Introduction to unitary $t$-designs

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March 25, 2010
Outline

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Trace double sum inequality

Symmetries and minimal designs

1-designs

Structure of designs

Conclusion
Outline

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Trace double sum inequality

Symmetries and minimal designs

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Structure of designs

Conclusion
Preliminaries: $U(d)$

- $U(d)$ is the topologically compact and connected group of norm preserving (unitary) operators on $\mathbb{C}^d$. 

$\langle f \rangle = \int_{U(d)} f(U) \, dU$. 

For convenience we normalize integration by assuming that $\int_{U(d)} dU = 1$. 

The goal of unitary $t$-designs is to evaluate averages of polynomials via a finite sum.
Preliminaries: $U(d)$

- $U(d)$ is the topologically compact and connected group of norm preserving (unitary) operators on $\mathbb{C}^d$.
- We can introduce the Haar measure and use it to integrate functions $f$ of $U \in U(d)$ to find their averages:

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Preliminaries: $\text{Hom}(r, s)$

**Definition**

$\text{Hom}(r, s)$ is the set of polynomials homogeneous of degree $r$ in entries of $U \in U(d)$ and homogeneous of degree $s$ in $U^*$.
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**Examples**

\[
\begin{align*}
U, V & \mapsto U^* V^* UV \quad \in \text{Hom}(2, 2) \\
U & \mapsto U^* V^* UV \quad \in \text{Hom}(1, 1)
\end{align*}
\]
Preliminaries: $\text{Hom}(r, s)$

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**Examples**

- $U, V \mapsto U^* V^* U V \in \text{Hom}(2, 2)$
- $U \mapsto U^* V^* U V \in \text{Hom}(1, 1)$
- $U \mapsto \frac{\text{tr}(U^* U)}{d} \in \text{Hom}(1, 1)$
Preliminaries: $\text{Hom}(r, s)$

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**Examples**

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Functional definition of unitary $t$-designs

**Definition**

A function $w : X \rightarrow (0, 1]$ is a weight function on $X$ if for all $U \in X$ we have $w(U) > 0$ and $\sum_{U \in X} w(U) = 1$.
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**Definition**

A tuple $(X,w)$ with finite $X \subset U(d)$ and weight function $w$ on $X$ is a **unitary $t$-design** if

$$\sum_{U \in X} w(U)f(U) = \int_{U(d)} f(U) \, dU$$

for all $f \in \text{Hom}(t,t)$.
# Functional definition of unitary $t$-designs

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for all $f \in \text{Hom}(t, t)$.

## Definition

A finite $X \subset U(d)$ is an **unweighted $t$-design** if it is a unitary $t$-design with a uniform weight function (i.e. $w(U) = \frac{1}{|X|}$ for all $U \in X$).
Functional definition is general enough

Proposition

*Every* \( t \)-design *is a* \( (t - 1) \)-design. \( \Box \)
Functional definition is general enough

Proposition

Every $t$-design is a $(t-1)$-design.

Proposition

For any $f \in \text{Hom}(r, s)$ with $r \neq s$

$$\int_{U(d)} f(U) \, dU = 0$$

Lemma

For any $f \in \text{Hom}(r, s)$, $U \in U(d)$, and $c \in \mathbb{C}$ we have $f(cU) = c^r \bar{c}^s f(U)$
Strengths and shortcomings of the functional definition

Strengths:

- Average of any polynomial with degrees in $U$ and $U^*$ less than $t$ can be evaluated one summand at a time.
- Multi-variable polynomials can be evaluated:

$$\int \cdots \int \frac{f(U_1, \ldots, U_n)}{U(d)} dU_1 \cdots dU_n$$

$$= \sum_{U_1 \in X} \cdots \sum_{U_n \in X} w(U_1) \cdots w(U_n) f(U_1, \ldots, U_n).$$
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$$

Shortcomings:

- Not clear how to test if a given $(X, \omega)$ is a $t$-design.
- If $(X, \omega)$ is not a design, then how far away is it?
Tensor product definition of unitary $t$-designs

Definition

A tuple $(X,w)$ with finite $X \subset U(d)$ and weight function $w$ on $X$ is a unitary $t$-design if

$$\sum_{U \in X} w(U) U^\otimes t \otimes (U^*)^\otimes t = \int_{U(d)} U^\otimes t \otimes (U^*)^\otimes t \, dU$$

▶ More tractable for checking if an arbitrary $(X,w)$ is a $t$-design.
▶ Literature has explicit formula for the RHS for many choices of $d$ and $t$ [Col03, CS06].
▶ Still not metric.
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\( \epsilon \)-approximate unitary \( t \)-designs

**Definition**

A tuple \((X, w)\) with finite \( X \subset U(d) \) and weight function \( w \) on \( X \) is an \( \epsilon \)-approximate unitary \( t \)-design if

\[
\| \sum_{U \in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t} - \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU \| < \epsilon
\]
ε-approximate unitary $t$-designs

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A tuple $(X, w)$ with finite $X \subset U(d)$ and weight function $w$ on $X$ is an $\epsilon$-approximate unitary $t$-design if

$$\left\| \sum_{U \in X} w(U) U \otimes^t (U^*) \otimes^t - \int_{U(d)} U \otimes^t (U^*) \otimes^t dU \right\| < \epsilon$$

- A glaring omission is a specification of which norm to use in the definition.
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- There are many choices of operator norms, important ones in QIT are Schatten norms. In particular the trace, Frobenius, and spectral norms.
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- A glaring omission is a specification of which norm to use in the definition.
- There are many choices of operator norms, important ones in QIT are Schatten norms. In particular the trace, Frobenius, and spectral norms.
- By modifying the definition slightly, we can also study super-operator norms. In particular, the diamond norm (most useful from a cryptographic and experimental point of view).
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Trace double sum inequality

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The trace double sum inequality

Theorem

A tuple \((X, w)\) is an \(\epsilon\)-approximate unitary \(t\)-design (with respect to the Frobenius norm) if and only if

\[
\sum_{U, V \in X} w(U)w(V)|\text{tr}(U^* V)|^{2t} - \int_{U(d)} |\text{tr}(U)|^{2t} \, dU \leq \epsilon^2
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- Proved earlier in the non-approximate case by Scott [Sco08].
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- Proved earlier in the non-approximate case by Scott [Sco08].
- The integral is the number of permutations of \(\{1, \ldots, t\}\) with no increasing subsequences of order greater than \(d\) [DS94, Rai98]. We will call this number \(\sigma\).
- If \(d \geq t\) then \(\sigma\) is \(t!\).
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- If \(d \geq t\) then \(\sigma\) is \(t!\).
- Limitation: no one really cares about the Frobenius norm. -_-
Metric definition of unitary $t$-designs

**Definition**

A weight function $w$ is an optimal weight function on $X$ if for all other choices of weight function $w'$ on $X$, we have:

$$
\sum_{U,V \in X} w(U)w(V)|\text{tr}(U^*V)|^{2t} \leq \sum_{U,V \in X} w'(U)w'(V)|\text{tr}(U^*V)|^{2t}.
$$

The trace double sum is a function $\Sigma$ defined for finite $X \subset U(d)$ as:

$$
\Sigma(X) = \sum_{U,V \in X} w(U)w(V)|\text{tr}(U^*V)|^{2t},
$$

**Definition**

A finite $X \subset U(d)$ is a unitary $t$-design if

$$
\Sigma(X) = \langle |\text{tr}(U)|^{2t} \rangle
$$
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Four symmetries of $t$-designs

Proposition

If $X = \{U_1, \ldots, U_n\}$ is a $t$-design then $Y = \{e^{i\phi_1}U_1, \ldots, e^{i\phi_n}U_n\}$ is also a $t$-design for all $\phi_1, \ldots, \phi_n \in [0, 2\pi]$. 
Four symmetries of \( t \)-designs

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If $X$ is a $t$-design then $X^* = \{U^* : U \in X\}$ is also a $t$-design.

**Proposition**

If $X \subset U(d)$ is a $t$-design then $\forall M \in U(d)$, $MX = \{MU : U \in X\}$ and $XM = \{UM : U \in X\}$ are also a $t$-design.
Minimal designs

Lemma

If \( X, Y \) are two \( t \)-designs then so is \( X \cup Y \).

- Designs can be arbitrarily large
Minimal designs

Lemma

If $X, Y$ are two $t$-designs then so is $X \cup Y$.

- Designs can be arbitrarily large
- We are interested in smaller designs

Definition

A minimal (unweighted) $t$-design $X$ is a $t$-design such that all $Y \subset X$ are not (unweighted) $t$-designs.
Characterization of minimal $t$-designs

**Theorem**

A $t$-design $X$ is minimal if and only if it has a unique optimal weight function $w$. 
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- Useful tool for proving minimality.
Characterization of minimal $t$-designs

Theorem

A $t$-design $X$ is minimal if and only if it has a unique optimal weight function $w$.

- Useful tool for proving minimality.
- Sadly, minimal designs are not necessarily minimum.
- Still working on finding correspondences between minimal and minimum designs.
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Orthonormal bases for $\mathbb{C}^{d \times d}$

Goal: find an orthonormal basis $|E_1\rangle, \ldots, |E_{d^2}\rangle$ of $\mathbb{C}^{d \times d}$ such that each $E_i \in U(d)$
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**Definition**

$X \subset U(d)$ is **pairwise traceless** if for every $U, V \in X$ with $U \neq V$ we have $\text{tr}(U^* V) = 0$.

A pairwise traceless $X \subset U(d)$ is **maximum** pairwise traceless if $|X| = d^2$.

Orthonormal bases of unitaries for $\mathbb{C}^{d \times d}$ are maximum pairwise traceless sets.
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**Proposition**

*For any $X \subset U(d)$, $X$ is maximum pairwise traceless if and only if $X$ is a minimum unweighted 1-design.*
Very brief introduction to MUBs

Definition

Two orthonormal bases \( \{|e_i\rangle : 1 \leq i \leq d\} \) and \( \{|e'_i\rangle : 1 \leq i \leq d\} \) of \( \mathbb{C}^d \) are \textbf{mutually unbiased} if \( |\langle e_i | e'_j \rangle|^2 = \frac{1}{d} \) for all \( 1 \leq i, j \leq d \).
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- Open question: determine the maximum number \( M(d) \) of pairwise mutually unbiased bases for \( \mathbb{C}^d \).
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- Open question: determine the maximum number \( M(d) \) of pairwise mutually unbiased bases for \( \mathbb{C}^d \).
- If we write the prime decomposition of \( d = p_1^{n_1} \ldots p_k^{n_k} \) such that \( p_i^{n_i} \leq p_{i+1}^{n_{i+1}} \) then \( p_1^{n_1} \leq M(d) \leq d + 1 \).
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Important features for us:

- \( M(d) \geq 2 \) for \( d \geq 1 \).
- Without loss of generality, can assume one of the bases to be the standard basis.

Example

\[
\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}
\]
Maximum pairwise traceless set construction

- Let $|e_1\rangle \ldots |e_d\rangle$ be an orthonormal basis of $\mathbb{C}^d$ that is mutually unbiased with the standard basis.
- Define $I_i = \sqrt{d} \text{diag}(|e_i\rangle)$ for $1 \leq i \leq d$. 


Maximum pairwise traceless set construction

- Let $|e_1\rangle...|e_d\rangle$ be an orthonormal basis of $\mathbb{C}^d$ that is mutually unbiased with the standard basis.
- Define $I_i = \sqrt{d} \text{diag}(|e_i\rangle)$ for $1 \leq i \leq d$.
- Consider the cyclic permutation group of order $d$, represented as $d$-by-$d$ matrices: $C^1...C^d$ where $C^d = C^0 = I$.
- Define $C_i^m = C^m I_i$
Maximum pairwise traceless set construction

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- Define $C_i^m = C^m I_i$

For any tuple $1 \leq i, j, m, n \leq d$ we have:

$$\text{tr}((C_i^m)^* C_j^n) = \text{tr}(I_i^* C^{d-m+n} I_j) = \begin{cases} d & \text{if } i = j \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$
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The center of $t$-designs is trivial

**Lemma**

For any $V \in U(d)$ and $[U, V] = U^*V^*UV$ we have:

$$\langle [\cdot, V] \rangle = \frac{\text{tr}(V^*)}{d} V$$
The center of $t$-designs is trivial

Lemma

For any $V \in U(d)$ and $[U, V] = U^*V^*UV$ we have:

$$\langle [\cdot, V] \rangle = \frac{\text{tr}(V^*)}{d} V$$

Proposition

If $X \subset U(d)$ is a minimal $t$-design then there is at most one element that commutes with all elements of $X$. In other words, $Z(X)$ is trivial.
Some other structural observations

Proposition

Every $t$-design of dimension $d$ spans $\mathbb{C}^{d \times d}$.
Some other structural observations

Proposition

**Every t-design of dimension d spans** $\mathbb{C}^{d \times d}$.

A group $t$-design is a unitary $t$-design that also happens to have group structure. Group designs were defined by Gross, Audenaert, and Eisert [GAE07], and all known constructions are via group designs.
Some other structural observations

Proposition

Every $t$-design of dimension $d$ spans $\mathbb{C}^{d \times d}$.

A group $t$-design is a unitary $t$-design that also happens to have group structure. Group designs were defined by Gross, Audenaert, and Eisert [GAE07], and all known constructions are via group designs.

Proposition

Every unitary irreducible representation of a finite group is a group $1$-design and vice versa.
A simple lower bound on the size of $t$-designs

Proposition

If $X \subset U(d)$ is a $t$-design then $|X| \geq \frac{d^{2t}}{\sigma}$.
A simple lower bound on the size of $t$-designs

**Proposition**

If $X \subset U(d)$ is a $t$-design then $|X| \geq \frac{d^{2t}}{\sigma}$.

- Best known bounds are by Roy and Scott [RS08]: $|X| \geq \binom{d^2 + t - 1}{t}$
- Asymptotically, for large $d$ and fixed $t$, both bounds are $\Theta(d^{2t})$
A simple lower bound on the size of $t$-designs

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- Best known bounds are by Roy and Scott [RS08]: $|X| \geq \binom{d^2 + t - 1}{t}$
- Asymptotically, for large $d$ and fixed $t$, both bounds are $\Theta(d^{2t})$
- By taking note of some structural observations, we can do a little better:

**Proposition**

If $X \subset U(d)$ is a $t$-design then $|X| \geq \frac{d^{2t}}{\sigma} + \frac{1}{2d^t} \left( \frac{\sigma}{2d^{2t}} \right)^2 (t-1)$.
Conjecture

If $X$ is a unitary $t$-design with $t \geq 2$, then for any $W \in X$ there exists some $Y \subset X - \{W\}$ such that $Y$ is a $t - 1$-design.
Conjecture

If $X$ is a unitary $t$-design with $t \geq 2$, then for any $W \in X$ there exists some $Y \subset X - \{W\}$ such that $Y$ is a $t - 1$-design.

If true, this conjecture can significantly improve our lower bounds:

Theorem

If $(X \subset U(d), w)$ is a unitary $t$-design and the conjecture is true, then:

$$|X| \geq \frac{d^{2t}}{\sigma_t} \left( 1 + 2 \frac{\sigma_t}{d^{2t}} \sigma^{\frac{t}{t-1}} \right)$$
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Concluding remarks

- Introduces 3 definitions of unitary $t$-designs and one for approximate ones.
- Showed the trace double sum inequality: $\Sigma(X) - \langle |tr(U)|^{2t} \rangle < \epsilon^2$ with equality if and if $X$ is a $\epsilon$ approximate $t$-design with respect to the Frobenius norm.
Concluding remarks

- Introduces 3 definitions of unitary $t$-designs and one for approximate ones.
- Showed the trace double sum inequality: $\Sigma(X) - \langle |\text{tr}(U)|^{2t} \rangle < \epsilon^2$ with equality if and if $X$ is a $\epsilon$ approximate $t$-design with respect to the Frobenius norm.
- Used an orthonormal basis of $\mathbb{C}^{d \times d}$ as a 1-design.
- Evaluated the average commutator on $U(d)$: $\langle [\cdot, V] \rangle = \frac{\text{tr}(V^*)}{d} V$
- Showed that $t$-designs are non-commuting
Concluding remarks

- Introduces 3 definitions of unitary $t$-designs and one for approximate ones.
- Showed the trace double sum inequality: $\Sigma(X) - \langle |tr(U)|^2 \rangle < \epsilon^2$ with equality if and if $X$ is a $\epsilon$ approximate $t$-design with respect to the Frobenius norm.
- Used an orthonormal basis of $\mathbb{C}^{d \times d}$ as a 1-design.
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