two vertices of G.

Note that for every u, A and k, we have $N_k^{A\cap N(u)}(u) = N_k^A(u)$.

Lemma 3.1. *Let s, d, m be positive integers and q be a positive real such that $qd^s/2$ is an integer larger than m. Let G be a d-regular bipartite graph of diameter larger than s with the following properties.*

(1) For every two vertices u, v of G of distance at most $s+1$, there are at most m distinct shortest (u, v)-paths.

(2) For every vertex u of G and every subset A of size at most m, $|N_s^A(u)| \geq qd^s$.

Then we have

$$c_s(G) \geq \frac{q^2 d^s}{24ms}.$$

Proof. Let us first define a few terms. A cop *controls* a vertex u if the cop is at u or at an adjacent vertex. A cop controls a path if it controls a vertex of the path. The cops control a path if there is a cop controlling it. A vertex r is *safe* if there is a subset $X \subseteq N_s(r)$ of size $qd^s/2$ such that for all $x \in X$, all shortest (r, x)-paths are uncontrolled.

Let the number of cops be c with $c < q^2d^s/24ms$, and we will show that the robber can evade them forever. If this many cops can capture the robber, then they can capture her from any starting configuration. Thus we may assume that the cops all start at one vertex u, and the robber starts at a vertex r at distance $s + 1$ from u. Such two vertices exist as G has diameter larger than s. Property (2) gives $N_s(r) \geq qd^s$, and by property (1), the cops control at most m vertices of $N_s(r)$. Since $qd^s - m > qd^s/2$, the robber is at a safe vertex at the starting configuration. Hence we just need to show that if the robber is at a safe vertex before the cops move, then she can move to a safe vertex after the cops move.

Suppose that the robber is at a safe vertex r, so by definition, there is a subset $X \subseteq N_s(r)$ of size $qd^s/2$ such that for all $x \in X$, all shortest (r, x)-paths are uncontrolled.

Denote by A the set of vertices of all shortest (r, x)-paths for all $x \in X$. In particular, $r \in A$ and $X \subseteq A$. Now, the cops move to new positions. At this moment there is no cop in A, so the robber is able to move to any vertex of X in her turn; thus to complete the proof, we need to show that there is a safe vertex in X.

Claim. Every vertex $u \notin A$ has at most m neighbours in X.

Proof of Claim. If u has no neighbour in X, then the claim is true, otherwise let $x \in X$ be adjacent to u. Note that as $d(r, x) = s$, we have $d(r, u) \in \{s-1, s, s+1\}$. The graph G is bipartite, so $d(r, u) \neq s$. If $d(r, u) = s - 1$ then u is on a shortest (r, x)-path, which contradicts the assumption $u \notin A$. Therefore, $d(r, u) = s + 1$, and x is on a shortest (r, u)-path. Hence by property (1), the number of neighbours of u in X is at most m. $\square$

Remark. It can be shown using a similar argument that every $x \in X$ has at most m neighbours in A.

By an *escaping pair* we mean a pair (x, y) of vertices with $x \in X$ and $y \in N_s^A(x)$. We call x the *head* and y the *tail* of the pair. (This terminology comes from the fact that x is the first coordinate and y is the second coordinate.) By the remark, the set $A \cap N(x)$ has at most m elements, and property (2) ensures that $|N_s^A(x)| = |N_s^{A \cap N(x)}(x)| \geq qd^s$. That is, every $x \in X$ is the head of at least qd^s distinct escaping pairs. We say that an escaping pair (x, y) is *free* if all shortest (x, y)-paths are uncontrolled. We just need to prove that there is an $x \in X$ such that x is the head of at least $qd^s/2$ free escaping pairs, because then x would be a safe vertex, and the robber, having speed s, can move to x in her turn. If (x, y) is an escaping pair, then every shortest (x, y)-path is called an *escaping path*. By definition, every escaping path can be written as $u_1 u_2 u_3 \ldots u_{s+1}$, where $u_1 \in X$ and $u_2 \notin A$.

Claim. Each cop controls at most $3msd^s$ escaping paths.

Proof of Claim. We first prove that every vertex v is on at most $d^s + msd^{s-1}$ escaping paths, and if $v \notin X$, then v is on at most msd^{s-1} escaping paths. Let $u_1 u_2 u_3 \ldots u_{s+1}$ be an escaping path with $u_1 \in X$ and $u_2 \notin A$, such that v is its i-th vertex, i.e. $v = u_i$.

Assume first that $i \neq 1$. There are at most d choices for each of $u_{i-1}, \ldots, u_2$, and for each of $u_{i+1}, u_{i+2}, \ldots, u_{s+1}$. By the previous claim, once u_2 is determined, there are at most m choices for u_1. Consequently, for each $2 \leq i \leq s+1$, v is the i-th vertex of at most md^{s-1} escaping paths, so if $v \notin X$ then v is on at most msd^{s-1} escaping paths.

If $i = 1$ then $v \in X$ and there are at most d choices for each of $u_2, u_3, \ldots, u_{s+1}$, thus each $v \in X$ is the first vertex of at most d^s escaping paths. This shows that v is on at most $d^s + msd^{s-1}$ escaping paths.

Recall that since the robber was at a safe vertex before the cops' move, no cop is in A at this moment. By the previous claim, each cop controls at most m vertices of X, through which he can control at most $m(d^s + msd^{s-1})$ escaping paths. Through every other neighbour he can control at most msd^{s-1} escaping paths. He controls $d+1$ vertices in total, so he controls no more than

$$m(d^s + msd^{s-1}) + (d+1-m)(msd^{s-1}) \leq 3msd^s$$

escaping paths. $\square$

Since there are c cops in the game, the cops control at most $3msd^s c$ escaping paths. By controlling each escaping path, the cops can decrease the number of free escaping pairs by at most 2 (as each path has two endpoints), hence the number of non-free escaping pairs is at most $6msd^s c$.

Now we prove that there is an $x \in X$ such that x is the head of at least $qd^s/2$ free escaping pairs, completing the proof. Recall that every $x \in X$ is the head of at least qd^s escaping pairs. Hence if there were no $x \in X$ such that x is the head of at least $qd^s/2$ free escaping pairs, then every $x \in X$ would be the head of at least $qd^s/2$ non-free escaping paths. As by definition of safeness, X has size $qd^s/2$, this would imply that the number of non-free escaping pairs is at least $(qd^s/2)^2$, which is larger than $6msd^s c$. This contradiction shows that the robber can evade the c cops forever. $\blacksquare$

Let k, s be positive integers and $d = 2^k$. Let $x_1, x_2, \ldots, x_d$ be the d elements of $\mathbb{GF}(2^k)$, the field with 2^k elements, represented as column vectors of length k over $\mathbb{Z}_2$. Let H be the following $1 + k(s+1)$ by d matrix over the field $\mathbb{Z}_2$.

$$H = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_d \\ x_1^3 & x_2^3 & \ldots & x_d^3 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2s+1} & x_2^{2s+1} & \ldots & x_d^{2s+1} \end{bmatrix}$$

Let $S = \{e_1, e_2, \ldots, e_d\} \subseteq \mathbb{Z}_2^{1+k(s+1)}$ be the set of columns of H. It is known that every set of $2s+3$ columns of H is linearly independent over $\mathbb{Z}_2$ (see page 281 of [3] for a proof), hence, in particular, every $(2s+2)$-subset of S is linearly independent over the field $\mathbb{Z}_2$. Let G be the graph with vertex set $\mathbb{Z}_2^{1+k(s+1)}$, and with vertices u, v adjacent if $u - v \in S$ (the Cayley graph of the additive group $\mathbb{Z}_2^{1+k(s+1)}$ with respect to S).

Lemma 3.2. *If $d \geq 2(s+1)!$, then the graph G has the following properties.*

(i) G is connected.

(ii) G is d-regular.

(iii) *G is bipartite.*

(iv) *For every two vertices u, v of G of distance at most $s+1$, there are at most $(s+1)!$ distinct shortest (u, v)-paths.*

(v) *For every vertex u of G and every subset A of size at most $(s+1)!$, $|N_s^A(u)| \geq (d/2s)^s$.*

(vi) *G has diameter larger than s.*

Proof. (i) To show connectivity one has to prove that every element of $\mathbb{Z}_2^{1+k(s+1)}$ can be written as a linear combination of members of S, which is equivalent to the matrix H having rank $1+k(s+1)$. Note that H has $1+k(s+1)$ rows, thus we need to show that no nontrivial linear combination of its rows over Z_2 is the zero vector. But it is known that the rows $2, 3, \ldots, 1+k(s+1)$ generate a dual BCH code, and every nontrivial linear combination of them has almost the same number of zeros and ones (see [38]).

(ii) This is clear as $|S| = d$.

(iii) This follows from the fact that each member of S has 1 as its first coordinate, hence there is no odd-size subset of S whose sum of members is zero.

(iv) Let u, v be two vertices of G of distance m, where $m \leq s+1$. Each shortest (u, v)-path has length m and thus corresponds to a unique ordered representation

$$u - v = s_1 + s_2 + \cdots + s_m,$$

with $s_1, \ldots, s_m \in S$. If some $s \in S$ appears more than once in this summation, then we can delete a pair of them (we are in $\mathbb{Z}_2$, so $s + s = 0$) and find a shorter representation (and a shorter (u, v)-path), which does not exist. So $s_1, \ldots, s_m$ are distinct. Any other shortest (u, v)-path gives another ordered representation

$$u - v = s'_1 + s'_2 + \cdots + s'_m,$$

in which $s'_1, \ldots, s'_m$ are distinct by a similar argument, and we have $s_1 + \cdots + s_m + s'_1 + \cdots + s'_m = 0$. By linear independence of every $(2s+2)$-subset of S, $(s'_1, \ldots, s'_m)$ is a permutation of $(s_1, \ldots, s_m)$. Therefore, the number of ordered representations of $u - v$ using m members of S is $m!$, so the number of shortest (u, v)-paths in G is also $m!$, which is not more than $(s+1)!$.

(v) Without loss of generality, we may assume that $A \subseteq N(u)$. Every $a \in A$ can be written as $a = u + e_i$ for some $e_i \in S$. There is a set $B \subseteq S$ of size at least $d - |A|$ such that for every $e \in B$, $u + e \notin A$. For every s-subset $\{e_{i_1}, \ldots, e_{i_s}\}$ of B, we have a vertex $u + e_{i_1} + \cdots + e_{i_s}$ of distance s from u. These vertices are all in $N_s^A(u)$ and are distinct, because of the linear independence of every $(2s+2)$-subset of S. Hence we have

$$|N_s^A(u)| \geq \binom{d-|A|}{s} \geq \left(\frac{d-|A|}{s}\right)^s \geq \frac{d^s}{(2s)^s},$$

where the last inequality follows from $d \geq 2(s+1)! \geq 2|A|$.

(vi) By linear independence of every $2s+2$ members of S, the distance between vertices 0 and $e_1 + \cdots + e_{s+1}$ is at least $s+1$. ■

Theorem 3.3. *Let s be a fixed positive integer. For every n, there exists a connected n-vertex graph G with $c_s(G) = \Omega(n^{s/s+1})$.*

Proof. Take k_0 large enough so that $d = 2^{k_0}$ satisfies

$$d \geq 2(s+1)! \quad \text{and} \quad d^s > 4(s+1)!(2s)^s.$$

We may assume that $n > 2^{1+k_0(s+1)}$. Let $k \geq k_0$ be the largest integer with $2^{1+k(s+1)} \leq n$, and let $n_0 = 2^{1+k(s+1)}$. By the way k is defined, we have

$$n_0 \leq n < 2^{s+1} n_0,$$

so $n = \Theta(n_0)$. Let G be the graph described above with parameters k, s. Let $m = (s+1)!$ and let q satisfy

$$qd^s = 2\left\lfloor \frac{d^s}{2(2s)^s} \right\rfloor.$$

By Lemma 3.2, G is a connected bipartite d-regular graph with $n_0 = O(d^{s+1})$ vertices and diameter larger than s. Also, for every two vertices u, v of G of distance at most $s+1$, there are at most m distinct shortest (u,v)-paths, and for every vertex u of G and every subset B of size at most m, we have

$$|N_s^B(u)| \geq (d/2s)^s \geq qd^s.$$

Moreover, $qd^s/2$ is an integer and

$$qd^s/2 = \left\lfloor \frac{d^s}{2(2s)^s} \right\rfloor \geq \frac{d^s}{4(2s)^s} > m.$$

Now by Lemma 3.1,

$$c_s(G) = \Omega(d^s) = \Omega({n_0}^{s/s+1}) = \Omega(n^{s/s+1}).$$

Let G' be the graph obtained by joining some vertex of G to an endpoint of a path with $n - n_0$ vertices. It is easy to check that G' is a connected graph on n vertices, whose cop number is the same as the cop number of G, which is $\Omega(n^{s/s+1})$. ■

Theorem 3.3 shows that for all $s \in \mathbb{N}$,

$$f_s(n) = \Omega(n^{s/s+1}).$$

We conjecture that this bound is asymptotically tight (see Chapter 11).

Chapter 4

The Relation Between Cop Number and Treewidth

Definition (tree decomposition, treewidth)**.** A *tree decomposition* of a graph G is a pair (T, W), where T is a tree and $W = (W_t : t \in V(T))$ is a family of subsets of $V(G)$ such that

(i) $\bigcup_{t \in V(T)} W_t = V(G)$, and every edge of G has both endpoints in some W_t, and

(ii) For every $v \in V(G)$, the set $\{t : v \in W_t\}$ induces a subtree of T.

We define the *width* of (T, W) to be

$$\max\{|W_t| - 1 : t \in V(T)\},$$

and the *treewidth* of G, written $tw(G)$, is the minimum width of a tree decomposition of G.

Example 4.1. Let m be a positive integer greater than one. The graph $\mathcal{A} = \mathcal{A}(m)$ is defined as follows: $\mathcal{A}$ has a total of $m + 2m\binom{m}{2}$ vertices, with a certain independent set $\{v_1, \ldots, v_m\}$, such that every two of the v_i's are connected by m disjoint paths of length 3, and $\mathcal{A}$ does not have any other edge. Thus $\mathcal{A}$ has a total of $3m\binom{m}{2}$ edges

We show that there exists a tree decomposition of $\mathcal{A}$ with width $\max\{m-1, 3\}$. Let T be the star with $1 + m\binom{m}{2}$ vertices, and let r be its dominating vertex. Define $W_r = \{v_1, \ldots, v_m\}$. To each path $v_i u_1 u_2 v_j$ assign a leaf l of the tree and set $W_l = \{v_i, u_1, u_2, v_j\}$. It is easy to verify that (T, W) is a tree decomposition of $\mathcal{A}$ with width $\max\{m-1, 3\}$.

We will use the following facts about tree decompositions, whose proofs can be found in Section 12.3 of [20].

Proposition 4.2. *Let (T, W) be a tree decomposition of a graph G.*

(a) Let A be the vertex set of a clique in G. Then there is a $t \in V(T)$ with $A \subseteq W_t$.

(b) Let t_1t_2 be an edge of T, and let T_1 and T_2 be the components of $T - t_1t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$. Define $X = W_{t_1} \cap W_{t_2}$, $U_1 = \cup_{t \in T_1} W_t$ and $U_2 = \cup_{t \in T_2} W_t$. Then X is a cut-set in G, and there is no edge between $U_1 \setminus X$ and $U_2 \setminus X$.

Joret et al. [35] proved that for every G,

$$c_1(G) \leq \frac{tw(G)}{2} + 1.$$

In this chapter we prove that for every G,

$$\frac{tw(G)+1}{\Delta(G)+1} \leq c_\infty(G) \leq tw(G) + 1.$$

Moreover, we prove that these bounds are tight. To prove the lower bound, we relate our Cops and Robber game with another pursuit-evasion game, called the Helicopter Cops and Robber game. This game, introduced in [50], has two different versions, and the one we define here is called jump-searching.

Definition (Helicopter Cops and Robber game (the jump-searching version))**.** For $X \subseteq V(G)$, an *X-flap* is the vertex set of a connected component of $G - X$. Two subsets $X, Y \subseteq V(G)$ *touch* if either $X \cap Y \neq \emptyset$ or some vertex in X has a neighbour in Y. A *position* is a pair (X, R), where $X \subseteq V(G)$ and R is an X-flap. (X is the set of vertices currently occupied by the cops and R tells us where the robber is — since she can run arbitrarily fast, all that matters is which component of $G - X$ contains her.) At the start, the cops choose a subset X_0, and the robber chooses an X_0-flap R_0. Note that if there are k cops in the game, then $|X_0| \leq k$. At the start of round i, we have some position (X_{i-1}, R_{i-1}). The cops choose a new set $X_i \subseteq V(G)$ with $|X_i| \leq k$ (and no other restriction), and announce it. Then the robber, knowing X_i, chooses an X_i-flap R_i which touches R_{i-1}. If this is not possible then the cops have won. Otherwise, i.e. if the robber never runs out of valid moves, the robber wins.

The following lemma establishes a link between the two games.

Lemma 4.3. *Let G be a graph. If k cops can capture an infinitely fast robber in the Cops and Robber game in G, then $k(\Delta + 1)$ cops can capture the robber in the Helicopter Cops and Robber game in G.*

Proof. We consider two games played in two copies of G: the first one, which we call the *real game*, is a game of Helicopter Cops and Robber with $k(\Delta + 1)$ cops; and the second one, the *virtual game*, is the usual Cops and Robber game with k cops and an infinitely fast robber. Given a winning strategy for the cops in the virtual game, we need to give a capturing strategy for the cops in the real game. We translate the moves of the cops from the virtual game to the real game, and translate the moves of the robber from the real game to the virtual game, in such a way that all the translated moves are valid, and if the robber is captured in the virtual game, then she is captured in the real game as well. Hence, as the cops have a winning strategy in the virtual game, they have a winning strategy in the real game, too.

In the virtual game, initially the cops choose a subset C_0 of vertices. Then the real cops choose $X_0 = \overline{N}(C_0)$. Recall that $|C_0| \leq k$ so that $|X_0| \leq k(\Delta+1)$. The real robber chooses R_0, which is an X_0-flap, and the virtual robber chooses an arbitrary vertex $r_0 \in R_0$. In general, at the end of round $i-1$ we have $X_{i-1} = \overline{N}(C_{i-1})$ and $r_{i-1} \in R_{i-1}$.

Suppose the virtual robber is not captured in round i. In round i, first the virtual cops move to a new set C_i. Each cop either stays still or moves to a neighbour, thus $C_i \subseteq \overline{N}(C_{i-1}) = X_{i-1}$ and since R_{i-1} was an X_{i-1}-flap, $C_i \cap R_{i-1} = \emptyset$. The real cops choose $X_i = \overline{N}(C_i)$ and announce it. The real robber, knowing X_i, chooses an X_i-flap R_i that touches R_{i-1}. If she cannot find a valid move then she is captured and the lemma is proved. Otherwise, note that by definition $C_i \cap R_i = \emptyset$. Let r_i be an arbitrary vertex of R_i. The virtual robber moves from r_{i-1} to r_i. Since R_{i-1} and R_i touch, and both of them are connected, $R_{i-1} \cup R_i$ is connected. Moreover, C_i does not intersect $R_{i-1} \cup R_i$, so this is a valid move in the virtual game.

Now, suppose the virtual robber is captured in round i. We claim that if this happens then the real robber has already been captured in one of previous rounds. If this is not the case, then in round i, the virtual cops move to a new set C_i such that $r_{i-1} \in C_i$. Each cop either stays still or moves to a neighbour, thus $C_i \subseteq \overline{N}(C_{i-1}) = X_{i-1}$ and since R_{i-1} was an X_{i-1}-flap, $C_i \cap R_{i-1} = \emptyset$. But $r_{i-1} \in C_i$ because the virtual robber has been captured in round i, and $r_{i-1} \in R_{i-1}$, thus $r_{i-1} \in C_i \cap R_{i-1}$, which is a contradiction. This shows that the real robber will be captured even before the virtual robber, and the proof is complete. ■

Seymour and Thomas [50] proved the following theorem.

Theorem 4.4. *The minimum number of cops needed to capture a robber in Helicopter Cops and Robber game is equal to the treewidth of the graph plus one.*

Using this we have

Theorem 4.5. *For every graph G we have*

$$\frac{tw(G)+1}{\Delta(G)+1} \leq c_\infty(G) \leq tw(G)+1,$$

and these bounds are tight.

Proof. The lower bound follows from Lemma 4.3 and Theorem 4.4. To prove tightness of the lower bound, let G be the complete graph on n vertices. Then the set of all vertices of G forms a clique in G, and by part (a) of Proposition 4.2, in any decomposition (T, W) of G, there exists a $t \in T$ with $V(G) \subseteq V_t$. Thus the treewidth of G is at least $n-1$. If we define T to be a tree with a single node t and $W_t = V(G)$, then (T, W) is a tree decomposition of G with width $n-1$. Therefore G has treewidth exactly $n-1$. A single cop can capture the robber in G, since G has domination number one. Hence, the complete graph on n vertices has treewidth $n-1$, maximum degree $n-1$, and cop number 1, so the lower bound is tight.

Now we prove the upper bound. Consider a tree decomposition (T, W) of G having minimum width. Assume that there are $tw(G)+1$ cops in the game, so for every $t \in V(T)$, there are at least $|W_t|$ cops in the game. The cops start at W_{t_1} for some arbitrary $t_1 \in V(T)$. Assume that the robber starts at r_0, and let t be such that $r_0 \in W_t$. Let t_2 be the neighbour of t_1 in the unique (t_1, t)-path in T. Let T_1 and T_2 be the components of $T - t_1t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$. Define $X = W_{t_1} \cap W_{t_2}$, $U_1 = \cup_{t \in T_1} W_t$, and $U_2 = \cup_{t \in T_2} W_t$. So the cops are all in U_1 and the robber is at a vertex in $U_2 \setminus X$. Note that the number of cops is at least $|W_{t_2}|$. Now the cops move in order to occupy W_{t_2}, in such a way that the cops in X stay still. After some rounds, the cops will be located at W_{t_2}, and during those rounds the robber could not escape from $U_2 \setminus X$, because by part (b) of Proposition 4.2, there is no edge between $U_1 \setminus X$ and $U_2 \setminus X$. When the cops have established in W_{t_2}, the total space available to the robber has been decreased. Continuing similarly the cops will eventually capture the robber.

Next we prove that the upper bound is tight. We actually prove that there exist graphs with $c_3(G) \geq tw(G)+1$. Let $m \geq 4$ be a positive integer. The graph G, which is the same as the graph $\mathcal{A}(m)$ introduced in Example 4.1, is defined as follows: G has a total of $m+2m\binom{m}{2}$ vertices, with a certain independent set $\{v_1, \ldots, v_m\}$, such that every two of the v_i's are connected by m disjoint paths of length 3, and G does not have any other edge. In Example 4.1 we proved that $tw(G) \leq \max\{3, m-1\}$. Note that $m \geq 4$, so $tw(G) \leq m-1$. Here we show that $c_3(G) \geq m$, which completes the proof.

It suffices to show that $m-1$ cops cannot capture a robber with speed 3. Say a cop *controls* a vertex u if the cop is at u or at an adjacent vertex. If there are $m-1$ cops in the game, we show that the robber can play such that at the end of each round, if the cops are in $C \subseteq V(G)$, then the robber is at a vertex $r \in \{v_1, \ldots, v_m\} \setminus \overline{N}(C)$. The robber can choose such a vertex at the beginning, because the distance between any two of the v_i's is 3, so each cop can control at most one of the v_i's. Assume that at the end of round i the cops are in C_i and the robber is at $r_i \in \{v_1, \dots, v_m\} \setminus \overline{N}(C_i)$. In round $i+1$, first the cops move to $C_{i+1} \subseteq \overline{N}(C_i)$. So the robber is not captured. There exists a vertex $r_{i+1} \in \{v_1, \dots, v_m\} \setminus \overline{N}(C_{i+1})$, because every cop controls at most one of the v_i's. If $r_{i+1} = r_i$ then the robber does not move at all. Otherwise, there are m disjoint (r_i, r_{i+1})-paths in G, and $m-1$ cops, so that at least one of these paths is cop-free, and the robber moves along this path (whose length is 3) to r_{i+1}. ∎

Theorem 4.5 is especially useful for giving lower bounds for the cop number, when the graph has small maximum degree. To illustrate this, we use it to give a short proof for

$$f_\infty(n) = \Omega(n),$$

which is proved in [27] using other ideas. Recall that $f_\infty(n)$ denotes the maximum cop number of a connected graph on n vertices, assuming the robber is infinitely fast. In the proof we use a theorem from an unpublished paper [28].

Corollary 4.6. *We have $f_\infty(n) = \Omega(n)$.*

Proof. Let G be an Erdös-Rényi random graph with n vertices and $2n$ edges. Gao [28] has proved that there is a positive constant β such that we have $tw(G) > \beta n$ with probability approaching one, as n goes to infinity.

Each vertex of G has average degree $2|E(G)|/|V(G)| = 4$. Hence by Markov's inequality, the probability that a fixed vertex has degree larger than $16/\beta$ is less than $\beta/4$. By linearity of expectation, the expected number of vertices of degree larger than $16/\beta$ is less than $n\beta/4$. Therefore by Markov's inequality, with probability at least $1/2$, G has at most $n\beta/2$ vertices of degree larger than $16/\beta$.

Consequently, for n large enough, there exists a graph G_n such that

- $tw(G_n) > \beta n$, and
- G_n has at most $n\beta/2$ vertices of degree larger than $16/\beta$.

Let H_n denote the graph obtained from G_n by deleting all vertices of degree larger than $16/\beta$. Deleting each vertex does not decrease treewidth by more than 1. Thus we have

$$(1-\beta/2)n \le |V(H_n)| \le n, \qquad tw(H_n) \ge n\beta/2, \text{ and } \qquad \Delta(H_n) \le 16/\beta.$$

By Theorem 4.5,

$$c_\infty(H_n) \ge \frac{tw(H_n)+1}{\Delta(H_n)+1} \ge \frac{tw(H_n)}{2\Delta(H_n)}$$

and so

$$\frac{|V(H_n)|}{c_\infty(H_n)} \leq \frac{2|V(H_n)|\Delta(H_n)}{tw(H_n)} \leq 2n \times \frac{16}{\beta} \times \frac{2}{n\beta} = 64/\beta^2 = O(1),$$

completing the proof. ∎

See Chapter 9 for other applications of Theorem 4.5.

Chapter 5

An Approximation Algorithm for Interval Graphs

Definition (interval graph)**.** Graph G is called an *interval graph* if there is a correspondence between its vertices and a set of closed intervals on the real line, such that two vertices are adjacent in G if and only if their corresponding intervals intersect.

Let $s \in \mathbb{N}$. Fomin et al. [23] proved that if G is an interval graph then $c_s(G) \leq 5s - 1$, and this leads to a polynomial time algorithm for computing $c_s(G)$ for fixed s. However, the complexity of computing $c_\infty(G)$ was left open. As a partial answer, in this chapter we prove that this problem is 3-approximable. As a consequence of our algorithm, we prove that $c_\infty(G) = O(\sqrt{n})$ for all connected interval graphs G, and provide examples for which this bound is tight.

Definition (k-wide)**.** For a subgraph H of G, say H is k-wide if

(i) H is k-connected, and

(ii) for any $S \subseteq V(G)$ with $|S| < k$ we have $V(H) \not\subseteq \overline{N}(S)$.

Lemma 5.1. *If G has a k-wide subgraph H then $c_\infty(G) \geq k$.*

Proof. Say a cop *controls* a vertex u if the cop is at u or at an adjacent vertex. Suppose that there are less than k cops in the game, and they initially start at a subset S of vertices. By condition (ii), there is a vertex $v \in V(H) \setminus \overline{N}(S)$, i.e. v is controlled by no cop. The robber starts at v, and will always remain in H. After each move of the cops, the set of vertices occupied by them has size less than k. Hence by condition (ii), there exists a vertex x of H that is not controlled by any of the cops. By condition (i), H is k-connected, so as the robber is currently in H, and the number of cops is less than k, there is a cop-free path to x. The robber moves there and will not be captured in the next round. Since she can elude forever by using this strategy, at least k cops are needed to capture her. ■

In the rest of this chapter, G is an interval graph. We will also assume that G is connected, since, as mentioned in the introduction, the cop number of a disconnected graph equals the sum of the cop numbers for each connected component. Consider a set of closed intervals whose intersection graph is G, and denote by I_v the interval corresponding to the vertex $v \in V(G)$. We may assume without loss of generality that none of the intervals have zero length. Such a representation can be found in polynomial time (see [33] for instance). Let $x_1 < x_2 < \cdots < x_{l+1}$ be the set of distinct endpoints of the intervals, and let $y_1, y_2, \ldots, y_l$ be points satisfying $x_i < y_i < x_{i+1}$ for $1 \leq i \leq l$. Also, define $V_i = \{v \in V(G) : y_i \in I_v\}$ for all $1 \leq i \leq l$. It is clear that the subgraph induced by $V_i \cup V_{i+1}$ is a clique (for $1 \leq i \leq l-1$), and in particular, it follows that each $G[V_i]$ is a clique for $1 \leq i \leq l$ (recall that $G[V_i]$ denotes the subgraph induced by V_i). Furthermore, $l \leq 2n$ and the sets $V_1, \ldots, V_l$ cover the vertices of G.

Lemma 5.2. *Every minimal cut-set X of G is one of the V_i's. Moreover, if $X = V_i$ is a cut-set, then for each $u_1 \in V_{i_1} \setminus X$ and $u_2 \in V_{i_2} \setminus X$ satisfying $i_1 < i < i_2$, u_1 and u_2 lie in different components of $G - X$.*

Proof. For an index $1 \leq i \leq l$, say point y_i is a *cut-point* if there exists a vertex $v \in V(G)$ with both endpoints of I_v lying strictly on the left of y_i, and also a vertex $v' \in V(G)$ with both endpoints of $I_{v'}$ lying strictly on the right of y_i. If y_i is a cut-point then clearly V_i is a cut-set of G.

Now, let X be a minimal cut-set of G. Let u_1, u_2 be vertices in different components of $G - X$, with $I_{u_1} = [x_a, x_b], I_{u_2} = [x_c, x_d]$, and assume by symmetry that $a < b < c < d$. For each i with $b \leq i \leq c - 1$, y_i is a cut-point. If for all of the i's in this range, there was a vertex $v_i \in V_i \setminus X$, then $u_1 v_b v_{b+1} \ldots v_{c-1} u_2$ would be a (u_1, u_2)-path in $G - X$. As such a path does not exist, there is an i in this range such that $V_i \subseteq X$. But then V_i is a cut-set of G, hence $X = V_i$.

For the second statement, let $X = V_i$ be a cut-set, $u_1 \in V_{i_1} \setminus X$ and $u_2 \in V_{i_2} \setminus X$ such that $i_1 < i < i_2$. Let $I_{u_1} = [x_a, x_b], I_{u_2} = [x_c, x_d]$, and so $x_a < x_b < y_i < x_c < x_d$. Every (u_1, u_2)-path contains a vertex whose corresponding interval contains y_i, but all such vertices are in X. Hence there is no (u_1, u_2)-path in $G - X$. ■

Definition ($G[a, b]$, interval subgraph, $w(G)$)**.** We write $G[a, b]$ for the subgraph induced by $\bigcup_{a \leq i \leq b} V_i$ (for $1 \leq a \leq b \leq l$), and we call each of these an *interval subgraph*. Let $w(G)$ be the maximum number M such that G has an M-wide interval subgraph.

Lemma 5.3. *$w(G)$ can be computed in polynomial time.*

Proof. Fix an interval subgraph $G[a,b]$. It is easy to see that there is an $S \subseteq V(G)$ with $V(G[a,b]) \subseteq \overline{N}(S)$ if and only if the domination number of $G[a,b]$ is at most $|S|$, that is, if there is a set of $|S|$ vertices of G dominating the vertices of $G[a,b]$, then there exists such a set inside $G[a,b]$. Moreover, $G[a,b]$ is an interval graph so its domination number can be found in polynomial time (using a greedy algorithm). The connectivity of $G[a,b]$ can also be computed in polynomial time (see [21] for example). Therefore, the largest M such that $G[a,b]$ is M-wide can be computed in polynomial time. Recall that $w(G)$ is the maximum number M such that G has an M-wide interval subgraph. The total number of interval subgraphs is $O(l^2) = O(n^2)$, so $w(G)$ can be computed in polynomial time. ■

The following lemma gives an appropriate upper bound for $c_\infty(G)$.

Lemma 5.4. *We have $c_\infty(G) \leq 3w(G)$.*

Proof. We just need to give a strategy for $3w(G)$ cops to capture the robber. Let $M = w(G)$. There are three teams of cops, each of size M. At the beginning the first team starts at a vertex in V_1, the second team starts at a vertex in V_l, and the third team starts at an arbitrary vertex. Suppose that the robber starts at a vertex r. The cops' strategy consists of several (at most l) phases, in each of which they reduce the free space of the robber. The following invariant is true at the start of each phase: the j-th team ($j = 1, 2$) is in a subset $X_j \subseteq V_{i_j}$ such that they block the robber from escaping $G[i_1, i_2]$.

Note that during this phase, if the robber goes to a vertex in $V_{i_1} \cup V_{i_2}$ then she will be captured immediately by the first or second team (recall that each $G[V_i]$ is a clique). If $i_2 \leq i_1 + 1$ then she should move to a vertex in $V_{i_1} \cup V_{i_2}$ and will be captured immediately, so assume that $i_2 > i_1 + 1$. Since $G[i_1+1, i_2-1]$ is not $(M+1)$-wide, either $G[i_1+1, i_2-1]$ has a minimal cut-set X of size at most M, or $G[i_1 + 1, i_2 - 1]$ has a dominating set X of size at most M.

In the second case, the third team moves to X (while the first and second teams stay still and block the robber from escaping $G[i_1, i_2]$), and the robber will be captured in the next move.

In the first case, the third team moves to X, and suppose that $X = V_{i_3}$ (by Lemma 5.2, X is of this form). Suppose that the robber moves to r right after the third team has settled in X and j be an index such that $r \in V_j$. If $j = i_3$ then the third team immediately captures her (since $G[V_{i_3}]$ is a clique), so assume, by symmetry, that $i_1 < j < i_3$. Now, the first team together with the third team block the robber from escaping the subgraph $G[i_1, i_3]$ (by the second statement in Lemma 5.2). The second and third team switch roles and this phase finishes. Note that $i_3 - i_1 < i_2 - i_1$ so the total number of phases is not larger than l. ■

Theorem 5.5. *There exists a polynomial-time 3-approximation algorithm for computing $c_\infty(G)$ when G is an interval graph.*

Proof. Given G, the sequence $(V_1, V_2, \ldots, V_l)$ can be found efficiently. Then $w(G)$ can be computed in polynomial time by Lemma 5.1. The value $3w(G)$ is a 3-approximation for $c_\infty(G)$ by Lemmas 5.1 and 5.4. ■

Next we will prove that $c_\infty(G) = O(\sqrt{n})$. Before doing so, we note that this bound is tight: let G be the strong product of the path on $3m$ vertices and the complete graph on m vertices. That is,

$$V(G) = \{1, 2, \ldots, 3m\} \times \{1, 2, \ldots, m\},$$

and

$$\{(i,j),(k,l)\} \in E(G) \text{ if } (i,j) \neq (k,l) \text{ and } |i-k| \leq 1.$$

Then G is an interval graph with $3m^2$ vertices, and is m-wide itself, hence

$$c_\infty(G) \geq m = \Omega(\sqrt{|V(G)|}).$$

We will need a lemma about minimum dominating sets in interval graphs, which may not be the best possible, but suffices for our purposes.

Lemma 5.6. *Let A be a minimum dominating set of G. Every vertex $v \in A$ is adjacent to at most two vertices of A, and every vertex $v \notin A$ is adjacent to at most five vertices of A.*

Proof. Let $I_v = [x, y]$ be the interval corresponding to vertex v. First, let $v \in A$. If there is a vertex $u \in A$ whose corresponding interval contains I_v, then $\overline{N}(v) \subseteq \overline{N}(u)$, which contradicts the minimality of A. If there is a vertex $u \in A$ whose corresponding interval is contained in I_v, then $\overline{N}(u) \subseteq \overline{N}(v)$, which contradicts the minimality of A. So for every $u \in A$ that is adjacent to v, the interval corresponding to u contains exactly one of x and y. If there are distinct $u_1, u_2 \in N(v) \cap A$ whose corresponding intervals contain x, then one can remove one of them (the one whose left-end-point of the corresponding interval is more to the right) from A, and still have a dominating set, which contradicts the minimality of A. Thus there exists at most one vertex in $N(v) \cap A$ whose corresponding interval contains x, and at most one vertex in $N(v) \cap A$ whose corresponding interval contains y, so $|N(v) \cap A| \leq 2$.

Second, let $v \notin A$. If there is a vertex $u \in A$ whose corresponding interval contains I_v, then since u is adjacent to at most two vertices of A, v is adjacent to at most two vertices of A as well. So we may assume that is not the case. If there are two distinct $u_1, u_2 \in A$ whose corresponding intervals are contained I_v, then $\overline{N}(u_1) \cup \overline{N}(u_2) \subseteq \overline{N}(v)$, which contradicts the minimality of A. Similarly, it can be shown that there are at most two distinct $u_1, u_2 \in A$ adjacent to v whose corresponding intervals contain x. Thus, v is adjacent to at most five vertices of A. ■

Theorem 5.7. *Let G be a connected interval graph with n vertices. Then $c_\infty(G) = O(\sqrt{n})$.*

Proof. By Lemma 5.4 it is enough to show that $w(G) = O(\sqrt{n})$. Let $G[a, b]$ be an arbitrary interval subgraph of G. We just need to prove that $G[a, b]$ is not $(\sqrt{5n} + 3)$-wide. Choose two arbitrary vertices $u_a \in V_a, u_b \in V_b$. Let a' be the smallest index in $[a, b]$ with $u_a \notin V_{a'}$, and b' be the largest index in $[a, b]$ with $u_b \notin V_{b'}$. If either of these indices does not exist or $a' > b'$, then $\{u_a, u_b\}$ is a dominating set for $G[a, b]$, so it is not $(\sqrt{5n} + 3)$-wide.

Consider the graph $G[a', b']$. Let n_1 be its number of vertices, T be a minimum dominating set for it, and δ be its minimum degree. Let $t = |T|$. Note that $T \cup \{u_a, u_b\}$ is a dominating set for $G[a, b]$, so the domination number of $G[a, b]$ is at most $t + 2$. Moreover, $G[a', b']$ is an interval graph, so by Lemma 5.6, every vertex $v \in V(G[a', b']) \setminus T$ is adjacent to at most five vertices of T, and every vertex $v \in T$ is adjacent to at most two vertices of T, hence (denoting the degree of u in $G[a', b']$ by $deg(u)$) we have

$$t(\delta + 1) \leq \sum_{u \in T} (deg(u) + 1) \leq 5n_1 \leq 5n,$$

so $\min\{t, \delta + 1\} \leq \sqrt{5n}$.

If $t \leq \sqrt{5n}$ then the domination number of $G[a, b]$ is at most $\sqrt{5n} + 2$ so it is not $(\sqrt{5n} + 3)$-wide. So we may assume that $\delta + 1 \leq \sqrt{5n}$. Let u be a vertex of minimum degree in $G[a', b']$, which is contained in some $V_i, a' \leq i \leq b'$. Thus $|V_i| \leq \delta + 1 \leq \sqrt{5n}$ and V_i is a cut-set in $G[a, b]$ (as it separates u_a, u_b), so $G[a, b]$ is not $(\sqrt{5n} + 1)$-connected, and hence not $(\sqrt{5n} + 3)$-wide. ■

Chapter 6

A Lower Bound for the Maximum Cop Number of Chordal Graphs

Definition (chordal graph)**.** Graph G is called *chordal* if it does not have an induced cycle with more than 3 vertices.

Note that any interval graph is chordal. It is well-known that every chordal graph G has a tree decomposition (T, W) such that for every $t \in V(T)$, the set W_t induces a clique in G. This fact can be used to show that if G is chordal, then $c_1(G) = 1$: the cop moves in T towards the robber, and the robber cannot go around the cop, since if at some moment for some $t \in V_t$, both the robber and the cop are in W_t, then the cop will capture the robber in the next round (see [47] for instance). However, when the robber has infinite speed the situation is quite different. In this chapter we prove that there exist chordal graphs G with $c_\infty(G) = \Omega(n/\log n)$. More precisely, it is shown that for every positive integer m, there exists a chordal graph G with $O(m \log m)$ vertices having $c_\infty(G) \geq m$.

Definition (access, accessible)**.** Say the robber *has access* to a subset $X \subseteq V(G)$ if there exists a cop-free path from the robber's vertex to a vertex in X. A pair (X, v) with $X \subseteq V(G)$ and $v \in V(G)$ is called *accessible* if

- $c_\infty(G) \geq |X|$,
- $N(v) = X$, and
- if there are $|X| - 1$ cops in the game, then there exists a strategy for the robber with the following properties: the robber has access to X in every round, but she never moves to a vertex in $X \cup \{v\}$.

In Figure 6.1, (X_i, v_i) is an accessible pair of G_i for $i = 1, 2$.

Figure 6.1: Examples of accessible sets

Lemma 6.1. *Let G_1, G_2 be graphs on disjoint vertex sets, and for $i = 1, 2$, $(X_i, v_i) \subseteq V(G_i)$ be an accessible pair for G_i with $|X_i| = k$. Let G be a graph with vertex set $V(G) = V_1 \cup U_1 \cup X \cup U_2 \cup V_2 \cup \{v\}$, and such that*

- *For $i = 1, 2$, $V_i = V(G_i) \setminus \{v_i\}$.*
- *We have $|U_1| = |X| = |U_2| = 2|X_1| = 2|X_2| = 2k$ and V_1, U_1, X, U_2, V_2 are disjoint.*
- *The following pairs induce complete bipartite subgraphs of G: (X_1, U_1), (U_1, X), (X, U_2), (U_2, X_2).*
- *There is no other edge between any two of V_1, U_1, X, U_2, V_2, but there can be arbitrary edges inside U_1, X, U_2.*
- *The set of neighbours of v is precisely X.*

Then X is an accessible subset of G.

In Figure 6.2 you see an example of such a G, where G_1 and G_2 are graphs shown in Figure 6.1.

Proof. Assume that there are $2k - 1$ cops in the game. We prove that the robber has an escaping strategy that evades the cops forever, and is such that she has access to X in every round, but never moves to $X \cup \{v\}$. Let $A_i = V_i \cup U_i$ for $i = 1, 2$. The strategy has the following invariant: at the end of each round, the robber is at a vertex of V_j for some $1 \leq j \leq 2$, such that there are less than k cops in A_j, and the robber has access to X_j. If we provide such a strategy, then since the robber has access to X_j and there are k disjoint paths from X_j to X, the robber has access to X in every round. We may assume without

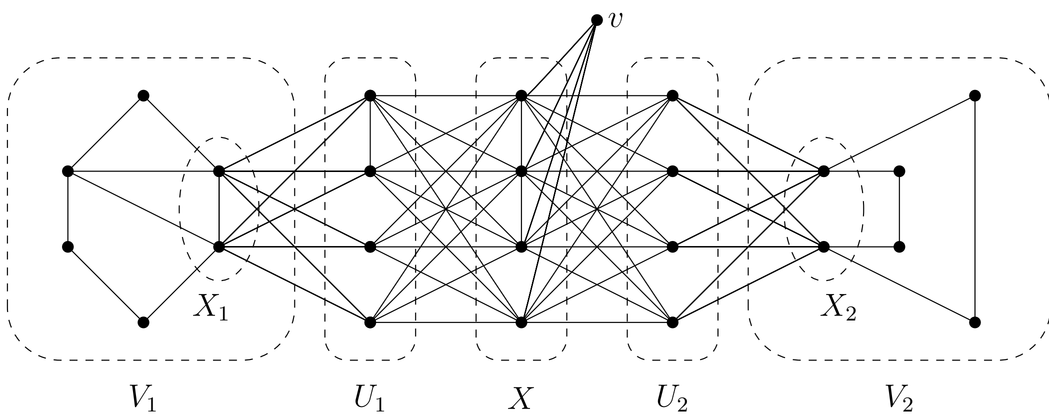

Figure 6.2: An example for Lemma 6.1

loss of generality that all the cops start at some vertex in V_2, and the robber starts at some vertex in V_1, so the invariant holds at the beginning (with $j = 1$).

Assume that the invariant holds at the end of the previous round, say with $j = 1$. This means that the robber is at a vertex of V_1, has access to X_1, and there are less than k cops in A_1. In the next round, first the cops move. If after their move, there are still less than k cops in A_1, then the robber assumes the game is actually played in G_1, where she considers all cops in V_1 as they are in G_1, and she considers all cops in $V(G) \setminus V_1$ as if they are at v_1; then she just plays her escaping strategy in G_1, thus she will not go to $X_1 \cup \{v_1\}$ and will not be captured in the next round. Recall that v_1 is the vertex in G_1 whose set of neighbours is X_1.

Now, assume that after the cops have moved, there are at least k cops in A_1. There are at most $k - 1$ cops in A_2 at this moment, and in particular, at most $k - 1$ cops in V_2. Recall that X_2 is an accessible subset of G_2, which means, in particular, that there exists a vertex $u \in V_2$ such that at this moment there is a cop-free path P from X_2 to u. (To see this, note that if one just considers the graph induced by V_2 and assumes that the game is played only in this subgraph, then the robber can choose a vertex that has access to X_2.) Since at the end of previous round there were less than k cops in A_1, there are less than k cops in V_1 at this moment. Hence the robber has access to X_1 (note that cops in $V(G) \setminus V_1$ will not block the robber's access to X_1), through which she can pass through U_1, X, U_2, X_2 (notice that each of these has at least one cop-free vertex), and finally go to u along the path P. ■

It is easy to verify that if both G_1 and G_2 are chordal graphs and the subgraphs induced by U_1, X, and U_2 are complete graphs, then the resulting graph G is chordal as well. This lets us deduce the following lower bound.

Theorem 6.2. *For every positive integer m, there exists a chordal graph G with $O(m \log m)$ vertices and $c_\infty(G) \geq m$.*

Proof. For every m, let $g(m)$ denote the number of vertices of the smallest connected chordal graph that has an accessible set of size m. Then, by Lemma 6.1 and the discussion above,

$$g(2) \leq 7, \quad g(m) \leq 2(g(\lceil m/2 \rceil) - 1) + 6\lceil m/2 \rceil + 1,$$

which gives $g(m) = O(m \log m)$ (one can show by induction that, for instance, $g(m) \leq 10m \log_2 m - 7$). ∎

Chapter 7

Lower Bounds for Expander Graphs

Definition (edge-isoperimetric number, vertex-isoperimetric number)**.** Let G be a graph. For a subset S of vertices of G, write ∂S for the set of edges with exactly one endpoint in S. Define the *edge-isoperimetric* and *vertex-isoperimetric* numbers of G as

$$\iota_e(G) = \min_{|S| \le n/2} \frac{|\partial S|}{|S|},$$

$$\iota_v(G) = \min_{|S| \le n/2} \frac{|N(S) \setminus S|}{|S|}.$$

Note that for any graph G we have $\iota_e(G) \le \Delta(G)$ (by taking S to be any single vertex) and $\iota_v(G) \le 1$ (by taking S to be any subset with $n/2$ vertices).

The idea of using expansion properties of graphs to bound the cop numbers, first appeared in [27], where the authors proved the following.

Theorem 7.1. *Let $s \in \mathbb{N}$ and $\alpha = 1+1/s$. There is a function $p = p(n) = \alpha^{-(1-o(1))\sqrt{\log_\alpha n}}$ for which the following holds. In every graph G on n vertices with $\Delta(G) < 1/p$ and $\iota_v(G) \ge p$, we have $c_s(G) \le (1 + o(1))2pn\sqrt{\log_\alpha n}$.*

Using the above theorem, they proved that

$$f_s(n) \le \alpha^{-(1-o(1))\sqrt{\log_\alpha n}},$$

where $s \in \mathbb{N}$ and $\alpha = 1 + 1/s$.

In this chapter we prove that for every graph G, we have

$$c_\infty(G) \ge \frac{\iota_e n}{\Delta^2 - \Delta + \iota_e(\Delta + 1)} \ge \frac{\iota_e n}{2\Delta^2},$$

and

$$c_\infty(G) \geq \max\left\{ \frac{\iota_v n}{3\Delta + \iota_v(\Delta+1)}, \frac{\iota_v n}{4\Delta} \right\}.$$

Lemma 7.2. *Let m be a positive integer such that for every subset S of at most m vertices, $G - \overline{N}(S)$ has a connected component of size larger than $n/2$. Then $c_\infty(G) > m$.*

Proof. Assume that there are m cops in the game, and we give an escaping strategy for the robber. The strategy has the following invariant: at the end of each round, if the cops are positioned in a subset S of vertices, then the robber is at a vertex of the unique component of $G - \overline{N}(S)$ that has size larger than $n/2$. Let S_0 be the subset of vertices that the cops occupy when the game starts. By hypothesis, $G - \overline{N}(S_0)$ has a connected component C_0 of size larger than $n/2$, and the robber starts at an arbitrary vertex of C_0.

Suppose that at the end of round i, the cops are in S_i, and the robber is in a component C_i of $G - \overline{N}(S_i)$ of size larger than $n/2$. In round $i+1$, the cops move to a new set $S_{i+1} \subseteq \overline{N}(S_i)$, so the robber is not captured. Let C_{i+1} be the connected component of $G - \overline{N}(S_{i+1})$ that has size larger than $n/2$. As both C_i and C_{i+1} have size larger than $n/2$, they intersect. Let $v \in C_i \cap C_{i+1}$. Since C_i is disjoint from $\overline{N}(S_i)$, at this moment there is no cop in C_i. Moreover, C_i is connected and the robber is in C_i, so she can move to v in this round. Hence at the end of round $i+1$, the robber is in C_{i+1}, the connected component of $G - \overline{N}(S_{i+1})$ of size larger than $n/2$, and the proof is complete. ■

Remark. The idea in the proof was first used in [27] to prove the existence of graphs with large cop number (in the infinitely-fast robber case).

Before proving the main result of this chapter, we need a technical lemma.

Lemma 7.3. *Let n, t be positive integers with $t \leq n$. Let $a_1, a_2, \ldots, a_m$ be positive integers such that each of them is at most $n/2$, and their sum is t. Then we have the following.*

(a) One can choose a subset of $\{a_1, \ldots, a_m\}$ whose sum is between $t/3$ and $n/2$ (inclusive).

(b) If $t \geq n/4$ then one can choose a subset of $\{a_1, \ldots, a_m\}$ whose sum is between $n/4$ and $n/2$ (inclusive).

Proof. We may assume without loss of generality that

$$a_1 \geq a_2 \geq \cdots \geq a_m.$$

(a) We use induction on m. If $m \leq 3$, then a_1 is between $t/3$ and $n/2$, so we may assume that $m \geq 4$. Since $a_m + a_{m-1} \leq a_{m-2} + a_{m-3}$ and the sum of the a_i's is t, which is not more than n, we have $a_m + a_{m-1} \leq n/2$. Define

$$b_1 = a_1, b_2 = a_2, \ldots, b_{m-2} = a_{m-2}, b_{m-1} = a_{m-1} + a_m.$$

Then each of the b_i's is at most $n/2$, and their sum is t. Thus by the induction hypothesis, there exists a subset of them whose sum is between $t/3$ and $n/2$. This gives a subset of the a_i's with the same sum, and the proof is complete.

(b) We use induction on m. If $m = 1$ then we have $a_1 = t \geq n/4$ so a_1 is between $n/4$ and $n/2$, and we are done. So, we may assume that $m \geq 2$. If a_{m-1} is at least $n/4$, then a_{m-1} is between $n/4$ and $n/2$, and we are done. So, we may assume that $a_{m-1} < n/4$, thus $a_{m-1} + a_m < n/2$. Define

$$b_1 = a_1, b_2 = a_2, \ldots, b_{m-2} = a_{m-2}, b_{m-1} = a_{m-1} + a_m.$$

Then each of the b_i's is at most $n/2$, and their sum is t. Thus by induction hypothesis, there exists a subset of them whose sum is between $n/4$ and $n/2$. This gives a subset of the a_i's with the same sum, and the proof is complete. ■

Now we are ready to prove the main result of this chapter.

Theorem 7.4. *For every graph G we have*

(a) $c_\infty(G) \geq \dfrac{\iota_e n}{\Delta^2 - \Delta + \iota_e(\Delta + 1)} \geq \dfrac{\iota_e n}{2\Delta^2}$,

(b) $c_\infty(G) \geq \dfrac{\iota_v n}{3\Delta + \iota_v(\Delta + 1)}$,

(c) $c_\infty(G) \geq \dfrac{\iota_v n}{4\Delta}$.

Proof. Let $c = c_\infty(G)$. By Lemma 7.2 there exists a subset S of at most c vertices such that $G - \overline{N}(S)$ has no component of size larger than $n/2$. We have

$$|\overline{N}(S)| \leq c(\Delta + 1), \qquad |\overline{N}(S) \setminus S| \leq c\Delta, \quad \text{and} \qquad |\partial\overline{N}(S)| \leq c\Delta(\Delta - 1),$$

where the last inequality holds since at most $c\Delta$ vertices of $\overline{N}(S)$ have a neighbour out of $\overline{N}(S)$, and each has at most $\Delta - 1$ such neighbours. Let $T = V(G) \setminus \overline{N}(S)$, and let $A_1, A_2, \ldots, A_m$ be the connected components of $G[T]$. As $G[T]$ has no component of size larger than $n/2$, we have $|A_i| \leq n/2$ for all i.

(a) Since all of the $|A_i|$'s are at most $n/2$, for all $1 \leq i \leq m$ we have $|\partial A_i| \geq \iota_e |A_i|$. Thus

$$|\partial T| = \sum_{i=1}^{m} |\partial A_i| \geq \sum_{i=1}^{m} \iota_e |A_i| = \iota_e \sum_{i=1}^{m} |A_i| = \iota_e |T|.$$

This gives

$$c\Delta(\Delta - 1) \geq |\partial\overline{N}(S)| = |\partial T| \geq \iota_e|T| = \iota_e(n - |\overline{N}(S)|) \geq \iota_e(n - c(\Delta + 1)).$$

Part (a) now results by simplifying and noting that $\iota_e \le \Delta$.

(b) By Lemma 7.3 part (a), one can pick some components of $G[T]$ such that their union, T', has size at least $|T|/3$ and at most $n/2$. Then the set $N(T') \setminus T'$ has size at least $\iota_v |T'|$ and at most $|\overline{N}(S) \setminus S|$. Thus

$$c\Delta \ge |\overline{N}(S) \setminus S| \ge \iota_v |T'| \ge \iota_v |T|/3 = \iota_v (n - |\overline{N}(S)|)/3 \ge \iota_v (n - c(\Delta + 1))/3,$$

and part (b) follows after simplification.

(c) If $|T| < n/4$, then we have $|\overline{N}(S)| > 3n/4$ so that $c(\Delta + 1) > 3n/4$ and

$$c > \frac{3n}{4(\Delta + 1)} > \frac{\iota_v n}{4\Delta},$$

as $\iota_v \le 1$ and $\Delta \ge 1$.

If $|T| \ge n/4$, then by Lemma 7.3 part (b), one can pick some components of T such that their union has size at least $n/4$ and at most $n/2$. Let T' be their union. Then the set $N(T') \setminus T'$ has size at least $\iota_v |T'|$ and at most $|\overline{N}(S) \setminus S|$, thus

$$c\Delta \ge |N(S) \setminus S| \ge \iota_v |T'| \ge \iota_v n/4,$$

and part (c) follows. ■

A *sequence of bounded-degree expanders* is a sequence $\{G_i\}_{i=1}^{\infty}$ of graphs, where each G_i has maximum degree $O(1)$ and vertex expansion $\Omega(1)$. Theorem 7.4 shows that every family of bounded-degree expanders have cop number $\Omega(n)$. It also provides lower bounds for graphs with high expansion, for example random graphs (see Chapter 8) and Cartesian products of graphs (see Chapter 9).

Chapter 8

Bounds for Random Graphs

In this chapter we study $c_s(G)$ when G is a random graph. The parameter $c_1(G)$ for random graphs G has been studied by many authors, see [11, 14, 45, 37].

Definition ($\mathcal{G}(n,p)$, asymptotically almost surely (a.a.s.))**.** Let n be a positive integer and $p \in [0,1]$. The space $\mathcal{G}(n,p)$ is a probability space over all labelled graphs on n vertices, such that for a randomly chosen $G \in \mathcal{G}(n,p)$ and a labelled graph H on n vertices,

$$\mathbf{Pr}(G = H) = p^{|E(H)|}(1-p)^{\binom{n}{2}-|E(H)|}.$$

This space may be viewed as $\binom{n}{2}$ independent coin flips, one for each pair of vertices, where the probability of drawing an edge between that pair is equal to p. An Erdös-Rényi random graph with parameters n and p, is a sample G from the space $\mathcal{G}(n,p)$. All asymptotics throughout are as $n \to \infty$. We say that an event in a probability space holds *asymptotically almost surely (a.a.s.)* if the probability that it holds approaches 1 as n goes to infinity. Note that $p = p(n)$ can tend to zero as n tends to infinity.

The main results of this chapter are the following (recall that $\gamma(G)$ is the domination number of G).

- Let $pn \geq 20 \log n$ and $p = 1 - \Omega(1)$. Then there exist positive constants η_1, η_2 such that a random graph $G \in \mathcal{G}(n,p)$ a.a.s. has
$$\frac{\eta_1}{p} \leq c_\infty(G) \leq \frac{\eta_2 \log(np)}{p}.$$
- If $np = \omega(\sqrt{n \log n})$ and $p = 1 - \Omega(1)$, then a.a.s.
$$c_\infty(G) = \gamma(G) = \Theta\left(\frac{\log n}{p}\right).$$

- If $np = n^{\alpha+o(1)}$, where $1/2 < \alpha < 1$, and $p = 1 - \Omega(1)$, then a.a.s.

$$c_s(G) = \Theta\left(\frac{\log n}{p}\right) \qquad \forall s \in \mathbb{N} \cup \{\infty\}.$$

- If $np = n^{1-o(1)}$, then a.a.s.

$$c_s(G) = (1+o(1))\frac{\log n}{\log \frac{1}{1-p}} \qquad \forall s \in \mathbb{N} \cup \{\infty\}.$$

- Let $d \geq 3$ be fixed. Then a.a.s. a randomly chosen labelled d-regular graph G on n vertices has
$$c_\infty(G) = \Theta(n).$$

First we will prove a large deviation inequality.

Lemma 8.1. *Let $X_1, X_2, \ldots, X_m$ be independent identically distributed indicator random variables with $\mathbb{E}X_i \geq q$ for all $1 \leq i \leq m$. Then for any $0 < b < 1$,*

$$\mathbf{Pr}[X_1 + \cdots + X_m \leq bm] \leq \left(2(1-q)^{1-b}\right)^m.$$

Proof. Let $p = \mathbb{E}X_i \geq q$. We have

$$\begin{aligned}
\mathbf{Pr}[X_1 + \cdots + X_m \leq bm] &= \sum_{i=0}^{\lfloor bm \rfloor} \binom{m}{i} p^i (1-p)^{m-i} \\
&\leq \sum_{i=0}^{\lfloor bm \rfloor} \binom{m}{i} (1-p)^{m-i} \\
&\leq \sum_{i=0}^{\lfloor bm \rfloor} \binom{m}{i} (1-p)^{m-mb} \\
&\leq 2^m (1-p)^{m-mb} = \left(2(1-p)^{1-b}\right)^m \leq \left(2(1-q)^{1-b}\right)^m. \qquad \blacksquare
\end{aligned}$$

Next we give a lower bound for vertex-expansion of random graphs, which is of independent interest.

Theorem 8.2. *Let $0 < b < 1$ be fixed. There is a constant $k(b)$ such that for every fixed $k > k(b)$ if $pn - k\log n \to \infty$, then the random graph $G \in \mathcal{G}(n,p)$ a.a.s. has $\iota_v(G) \geq b$.*

Proof. Assume that $pn - k\log n \to \infty$, where k is a constant that will be defined later. Let $V(G) = \{v_1, \dots, v_n\}$. For $1 \le r \le n/2$, define

$$A^{(r)} = \{v_{n-r+1}, \dots, v_n\}, \quad X^{(r)} = |N(A^{(r)}) \cap \{v_1, \dots, v_{n/2}\}|.$$

Note that $|A^{(r)}| = r$ and $X^{(r)} = X_1^{(r)} + \dots + X_{n/2}^{(r)}$, where $X_i^{(r)}$ is the indicator random variable for $v_i \in N(A^{(r)})$. For all $1 \le i \le n/2$ we have

$$\mathbb{E}X_i^{(r)} = \mathbf{Pr}[v_i \in N(A^{(r)})] = 1 - (1-p)^r \ge 1 - e^{-pr}.$$

By symmetry (among the subsets of size r) and the union bound (which states that the probability that at least one of a certain set of events happen is at most the sum of their probabilities) it suffices to prove that

$$\sum_{r=1}^{n/2} \binom{n}{r} \mathbf{Pr}\left[X^{(r)} < br\right] = o(1).$$

Let t be a constant satisfying $e^{t(1-b)} > 8$. We split this sum into two parts: $1 \le r \le \lfloor t/p \rfloor$ and $\lceil t/p \rceil \le r \le n/2$.

First, let $\lceil t/p \rceil \le r$. As $\mathbb{E}X_i^{(r)} \ge 1 - \exp(-pr) \ge 1 - \exp(-t)$ for all $1 \le i \le n/2$, Lemma 8.1 (with $m = n/2$ and $q = 1 - \exp(-t)$) gives

$$\mathbf{Pr}[X^{(r)} < bn/2] \le \left(2e^{-t(1-b)}\right)^{n/2}.$$

Thus, as $\mathbf{Pr}\left[X^{(r)} < br\right] \le \mathbf{Pr}[X^{(r)} < bn/2]$, we have

$$\sum_{r=\lceil t/p \rceil}^{n/2} \binom{n}{r} \mathbf{Pr}\left[X^{(r)} < br\right] \le 2^n \left(2e^{-t(1-b)}\right)^{n/2} = \left(8e^{-t(1-b)}\right)^{n/2}$$

which is $o(1)$ as t satisfies $e^{t(1-b)} > 8$.

For the other part, $1 \le r \le \lfloor t/p \rfloor$, let $\alpha \le (1-e^{-t})/t$ be fixed. Since $1 - e^{-x}$ is concave, we have

$$1 - e^{-x} \ge \alpha x \quad \forall\, 0 \le x \le t.$$

When $r \le \lfloor t/p \rfloor$, we have $pr \le t$, thus

$$\mathbb{E}X^{(r)} = \frac{n}{2}\mathbb{E}X_1^{(r)} \ge \frac{n}{2}(1 - e^{-pr}) \ge \frac{n\alpha pr}{2}.$$

Let $\epsilon = 1 - \frac{2b}{n\alpha p}$. Note that when pn is sufficiently large, ϵ gets arbitrarily close to 1. We have

$$\mathbf{Pr}\left[X^{(r)} < br\right] = \mathbf{Pr}\left[X^{(r)} < (1-\epsilon)\frac{n\alpha pr}{2}\right] \le \mathbf{Pr}\left[X^{(r)} < (1-\epsilon)\mathbb{E}X^{(r)}\right].$$

Since $X^{(r)}$ is a sum of indicator variables, by the one-sided Chernoff bound (Theorem 4.2 of [40]),

$$\mathbf{Pr}\left[X^{(r)} < (1-\epsilon)\mathbb{E}X^{(r)}\right] < \exp\left(-\mathbb{E}X^{(r)}\frac{\epsilon^2}{2}\right).$$

Thus we find

$$\mathbf{Pr}\left[X^{(r)} < br\right] \leq \mathbf{Pr}\left[X^{(r)} < (1-\epsilon)\mathbb{E}X^{(r)}\right] \leq \exp\left(-\mathbb{E}X^{(r)}\frac{\epsilon^2}{2}\right) \leq \exp\left(-\frac{n\alpha pr\epsilon^2}{4}\right).$$

Let k be a constant larger than $k(b) = 8/\alpha$. Assume that $pn - k\log n \to \infty$, as hypothesized in the theorem. For pn sufficiently large, ϵ becomes arbitrarily close to 1, and $\frac{8}{\alpha\epsilon^2}$ gets arbitrarily close to $\frac{8}{\alpha}$, so $pn - \frac{8}{\alpha\epsilon^2}\log n \to \infty$ as well. Thus we find

$$\lim_{n\to\infty} 2\log n - \frac{n\alpha p\epsilon^2}{4} = -\infty.$$

For n sufficiently large, $\log\frac{1}{p} < \log n$, and thus (as $r \geq 1$),

$$\lim_{n\to\infty} \log\frac{1}{p} + r\log n - \frac{n\alpha p\epsilon^2 r}{4} \leq \lim_{n\to\infty} r\left(2\log n - \frac{n\alpha p\epsilon^2}{4}\right) = -\infty.$$

Therefore, we have

$$\frac{n^r}{p}\exp\left(-\frac{n\alpha pr\epsilon^2}{4}\right) = o(1).$$

Consequently, as t is fixed,

$$\sum_{r=1}^{\lfloor t/p\rfloor}\binom{n}{r}\mathbf{Pr}\left[X^{(r)} < br\right] \leq \sum_{r=1}^{\lfloor t/p\rfloor} n^r\exp\left(-\frac{n\alpha pr\epsilon^2}{4}\right) = \frac{t}{p}o(p) = o(1). \quad \blacksquare$$

For upper bounds, we will use some known bounds on the domination number $\gamma(G)$ of random graphs. The following theorem has been proved in page 4 of [3].

Theorem 8.3. *Every graph G has*

$$\gamma(G) \leq n\frac{1+\log(\delta+1)}{\delta+1}.$$

Corollary 8.4. *If $np > 2\log n$ then a random graph $G \in \mathcal{G}(n,p)$ a.a.s. has*

$$\gamma(G) = O\left(\frac{n\log\delta}{\delta}\right) = O\left(\frac{\log(np)}{p}\right).$$

Proof. For $np > 2\log n$, a.a.s. δ is $\Theta(np)$. ∎

The following theorem has been proved in [14] when $p = o(1)$, and in [52] when $p = \Omega(1)$.

Theorem 8.5. *A random graph $G \in \mathcal{G}(n,p)$ a.a.s has*

$$\gamma(G) \leq (1+o(1))\frac{\log n}{\log \frac{1}{1-p}}.$$

Note that if $p = 1 - \Omega(1)$, then the right-hand-side is $\Theta(\frac{\log n}{p})$.

We are ready to prove the main theorem of this chapter, which provides bounds for cop numbers of the random graph $\mathcal{G}(n,p)$ for different speeds and various ranges of p.

Theorem 8.6. *(a) For all ranges of np and any $s \in \mathbb{N} \cup \{\infty\}$ we have*

$$c_s(G) \leq \gamma(G).$$

For specific ranges of np, we have the following results.

(b)

$$np = o(\log n), G \text{ connected } \Rightarrow \text{ a.a.s. } c_\infty(G) = \Omega\left(\frac{\delta n}{\Delta^2}\right).$$

(c)

$$np = 20\log n + \omega(1) \Rightarrow \text{a.a.s. } c_\infty(G) = \Omega\left(\frac{n}{\Delta}\right) = \Omega\left(\frac{1}{p}\right), \text{and}$$
$$\text{a.a.s. } c_\infty(G) = O\left(\frac{n\log\delta}{\delta}\right) = O\left(\frac{\log(np)}{p}\right).$$

(d)

$$np = \omega\left(\sqrt{n\log n}\right) \Rightarrow \text{ a.a.s. } c_\infty(G) = \gamma(G).$$

(e)

$$np = n^{\alpha+o(1)}, \frac{1}{2} < \alpha < 1 \Rightarrow \text{ a.a.s. } c_s(G) = \Theta\left(\frac{\log n}{p}\right) = n^{1-\alpha+o(1)} \quad \forall s \in \mathbb{N} \cup \{\infty\}.$$

(f)

$$np = n^{1-o(1)} \Rightarrow \text{ a.a.s. } c_s(G) = (1+o(1))\frac{\log n}{\log\frac{1}{1-p}} \quad \forall s \in \mathbb{N} \cup \{\infty\}.$$

Remark. Notice the gap between ranges of parts (a) and (b): when $np = \Omega(\log n)$ but $np < 20 \log n$, we do not give a lower bound for c_∞.

Proof. (a) For every $s \in \mathbb{N} \cup \{\infty\}$ we have $c_s(G) \leq \gamma(G)$, since if the cops start by occupying a dominating set, they will capture the robber in the first round.

(b) It is known that if $\delta(G) = o(\log n)$, then a.a.s. $\iota_e(G) = \delta(G)$ [9]. Part (b) thus follows from part (a) of Theorem 7.4.

(c) Let $b = 0.001, t = 2.1, \alpha = 0.41$ and $k = 20$. It follows from the proof of Theorem 8.2 that if $pn \geq k \log n$ then G a.a.s. has $\iota_v(G) \geq b$, and the lower bound follows from part (c) of Theorem 7.4, and noting that in this range we have $\Delta = \Theta(np)$. The upper bound follows from Corollary 8.4.

(d) Clearly $c_\infty(G) \leq \gamma(G)$. Since $np = \omega\left(\sqrt{n \log n}\right)$, we have $n \log n = o((np)^2) = o(\delta^2)$. By Corollary 8.4 we have

$$\gamma(G) = O\left(\frac{n \log \delta}{\delta}\right) = O\left(\frac{n \log n}{\delta}\right) = o(\delta).$$

So we may assume that $\gamma(G) < \delta(G)$.

A.a.s. G is $\delta(G)$-connected so a.a.s. it is $\gamma(G)$-connected. For proving that $\gamma(G) \leq c_\infty(G)$ a.a.s., we need to show that at least $\gamma(G)$ cops are needed to capture an infinitely fast robber if G is a $\gamma(G)$-connected graph. Indeed if there are less than $\gamma(G)$ cops in the game, there exists a non-dominated vertex in every round, and there exists an unblocked path to that vertex since G is $\gamma(G)$-connected, so the robber can move there, and will never be captured.

(e) Bonato et al. [14] proved that if $np = n^{\alpha + o(1)}$, where $1/2 < \alpha < 1$, then a.a.s. $G \in \mathcal{G}(n, p)$ satisfies

$$c_1(G) = \Theta\left(\frac{\log n}{p}\right) = n^{1-\alpha+o(1)}.$$

For other s, note that we have

$$c_1(G) \leq c_2(G) \leq \cdots \leq c_\infty(G) \leq \gamma(G).$$

And by Corollary 8.4, $\gamma(G)$ is a.a.s at most

$$O\left(\frac{\log np}{p}\right) = O\left(\frac{\log n}{p}\right) = n^{1-\alpha+o(1)}.$$

(f) Bonato et al. [14] proved that if $np = n^{1-o(1)}$ then a.a.s. $G \in \mathcal{G}(n,p)$ satisfies

$$c_1(G) = (1+o(1))\frac{\log n}{\log \frac{1}{1-p}}.$$

For other s, note that we have

$$c_1(G) \le c_2(G) \le \cdots \le c_\infty(G) \le \gamma(G).$$

And by Theorem 8.5, $\gamma(G)$ is a.a.s. at most

$$(1+o(1))\frac{\log n}{\log \frac{1}{1-p}}. \quad \blacksquare$$

Finally, we give bounds for c_∞ of random regular graphs, using the following theorem for their edge expansion, which has been proved by Bollobás [10].

Theorem 8.7. *Let $d \ge 3$ be fixed. Then a.a.s. a randomly chosen d-regular labelled graph G on n vertices has*

$$\iota_e(G) \ge d/2 - \sqrt{d \log 2}.$$

Corollary 8.8. *Let $d \ge 3$ be fixed. Then a.a.s. a randomly chosen d-regular labelled graph G on n vertices has*

$$\frac{d - 2\sqrt{d\log 2}}{4d^2}\, n \le c_\infty(G) \le \gamma(G) \le \frac{1+\log(d+1)}{d+1}\, n.$$

Proof. The lower bound follows from the above bound for $\iota_e(G)$ and part (a) of Theorem 7.4. The upper bound for $\gamma(G)$ follows from Corollary 8.4. $\blacksquare$

Chapter 9

Bounds for Cartesian Products of Graphs

Neufeld and Nowakowski [41] gave bounds for $c_1(G)$ when G is a product graph (see Section 1.3.1). In this chapter we study $c_\infty(G)$ when G is a Cartesian product of graphs.

Definition (Cartesian product)**.** Let $G_1, G_2, \ldots, G_m$ be graphs. Define G to be the graph with vertex set $V(G_1) \times V(G_2) \times \cdots \times V(G_m)$ with vertices $(u_1, u_2, \ldots, u_m)$ and $(v_1, v_2, \ldots, v_m)$ being adjacent if there exists an index $1 \leq j \leq m$ such that

- $u_i = v_i$ for all $i \neq j$, and
- u_j and v_j are adjacent in G_j.

Then G is called the *Cartesian product* of $G_1, G_2, \ldots, G_m$.

Remark. If every G_i is isomorphic to an edge, then the graph G is actually the m-dimensional hypercube, which is denoted by $\mathcal{H}_m$.

The following theorem is the main result of this chapter.

Theorem 9.1. *Let $G_1, G_2, \ldots, G_m$ be graphs and let n_i denote the number of vertices of G_i for $1 \leq i \leq m$. Let G be the Cartesian product of $G_1, G_2, \ldots, G_m$, and $n = |V(G)| = n_1 n_2 \ldots n_m$. Then we have*

(a)

$$\frac{\min\{\iota_e(G_i) : 1 \leq i \leq m\} n}{4(\Delta(G_1) + \cdots + \Delta(G_m))^2} \leq c_\infty(G) \leq \frac{n c_\infty(G_1)}{n_1}.$$

Since the upper bound holds for every ordering of the graphs, one can choose the graph with the smallest c_∞ as G_1, in order to optimize the upper bound.

(b) *If every G_i is a path and $n_1 = \min\{n_i : 1 \le i \le m\}$, then*

$$\frac{n}{4n_1 m^2} \le c_\infty(G) \le \frac{n}{n_1}.$$

(c) *If every G_i is a cycle, $n_1 = \min\{n_i : 1 \le i \le m\}$ and n_1 is even, then*

$$\frac{n}{2n_1 m^2} \le c_\infty(G) \le \frac{2n}{n_1}.$$

(d) *There exist positive constants κ_1, κ_2 such that if every G_i is isomorphic to the complete graph on k vertices, then*

$$\frac{\kappa_1 n}{m^{3/2} k} \le c_\infty(G) \le \min\left\{\frac{n}{k}, \frac{\kappa_2 n}{\sqrt{m}}\right\}.$$

(e) *If every G_i is isomorphic to an edge, i.e. if G is the m-dimensional hypercube $\mathcal{H}_m$, then there exist constants $\eta_1, \eta_2 > 0$ such that*

$$\frac{\eta_1 n}{m\sqrt{m}} \le c_\infty(G) \le \frac{\eta_2 n}{m}.$$

Proof. (a) Chung and Tetali [17] have proved that

$$\iota_e(G) \ge \min\{\iota_e(G_i) : 1 \le i \le m\}/2.$$

Noting that $\Delta(G) = \Delta(G_1) + \cdots + \Delta(G_m)$, the lower bound thus follows from part (a) of Theorem 7.4.

For the upper bound we give a strategy for $nc_\infty(G_1)/n_1$ cops to capture a robber in G. Let $k = c_\infty(G_1)$. By definition, there is a winning strategy for k cops when the game is played in G_1. We consider a *virtual game*, in which k virtual cops are capturing a virtual robber in G_1. (Using a virtual game for bounding the cop number is also used in the proof of Lemma 4.3, where it has been explained in more detail.) For every virtual cop, we put $n/n_1 = n_2 n_3 \dots n_m$ real cops in the real game, such that if the virtual cop is in $u_1 \in V(G_1)$, then the real cops occupy $\{u_1\} \times V(G_2) \times \cdots \times V(G_m)$. Also, if the real robber is at $(v_1, \dots, v_m) \in G$, then the virtual robber is at $v_1 \in G_1$. It is not hard to see that the real cops can move in such a way that these constraints hold throughout the games. Hence, once the virtual robber has been captured, the real robber has also been captured, and the proof is complete.

(b) Azizoğlu and Eğecioğlu [8] have proved that

$$\iota_e(G) = \left\lfloor \frac{n_1}{2} \right\rfloor^{-1} \ge \frac{2}{n_1}.$$

As G has n vertices and maximum degree $2m$, the lower bound follows from part (a) of Theorem 7.4. The upper bound follows from part (a) of the present theorem, since G_1 is a path and has $c_\infty(G_1) = 1$.

(c) Azizoğlu and Eğecioğlu [7] have proved that

$$\iota_e(G) = \frac{4}{n_1}.$$

As G has n vertices and maximum degree $2m$, the lower bound follows from part (a) of Theorem 7.4. The upper bound follows from part (a) of the present theorem, since G_1 is a cycle and has $c_\infty(G_1) = 2$.

(d) Sunil Chandran and Kavitha [16] have proved that

$$tw(G) = \Theta\left(\frac{n}{\sqrt{m}}\right).$$

As G has maximum degree $O(mk)$, the lower bound follows from Theorem 4.5. The upper bound $c_\infty(G) = O\left(n/\sqrt{m}\right)$ follows from the same theorem, and the bound $c_\infty(G) \leq n/k$ follows from part (a) of the present theorem, since G_1 is a complete graph and has $c_\infty(G_1) = 1$.

(e)

Claim. For any positive m, the m-dimensional hypercube $\mathcal{H}_m$ has domination number at most $2^{m+1}/(m+1)$.

Proof of Claim. If for some positive integer k, $m = 2^k - 1$, then it is well-known that $\mathcal{H}_m$ has domination number exactly $2^m/(m+1)$ (see [44] for example). Otherwise, let k be the largest integer with $2^k - 1 \leq m$. Thus $m < 2^{k+1} - 1$. It is easy to see that for every graph G with domination number r, the Cartesian product of G and an edge has domination number at most $2r$. Hence one can prove using induction that for $i \geq 2^k - 1$, the domination number of $\mathcal{H}_i$ is at most

$$\frac{2^{2^k-1}}{2^k} 2^{i-(2^k-1)} = 2^{i-k}.$$

In particular, the domination number of $\mathcal{H}_m$ is at most $2^{m-k} < \frac{2^{m+1}}{m+1}$. □

The upper bound follows from the above claim (recall that $n = 2^m$).

Sunil Chandran and Kavitha [16] have proved that $tw(\mathcal{H}_m) = \Theta(2^m/\sqrt{m})$. Since $\mathcal{H}_m$ has maximum degree m, the lower bound follows from Theorem 4.5. ■

Chapter 10

The Same-Speed Variation

In the concluding remarks of [27] a variation is proposed where the cops and the robber have the same speed. In this short chapter we prove that the cop number of a graph in this variation equals the cop number of a related graph in the usual setting.

Definition ($c_{a,b}(G)$, G_t)**.** Let a and b be positive integers. Let $c_{a,b}(G)$ denote the cop number of G when the robber has speed a and the cops have speed b. Note that in fact $c_s(G) = c_{s,1}(G)$. Let t be a positive integer, and let G_t be the graph with vertex set $V(G)$ with $u, v \in V(G_t)$ being adjacent if their distance in G is at most t.

Theorem 10.1. *For any graph G and any positive integer t we have*

$$c_{t,t}(G) = c_1(G_t).$$

Proof. Consider the usual game played in G_t with both players having speed one. Call this game the *original game*, and consider a game played in G with all players having speed t, and call this game the *alternative game*. The set of possible moves for each player is almost the same in the two games, the only difference is that there could be a possible move for the robber in the original game, which is not possible in the alternative game: if the robber is at u, and v is a vertex at distance at most t from u (in G), then she can always move from u to v in the original game, but, in the alternative game, all of the (u, v)-paths of length at most t may be blocked by a cop.

But, notice that if in some round, the robber moves from u to v in her turn, such that there is a (u, v)-path of length at most t in G with a cop standing at one of its internal vertices, then the robber will be captured in the next round. This is because that condition implies that the cop's vertex is at distance at most t from v (in G), hence he can capture the robber in the next round. We deduce that such a move results in an immediate capture in the original game, and the robber better not do it. Apart from that kind of move, which we saw does not really give an advantage to the robber, the set of moves for the players are the same in the two games, and the equality follows. ■

Chapter 11

Future Work

In this chapter we present a few open questions and research directions on this game.

1. It is known that $f_1(n) = \Omega(\sqrt{n})$. We proved that $f_s(n) = \Omega(n^{s/s+1})$ for every $s \in \mathbb{N}$ (see Theorem 3.3). Meyniel conjectured that $f_1(n) = \Theta(\sqrt{n})$. Generalizing this conjecture, we conjecture that

   $$f_s(n) = \Theta(n^{s/s+1})$$

   for every $s \in \mathbb{N}$. In other words, we conjecture that for any $s \in \mathbb{N}$, just $O(n^{s/s+1})$ cops are enough to capture a robber having speed s, in a connected graph with n vertices. This conjecture also appears in [39].

   This seems to be a difficult problem (even for the $s = 1$ case the best known asymptotic bound is $f_1(n) \leq n^{1-o(1)}$, which is far from the conjectured $O(\sqrt{n})$ bound). The best upper bound so far (for general s), given by Frieze et al. [27], is the following:

   $$\text{If } \alpha = 1 + s^{-1}, \text{ then } f_s(G) \leq n\alpha^{-(1-o(1))\sqrt{\log_\alpha n}}.$$

2. Fomin et al. [23] asked about the complexity of computing $c_\infty(G)$ when G is an interval graph. We proved that this problem is 3-approximable (see Theorem 5.5), but it is still not known if it is $\mathcal{NP}$-hard or not.

3. We proved that there exist chordal graphs G with $c_\infty(G) = \Omega(n/\log n)$ (see Theorem 6.2). Are there chordal graphs G with $c_\infty(G) = \Omega(n)$?

4. When $np = 20 \log n + \omega(1)$, in part (c) of Theorem 8.6, we have determined $c_\infty(G)$ for a random $G \in \mathcal{G}(n, p)$ up to an $O(\log(np))$ factor. Can one close this gap?

5. In part (b) of Theorem 9.1 we have determined c_∞ for the Cartesian product of m paths, up to an $O(m^2)$ factor. Can one close the gap? In part (e) of the same theorem, we have determined c_∞ for the m-dimensional hypercube (which can be considered as the Cartesian product of m paths of length one) up to an $O(\sqrt{m})$ factor. The same question can be asked here: what is the correct value?

6. We have proved several bounds for c_∞ in various classes of graphs. Can one find similar (perhaps weaker) bounds for c_s, s fixed or $s < n$, by using the same ideas?

7. Fomin et al. [23] proved that computing $c_s(G)$ is $\mathcal{NP}$-hard for every $s \in \mathbb{N} \cup \{\infty\}$. From their proof it follows that there is a constant $k > 0$ such that there is no polynomial time algorithm to approximate $c_s(G)$ within a multiplicative factor $k \log n$ unless $\mathcal{P} = \mathcal{NP}$. They ask whether there is an $O(n^{1-\epsilon})$-approximation algorithm for computing $c_s(G)$. They also ask whether for large s, say $s \geq \sqrt{n}$, the problem is in $\mathcal{NP}$. (Note that when a problem is $\mathcal{NP}$-hard, it is not necessarily in $\mathcal{NP}$.)

8. Let G be a planar graph. Aigner and Fromme [1] proved that $c_1(G) \leq 3$, and Fomin et al. [23] proved that $c_2(G)$ can be as large as $\Omega(\sqrt{\log n})$. As the $\sqrt{n} \times \sqrt{n}$ grid has treewidth $\sqrt{n}$ and maximum degree 4, from Theorem 4.5 it follows that $c_\infty(G)$ can be as large as $\Omega(\sqrt{n})$. Also, since every planar graph has treewidth $O(\sqrt{n})$ (see [24] for example), we have $c_\infty(G) = O(\sqrt{n})$. What can be said about $c_s(G)$ for other values of s? The authors of [23] ask if $c_s(G)$ can be computed in polynomial time for $s > 1$. This question is open even for grids.

9. Andreae [5] considered the game played in graphs for which a minor is excluded, and proved upper bounds for c_1 of such graphs. Joret et al. [35] considered the game played in graphs with forbidden (induced) subgraphs, and proved upper bounds for c_1 of such graphs. Frankl [25, 26] considered the game played in Cayley graphs and graphs with large girth, and proved lower and upper bounds for c_1 of such graphs. See Section 1.3.1 for more details. One may naturally ask if it is possible to obtain bounds for c_s of these classes of graphs, for larger values of s.

10. What happens if s is not a constant, but grows with n? When $s(n) = n$ then we get the infinitely fast robber version, but one may also consider functions growing slower than n.

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