Unitary t-designs

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- States are evolved by acting on them by matrices; i.e. $|\psi_{t+1}\rangle = M|\psi_t\rangle$
- However, we want the state to remain normalized. Thus, any M must be norm-preserving. In $\mathbb{C}^d$ this is the group $U(d)$ of d-by-d *unitary matrices*.

Preliminaries: $U(d)$

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- We can introduce the Haar measure and use it to integrate functions f of $U \in U(d)$ to find their averages:

$$\langle f \rangle = \int_{U(d)} f(U)\, dU.$$

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- The goal of unitary t-designs is to evaluate averages of polynomials via a finite sum.

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U & \mapsto & \dfrac{tr(U^* U)}{d} & \in Hom(1,1) \\
U, V & \mapsto & tr(U^* V) U^2 + VU^* VU & \in Hom(3,1) \\
U & \mapsto & \underbrace{tr(U^* V) U^2}_{Hom(2,1)} + \underbrace{VU^* VU}_{Hom(1,1)} & \notin Hom(2,1)
\end{array}
$$

Functional definition of unitary t-designs

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A function $w : X \to (0,1]$ is a weight function on X if for all $U \in X$ we have $w(U) > 0$ and $\sum_{U \in X} w(U) = 1$

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A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

$$\sum_{U\in X} w(U)f(U) = \int_{U(d)} f(U)\, dU$$

for all $f \in \mathrm{Hom}(t,t)$.

Functional definition is general enough

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Every t-design is a $(t-1)$-design.

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For any $f \in \mathrm{Hom}(r, s)$ with $r \neq s$

$$\int_{U(d)} f(U)\, dU = 0$$

Lemma

For any $f \in \mathrm{Hom}(r, s)$, $U \in U(d)$, and $c \in \mathbb{C}$ we have $f(cU) = c^r \bar{c}^s f(U)$

Strengths and shortcomings of the functional definition

Strengths:

- Average of any polynomial with degrees in U and U^* less than t can be evaluated one summand at a time.
- Multi-variable polynomials can be evaluated:

$$\int_{U(d)} \cdots \int f(U_1, ..., U_n) dU_1 ... dU_n$$

$$= \sum_{U_1 \in X} ... \sum_{U_n \in X} w(U_1) ... w(U_n) f(U_1, ..., U_n).$$

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Shortcomings:

- Not clear how to test if a given (X, w) is a t-design.
- If (X, w) is not a design, then how far away is it?

Tensor product definition of unitary t-designs

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A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

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- More tractable for checking if an arbitrary (X, w) is a t-design.
- Literature has explicit formula for the RHS for many choices of d and t [Col03, CS06].
- Still not metric.

Metric definition of unitary t-designs

Definition

A weight function w is a proper weight function on X if for all other choices of weight function w' on X, we have:

$$\sum_{U,V\in X} w(U)w(V)|tr(U^*V)|^{2t} \leq \sum_{U,V\in X} w'(U)w'(V)|tr(U^*V)|^{2t}.$$

The trace double sum is a function Σ defined for finite $X \subset U(d)$ as:

$$\Sigma(X) = \sum_{U,V\in X} w(U)w(V)|tr(U^*V)|^{2t},$$

Definition

A finite $X \subset U(d)$ is a unitary t-design if

$$\Sigma(X) = \langle |tr(U)|^{2t} \rangle$$

Strengths and shortcomings of the metric definition

Strengths:

- $\Sigma(X) > \langle |tr(U)|^{2t} \rangle$ if X is not a t-design. This gives us a useful metric to say how far a set with proper weight function is from being a design.

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- $\langle |tr(U)|^{2t} \rangle$ has a nice combinatorial interpertation: the number of permutations of $\{1, ..., t\}$ with no increasing subsequences of order greater than d [DS94, Rai98].
- If $d \geq t$ then RHS is $t!$.

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Shortcomings:

- Does not give any insight into what t-designs are useful for.

Characterization of minimal t-designs

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- Useful tool for proving minimality.
- Sadly, minimal designs are not necessarily minimum.
- Currently working on finding correspondences between minimal and minimum designs.

A lower bound on the size of t-designs

Proposition

If $X \subset U(d)$ is a t-design then $|X| \geq \frac{d^{2t}}{\langle |tr(U)|^{2t} \rangle}$.

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- Best known bounds are by Roy and Scott [RS08]: $|X| \geq \binom{d^2+t-1}{t}$
- Asymptotically, for large d and fixed t, both bounds are $\Theta(d^{2t})$

1-design construction

- Let $|e_1\rangle ... |e_d\rangle$ be an orthonormal basis of $\mathbb{C}^d$ that is mutually unbiased with the standard basis.
- Define $I_i = \sqrt{d}\mathrm{diag}(|e_i\rangle)$ for $1 \leq i \leq d$.

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- Consider the cyclic permutation group of order d, represented as d-by-d matrices: $C^1 ... C^d$ where $C^d = C^0 = I$.
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For any tuple $1 \leq i, j, m, n \leq d$ we have:

$$tr((C_i^m)^* C_j^n) = tr(I_i^* C^{d-m+n} I_j) = \begin{cases} d & \text{if } i = j \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

Evaluating the average commutator over $U(d)$

Theorem

For any $V \in U(d)$ and $[U, V] = U^* V^* U V$ we have:

$$\langle [\,\cdot\,, V] \rangle = \frac{tr(V^*)}{d} V$$

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^*DP$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^*V^*UV\ dU = \Big[\int_{U(d)} U^*V^*U\ dU\Big]V = \Big[\int_{U(d)} U^*P^*DPU\ dU\Big]V$$

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But we know a symmetry that allows substituting $PU \to U$ without changing the average.

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- Let $f(U) = U^* D U$.

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- Let $f(U) = U^* D U$.
- Look at the elements of the design: $f(C_i^m) = I_i^* (C^m)^* D C^m I_i$.

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- $(C^m)^* D C^m = \mathrm{diag}(\lambda_{c^m(1)}, ..., \lambda_{c^m(d)})$

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- Let $f(U) = U^*DU$.
- Look at the elements of the design: $f(C_i^m) = I_i^*(C^m)^*DC^mI_i$.
- $(C^m)^*DC^m = \mathrm{diag}(\lambda_{c^m(1)}, ..., \lambda_{c^m(d)})$

Thus, $\langle f \rangle = \frac{\lambda_1 + ... + \lambda_d}{d} I$ $\square$

t-designs are non-commuting

Theorem

For all $d \geq 2$ if $X \subset U(d)$ is a minimal t-design then X has a trivial center.

Supports our intuition that designs must be well 'spread out'.

Concluding remarks

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- Used an orthonormal basis of $\mathbb{C}^{d\times d}$ as a 1-design.
- Evaluated the average commutator on $U(d)$: $\langle[\,\cdot\,, V]\rangle = \frac{tr(V^*)}{d}V$
- Showed that t-designs are non-commuting

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Thank you for listening!

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