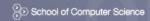
## Unitary t-designs

Artem Kaznatcheev

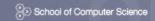
McGill University

February 13, 2010

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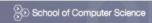


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- ▶ States are evolved by acting on them by matrices; i.e.  $|\psi_{t+1}\rangle = M|\psi_t\rangle$
- ▶ However, we want the state to remain normalized. Thus, any M must be norm-preserving. In  $\mathbb{C}^d$  this is the group U(d) of d-by-d unitary matrices.



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$$\langle f \rangle = \int_{U(d)} f(U) \ dU.$$

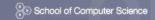
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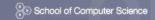
$$\langle f \rangle = \int_{U(d)} f(U) \ dU.$$

- For convenience we normalize integration by assuming that  $\int_{U(d)} dU = 1$ .
- ► The goal of unitary *t*-designs is to evaluate averages of polynomials via a finite sum.



#### Definition

 $\operatorname{Hom}(r,s)$  is the set of polynomials homogeneous of degree r in entries of  $U \in U(d)$  and homogeneous of degree s in  $U^*$ .



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$$\begin{array}{ccccc} U,V & \mapsto & U^*V^*UV & & \in \textit{Hom}(2,2) \\ & U & \mapsto & U^*V^*UV & & \in \textit{Hom}(1,1) \\ & U & \mapsto & \frac{\textit{tr}(U^*U)}{\textit{d}} & & \in \textit{Hom}(1,1) \end{array}$$

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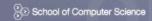
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### Examples

# Functional definition of unitary t-designs

### Definition

A function  $w: X \to (0,1]$  is a weight function on X if for all  $U \in X$  we have w(U) > 0 and  $\sum_{U \in X} w(U) = 1$ 



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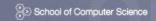
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$$\sum_{U \in X} w(U)f(U) = \int_{U(d)} f(U) \ dU$$

for all  $f \in \text{Hom}(t, t)$ .



# Functional definition is general enough

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### Proposition

For any  $f \in \text{Hom}(r, s)$  with  $r \neq s$ 

$$\int_{U(d)} f(U) \ dU = 0$$

#### Lemma

For any  $f\in \mathrm{Hom}(r,s),\ U\in U(d)$ , and  $c\in \mathbb{C}$  we have  $f(cU)=c^rar{c}^sf(U)$ 



# Strengths and shortcomings of the functional definition

### Strengths:

- Average of any polynomial with degrees in U and  $U^*$  less than t can be evaluated one summand at a time.
- ► Multi-variable polynomials can be evaluated:

$$\int_{U(d)} \cdots \int f(U_1, ..., U_n) dU_1 ... dU_n$$

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### Shortcomings:

- ▶ Not clear how to test if a given (X, w) is a t-design.
- ▶ If (X, w) is not a design, then how far away is it?



## Tensor product definition of unitary *t*-designs

#### Definition

A tuple (X,w) with finite  $X \subset U(d)$  and weight function w on X is a unitary t-design if

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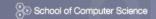
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- ▶ More tractable for checking if an arbitrary (X, w) is a t-design.
- ▶ Literature has explicit formula for the RHS for many choices of *d* and *t* [Col03, CS06].
- Still not metric.



## Metric definition of unitary *t*-designs

#### Definition

A weight function w is a proper weight function on X if for all other choices of weight function w' on X, we have:

$$\sum_{U,V \in X} w(U) w(V) |tr(U^*V)|^{2t} \leq \sum_{U,V \in X} w'(U) w'(V) |tr(U^*V)|^{2t}.$$

The trace double sum is a function  $\Sigma$  defined for finite  $X \subset U(d)$  as:

$$\Sigma(X) = \sum_{U,V \in X} w(U)w(V)|tr(U^*V)|^{2t},$$

#### Definition

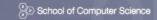
A finite  $X \subset U(d)$  is a unitary *t*-design if

$$\Sigma(X) = \langle |tr(U)|^{2t} \rangle$$



### Strengths:

▶  $\Sigma(X) > \langle |tr(U)|^{2t} \rangle$  if X is not a t-design. This gives us a useful metric to say how far a set with proper weight function is from being a design.



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- $\langle |tr(U)|^{2t} \rangle$  has a nice combinatorial interpertation: the number of permutations of  $\{1,...,t\}$  with no increasing subsequences of order greater than d [DS94, Rai98].
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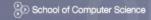
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- ▶ One of the easiest way to test if *X* is a *t*-design

### Shortcomings:

▶ Does not give any insight into what *t*-designs are useful for.



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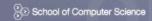
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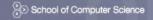
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- Useful tool for proving minimality.
- Sadly, minimal designs are not necessarily minimum.
- Currently working on finding correspondences between minimal and minimum designs.



# A lower bound on the size of *t*-designs

### Proposition

If 
$$X \subset U(d)$$
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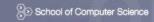
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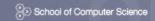
- ▶ Best known bounds are by Roy and Scott [RS08]:  $|X| \ge {d^2 + t 1 \choose t}$
- Asymptotically, for large d and fixed t, both bounds are  $\Theta(d^{2t})$

## 1-design construction

- ▶ Let  $|e_1\rangle...|e_d\rangle$  be an orthonormal basis of  $\mathbb{C}^d$  that is mutually unbiased with the standard basis.
- ▶ Define  $I_i = \sqrt{d} \operatorname{diag}(|e_i\rangle)$  for  $1 \le i \le d$ .



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- ▶ Define  $C_i^m = C^m I_i$

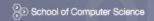


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For any tuple  $1 \le i, j, m, n \le d$  we have:

$$tr((C_i^m)^*C_j^n)=tr(I_i^*C^{d-m+n}I_j)=\begin{cases} d & \text{if } i=j \text{ and } m=n\\ 0 & \text{otherwise} \end{cases}$$



# Evaluating the average commutator over U(d)

#### Theorem

For any  $V \in U(d)$  and  $[U, V] = U^*V^*UV$  we have:

$$\langle [\,\cdot\,,V] \rangle = \frac{tr(V^*)}{d}V$$



Consider the diagonalization of  $V^*$ , i.e.  $V^* = P^*DP$ , with  $D = \operatorname{diag}(\lambda_1, ..., \lambda_d).$ 

$$\int_{U(d)} U^*V^*UV \ dU = \Big[\int_{U(d)} U^*V^*U \ dU\Big]V = \Big[\int_{U(d)} U^*P^*DPU \ dU\Big]V$$



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Artem Kaznatcheev (McGill University)

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Thus, 
$$\langle f \rangle = \frac{\lambda_1 + ... + \lambda_d}{d} I$$



## t-designs are non-commuting

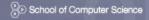
#### **Theorem**

For all  $d \ge 2$  if  $X \subset U(d)$  is a minimal t-design then X has a trivial center.

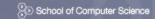
Supports our intuition that designs must be well 'spread out'.



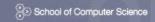
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- ► Classified minimal designs: a *t*-design is minimal if and only if it has a unique proper weight function.
- ▶ Used an orthonormal basis of  $\mathbb{C}^{d \times d}$  as a 1-design.
- ▶ Evaluated the average commutator on U(d):  $\langle [\cdot, V] \rangle = \frac{tr(V^*)}{d}V$
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Thank you for listening!



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