Calculus, Combinatorics, and (Quantum) Computation

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- However, we want the state to remain normalized. Thus, any M must be norm-preserving. In $\mathbb{C}^d$ this is the set of d-by-d *unitary matrices*.

Important features of unitary matrices

- The identity matrix I is unitary.
- Each unitary U has an inverse $U^{-1} = U^*$, where $(U_{ij})^* = \overline{U_{ji}}$.
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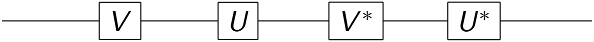
- Using the Haar measure we can define a calculus on the unitaries. In particular, we can perform integration.
- For convenience we normalize the Haar measure so that $\int_{U(d)} dU = 1$

Representing transformations

- A series of transformations can be represented visually as a circuit



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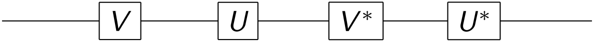

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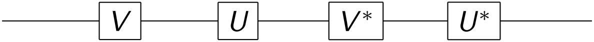

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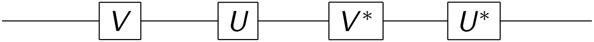

- Or, as an equation: $U^* V^* UV$
- In particular, many transformations we are interested in studying can be represented as *polynomials* in U.
- We will let $Hom(r, s)$ denote polynomials homogeneous of degree r in entries of $U \in U(d)$ and homogeneous of degree s in the entries of U^*.

Representing transformations

Examples

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\begin{aligned}
U, V &\mapsto U^* V^* U V &\in Hom(2,2) \\
U &\mapsto U^* V^* U V &\in Hom(1,1)
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$$U, V \mapsto U^* V^* U V \in Hom(2,2)$$

$$U \mapsto U^* V^* U V \in Hom(1,1)$$

$$U \mapsto \frac{tr(U^* U)}{d} \in Hom(1,1)$$

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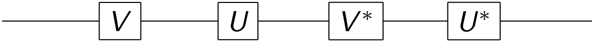
U & \mapsto & U^* V^* U V & \in Hom(1,1) \\
U & \mapsto & \dfrac{tr(U^* U)}{d} & \in Hom(1,1) \\
U, V & \mapsto & tr(U^* V) U^2 + V U^* V U & \in Hom(3,1) \\
U & \mapsto & \underbrace{tr(U^* V) U^2}_{Hom(2,1)} + \underbrace{V U^* V U}_{Hom(1,1)} & \notin Hom(2,1)
\end{array}
$$

A toy theoretical motivation

- Consider the commutator circuit we saw earlier:



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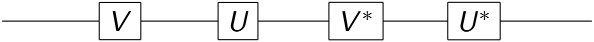

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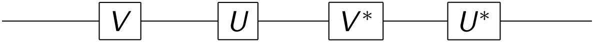

- What if you select U, but somebody else gets to choose V. Can we predict how an average transformation will behave?
- To find the average, we can integrate over $U(d)$.

$$\langle U^* V^* UV \rangle = \int_{U(d)} U^* V^* UV \; dV = \int_{U(d)} V^* UV \; dV$$

The difficulty is evaluating such integrals for arbitrary d.

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A toy 'experimental' motivation

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- Someone gives us a black box, one where we choose a unitary U and then we can observe some unknown but fixed polynomial $f(U)$.
- To find the average transformation, we would need to once again integrate $\int_{U(d)} f(U)dU$ - impossible with a finite number of experiments. It would be nice to replace the integral with something finite.

Unitary t-designs

Definition

A tuple (X,w) with finite $X \subset U(d)$ and $w : X \to \mathbb{R}$ is called a unitary t-design if

$$\sum_{U \in X} w(U) f(U) = \int_{U(d)} f(U)\, dU \qquad (1)$$

for all $f \in Hom(t,t)$, $\forall U \in X$ $w(U) > 0$, and $\sum_{U \in X} w(U) = 1$.

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Equivalently, we can replace eq. 1 by:

$$\sum_{U \in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t} = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU \qquad (2)$$

To avoid references to an arbitrary $f \in Hom(t,t)$.

Checking if X is a t-design

Theorem

For all finite $X \subset U(d)$ and weight functions $w : X \to \mathbb{R}$ we have:

$$\sum_{U,V \in X} w(U)w(V)|tr(U^* V)|^{2t} \geq \int_{U(d)} |tr(U)|^{2t} dU \qquad (3)$$

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Conveniently, the RHS of ineq. 3 has a relatively simple combinatorial interpertation and does not need to be evaluated through integration.

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- $X_M = \{V | U_kV = M\}$ for any $M \in U(d)$

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Lemma

If X, Y are two t-designs then so is $X \cup Y$.

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Important current questions:

- Are minimal designs the same size as minimum ones?
- If not, is there a max size for a minimal design?

Miscellaneous results

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- $|X| \in \Omega(d^{2t})$
- There is some $U, V \in X$ such that $U^*V^*UV \neq I$

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- Use commuting classes to construct small designs