### Properties of unitary *t*-designs

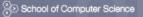
Artem Kaznatcheev

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October 7, 2009



Properties of unitary *t*-designs





#### Introduction

Trace double sum inequality

Symmetries and minimal designs

Greedy algorithms

Ouroboros application

#### Conclusion



Properties of unitary *t*-designs

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# Outline

#### Introduction

Trace double sum inequality

Symmetries and minimal designs

Greedy algorithms

**Ouroboros** application

#### Conclusion



Properties of unitary *t*-designs



# Preliminaries: U(d)

► U(d) is the topologically compact and connected group of norm preserving (unitary) operators on C<sup>d</sup>.



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# Preliminaries: U(d)

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- ► U(d) is the topologically compact and connected group of norm preserving (unitary) operators on C<sup>d</sup>.
- We can introduce the Haar measure and use it to integrate functions f of U ∈ U(d) to find their averages:

$$\langle f \rangle = \int_{U(d)} f(U) \, dU.$$

For convenience we normalize integration by assuming that  $\int_{U(d)} dU = 1.$ 

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$$\langle f \rangle = \int_{U(d)} f(U) \ dU.$$

- ► For convenience we normalize integration by assuming that  $\int_{U(d)} dU = 1.$
- The goal of unitary t-designs is to evaluate averages of polynomials via a finite sum.

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Definition

Hom(r, s) is the set of polynomials homogeneous of degree r in entries of  $U \in U(d)$  and homogeneous of degree s in  $U^*$ .



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#### Examples

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### Examples

U,V ⊦	$\rightarrow U^*V^*UV$	$\in$ Hom(2,2)
U ⊦		$\in \mathit{Hom}(1,1)$
U ⊦	$\rightarrow \frac{tr(U^*U)}{d}$	$\in \mathit{Hom}(1,1)$
U,V ⊦	$\rightarrow$ tr(U <sup>*</sup> V)U <sup>2</sup> + VU <sup>*</sup> VU	$\forall \in \mathit{Hom}(3,1)$
U ⊦	$\rightarrow \underbrace{tr(U^*V)U^2}_{} + \underbrace{VU^*VU}_{}$	$\notin$ Hom(2,1)
	Hom(2,1) Hom(1,1)	

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### Functional definition of unitary *t*-designs

#### Definition

A function  $w : X \to (0, 1]$  is a weight function on X if for all  $U \in X$  we have w(U) > 0 and  $\sum_{U \in X} w(U) = 1$ 

# Functional definition of unitary t-designs

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#### Definition

A tuple (X,w) with finite  $X \subset U(d)$  and weight function w on X is a unitary *t*-design if

$$\sum_{U \in X} w(U)f(U) = \int_{U(d)} f(U) \, dU$$

for all  $f \in Hom(t, t)$ .

## Functional definition of unitary *t*-designs

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#### Definition

A finite  $X \subset U(d)$  is an unweighted *t*-design if it is a unitary *t*-design with a uniform weight function (i.e.  $w(U) = \frac{1}{|X|}$  for all  $U \in X$ ).

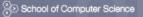
### Functional definition is general enough

Proposition

Every t-design is a (t-1)-design.



Properties of unitary *t*-designs



### Functional definition is general enough

Proposition

Every t-design is a (t-1)-design.

Proposition

```
For any f \in Hom(r, s) with r \neq s
```

 $\int_{U(d)} f(U) \, dU = 0$ 

#### Lemma

For any  $f \in \text{Hom}(r, s)$ ,  $U \in U(d)$ , and  $c \in \mathbb{C}$  we have  $f(cU) = c^r \bar{c}^s f(U)$ 

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# Strengths and shortcomings of the functional definition

Strengths:

- ► Average of any polynomial with degrees in U and U\* less than t can be evaluated one summand at a time.
- Multi-variable polynomials can be evaluated:

$$\int \cdots \int f(U_1, ..., U_n) dU_1 ... dU_n$$
$$= \sum_{U_1 \in X} ... \sum_{U_n \in X} w(U_1) ... w(U_n) f(U_1, ..., U_n).$$

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Shortcomings:

- Not clear how to test if a given (X, w) is a *t*-design.
- If (X, w) is not a design, then how far away is it?

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# Tensor product definition of unitary *t*-designs

### Definition

A tuple (X,w) with finite  $X \subset U(d)$  and weight function w on X is a unitary *t*-design if

$$\sum_{U\in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t} = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU$$

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## Tensor product definition of unitary t-designs

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- More tractable for checking if an arbitrary (X, w) is a *t*-design.
- Literature has explicit formula for the RHS for many choices of d and t [Col03, CS06].
- Still not metric.

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#### Introduction

#### Trace double sum inequality

Symmetries and minimal designs

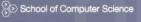
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### Trace double sum inequality

Theorem

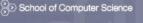
For all finite  $X \subset U(d)$  we have

$$\sum_{U,V \in X} w(U)w(V) |tr(U^*V)|^{2t} \ge \int_{U(d)} |tr(U)|^{2t} \ dU$$

With equality if and only if X is a t-design.



Properties of unitary *t*-designs



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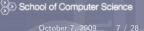
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Proved earlier by Scott [Sco08].



Properties of unitary *t*-designs



### Trace double sum inequality

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With equality if and only if X is a t-design.

Proved earlier by Scott [Sco08].

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RHS is the number of permutations of {1, ..., t} with no increasing subsequences of order greater than d [DS94, Rai98].

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- If  $d \ge t$  then RHS is t!.
- We will call the RHS  $\sigma$ .

Properties of unitary t-de

Consider an arbitrary finite  $X \subset U(d)$  with a weight function w, define matrices S and  $\Sigma$  as:

$$S = \sum_{U \in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t}$$
$$\Sigma = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU$$

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$$\Sigma = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU$$

Consider the matrix  $D = S - \Sigma$ :

$$tr(D^*D) = tr((S^* - \Sigma^*)(S - \Sigma))$$
  
=  $tr(S^*S) - tr(\Sigma^*S) - tr(S^*\Sigma) + tr(\Sigma^*\Sigma)$ 

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Trace is linear, thus can be brought past the integrals, summations and weights.



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Trace is linear, thus can be brought past the integrals, summations and weights.

$$tr((U^{\otimes t} \otimes (U^*)^{\otimes t})^* (V^{\otimes t} \otimes (V^*)^{\otimes t}))$$
  
=  $tr((U^*V)^{\otimes t} \otimes (UV^*)^{\otimes t})$   
=  $tr(U^*V)^t tr(UV^*)^t$   
=  $|tr(U^*V)|^{2t}$ 

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Consider the fourth summand  $tr(\Sigma^*\Sigma)$ :

$$tr(\Sigma^*\Sigma) = \int_{U(d)} \int_{U(d)} |tr(U^*V)|^{2t} dV dU$$



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Consider the fourth summand  $tr(\Sigma^*\Sigma)$ :

$$tr(\Sigma^*\Sigma) = \int_{U(d)} \int_{U(d)} |tr(U^*V)|^{2t} dV dU$$

Let  $f(U) = \int_{U(d)} |tr(U^*V)|^{2t} dV$  be the inner integral.

$$tr(\Sigma^*\Sigma) = \int_{U(d)} f(U)dU = \int_{U(d)} f(I)dU = \int_{U(d)} |tr(V)|^{2t}dV$$

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## Metric definition of unitary *t*-designs

#### Definition

A weight function w is a proper weight function on X if for all other choices of weight function w' on X, we have:

$$\sum_{U,V\in X} w(U)w(V)|tr(U^*V)|^{2t} \leq \sum_{U,V\in X} w'(U)w'(V)|tr(U^*V)|^{2t}.$$

The trace double sum is a function  $\Sigma$  defined for finite  $X \subset U(d)$  as:

$$\Sigma(X) = \sum_{U,V \in X} w(U)w(V)|tr(U^*V)|^{2t},$$

Definition

A finite  $X \subset U(d)$  is a unitary *t*-design if

$$\Sigma(X) = \langle |tr(U)|^{2t} \rangle$$

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#### Symmetries

## Four symmetries of *t*-designs

#### Proposition

If  $X = \{U_1, ..., U_n\}$  is a t-design then  $Y = \{e^{i\phi_1}U_1, ..., e^{i\phi_n}U_n\}$  is also a t-design for all  $\phi_1, ..., \phi_n \in [0, 2\pi]$ .



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#### Symmetries

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If X is a t-design then  $X^* = \{U^* : U \in X\}$  is also a t-design.



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#### Symmetries

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#### Proposition

If X is a t-design then 
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#### Proposition

If  $X \subset U(d)$  is a t-design then  $\forall M \in U(d), MX = \{MU : U \in X\}$  and  $XM = \{UM : U \in X\}$  are also a t-design.

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# Minimal designs

Lemma

If X, Y are two t-designs then so is  $X \cup Y$ .

Designs can be arbitrarily large



Properties of unitary *t*-designs



# Minimal designs

#### Lemma

If X, Y are two t-designs then so is  $X \cup Y$ .

- Designs can be arbitrarily large
- We are interested in smaller designs

#### Definition

A minimal (unweighted) *t*-design X is a *t*-design such that all  $Y \subset X$  are not (unweighted) *t*-designs.

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## Characterization of minimal *t*-designs

Theorem

A t-design X is minimal if and only if it has a unique proper weight function w.



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## Characterization of minimal *t*-designs

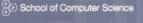
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Useful tool for proving minimality.



Properties of unitary *t*-designs



# Characterization of minimal *t*-designs

Theorem

A t-design X is minimal if and only if it has a unique proper weight function w.

- Useful tool for proving minimality.
- Sadly, minimal designs are not necessarily minimum.
- Currently working on finding correspondences between minimal and minimum designs.



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 $(\Rightarrow)$  Consider the contrapositive: if there are two distinct proper weight functions w and w' on X then X is not a minimal *t*-design.



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#### Minimal designs

# Proof of CMD, Part 1

 $(\Rightarrow)$  Consider the contrapositive: if there are two distinct proper weight functions w and w' on X then X is not a minimal t-design. Define:

$$\alpha = \min_{U \in X} \frac{w'(U)}{w(U)}.$$

Let  $Y = X - \{U \in X : w'(U) - \alpha w(U) = 0\}$ , with weight function

$$w'' = \frac{w' - \alpha w}{1 - \alpha}$$

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#### Minimal designs

# Proof of CMD, Part 1

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Let  $\langle f \rangle_X^w$  be the average of  $f \in \text{Hom}(t, t)$  over X with weight function w

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Let  $\langle f \rangle_X^w$  be the average of  $f \in Hom(t, t)$  over X with weight function w:

$$\langle f \rangle_Y^{w^{\prime\prime}} = \frac{\langle f \rangle_X^{w^\prime} - \alpha \langle f \rangle_X^w}{1 - \alpha} = \frac{\langle f \rangle - \alpha \langle f \rangle}{1 - \alpha} = \langle f \rangle$$

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( $\Leftarrow$ ) Consider a strengthened contrapositive: if (X, w), (Y, w') are *t*-designs such that  $Y \subset X$  then there are infinitely many proper weight functions on X.



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( $\Leftarrow$ ) Consider a strengthened contrapositive: if (X, w), (Y, w') are *t*-designs such that  $Y \subset X$  then there are infinitely many proper weight functions on X.

Assuming that w'(U) = 0 for  $U \notin Y$ , let w'' = pw + (1 - p)w' for any choice of  $p \in (0, 1)$ .



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# Introducing greedy 'algorithms'

Proposition

## A finite $X \subset U(d)$ is a t-design if and only if for all finite $Y \subset U(d)$ , $\Sigma(X) \leq \Sigma(X \cup Y)$



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#### Introduction

# Introducing greedy 'algorithms'

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#### Lemma

For every finite  $X \subset U(d)$  there is some t-design Z such that  $X \subseteq Z$ 





#### Introduction

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#### Lemma

For every finite  $X \subset U(d)$  there is some t-design Z such that  $X \subseteq Z$ 

#### Definition

The contribution of U to X is a function S defined as  $S(U;X) = \sum_{V \in X} w(V) |tr(U^*V)|^{2t}$ 

• 
$$\Sigma(X) = \sum_{U \in X} w(U)S(U; X).$$

• Total amount  $U \in X$  contributes to  $\Sigma(X)$  is  $d^{2t} + 2S(U; X - \{U\})$ .

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1. If X is not a t-design select a  $U \notin X$  that minimizes S(U; X).

- 1. If X is not a t-design select a  $U \notin X$  that minimizes S(U; X).
- 2. Let  $X' = X \cup \{U\}$  with w'(U) = p and for  $V \in X$ w'(V) = (1 - p)w(V)
- **3**. Repeat with  $X \leftarrow X'$  until we have a *t*-design

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3. Repeat with  $X \leftarrow X'$  until we have a *t*-design

If we adjust only p and our choice of U, then the new trace double sum is:

$$\Sigma(X') = (1-p)^2 \Sigma(X) + p^2 d^{2t} + 2p(1-p)S(U;X)$$

Which is minimized by:

$$p = \frac{\Sigma(X) - S(U; X)}{\Sigma(X) - 2S(U; X) + d^{2t}}$$

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Dangers:

- Weight function w' might not be proper weight function on X'.
- Might be able to lower the contribution of S(U; X) at the expense of small increase in  $\Sigma(X)$

• Note that  $S(U; X) \ge 0$  for any U and X.



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- Note that  $S(U; X) \ge 0$  for any U and X.
- If we assume that S(U, X) = 0 at each time step, then the p-adjustment algorithm produces a proper weight function w'.
- Use this observation to find lower bounds.

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# Proposition If $X \subset U(d)$ is a t-design then $|X| \ge \frac{d^{2t}}{\sigma}$ .



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### Proposition

If  $X \subset U(d)$  is a t-design then  $|X| \geq \frac{d^{2t}}{\sigma}$ .

- ▶ Best known bounds are by Roy and Scott [RS08]:  $|X| \ge {d^2+t-1 \choose t}$
- Asymptotically, for large d and fixed t, both bounds are  $\Theta(d^{2t})$

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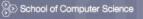
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Consider our algorithm with the best case of  $S(U_k; X_k) = 0$  for every time step k:

$$\Sigma(X_{k+1}) = rac{d^{2t}\Sigma(X_k)}{\Sigma(X_k) + d^{2t}}$$



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Making some changes of variable, we obtain the recurrence x(1) = 1 and:

$$x(k+1) = \frac{x(k)}{x(k)+1}$$

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Recurrence is solved by  $x(k) = \frac{1}{k}$ 

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Consider our algorithm with the best case of  $S(U_k; X_k) = 0$  for every time step k:

$$\Sigma(X_{k+1}) = rac{d^{2t}\Sigma(X_k)}{\Sigma(X_k) + d^{2t}}$$

Making some changes of variable, we obtain the recurrence x(1) = 1 and:

$$x(k+1) = \frac{x(k)}{x(k)+1}$$

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Recurrence is solved by  $x(k) = \frac{1}{k}$ Until  $d^{2t}x(k)$  falls below the value  $\sigma$  we know that there is no possible way to construct a *t*-design X with  $|X| \le k$ .

Properties of unitary t-o

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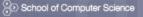
## Limitations of greedy algorithms

Theorem

A p-adjustment greedy algorithm cannot construct an unweighted t-design.



Properties of unitary *t*-designs



## Limitations of greedy algorithms

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### Proposition

For an unweighted  $X \subset U(d)$ , and all elements  $U, V \in X$ ,  $S(U; X) = S(V; X) \ge \sigma$  with equality if and only if X is a t-design.



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#### Lemma

For  $X \subset U(d)$  with proper weight function w, and any pair of elements  $U, V \in X$ , if  $w(U) \ge w(V)$  then  $S(U; X - \{U, V\}) \le S(V; X - \{U, V\})$ .

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# Outline

Introduction

Trace double sum inequality

Symmetries and minimal designs

Greedy algorithms

#### Ouroboros application

#### Conclusion



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# Orthonormal bases for $\mathbb{C}^{d \times d}$

Goal: find an orthonormal basis  $|E_1\rangle,...,|E_{d^2}\rangle$  of  $\mathbb{C}^{d\times d}$  such that each  $E_i\in U(d)$ 

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Definition

 $X \subset U(d)$  is pairwise traceless if for every  $U, V \in X$  with  $U \neq V$  we have  $tr(U^*V) = 0$ . A pairwise traceless  $X \subset U(d)$  is maximum pairwise traceless if  $|X| = d^2$ .

Orthonormal bases of unitaries for  $\mathbb{C}^{d \times d}$  are maximum pairwise traceless sets.



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Orthonormal bases of unitaries for  $\mathbb{C}^{d \times d}$  are maximum pairwise traceless sets.

Proposition

For any  $X \subset U(d)$ , X is maximum pairwise traceless if and only if X is a minimum unweighted 1-design.

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Properties of unitary *t*-designs

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#### Definition

Two orthonormal bases  $\{|e_i\rangle : 1 \le i \le d\}$  and  $\{|e'_i\rangle : 1 \le i \le d\}$  of  $\mathbb{C}^d$  are mutually unbiased if  $|\langle e_i | e'_j \rangle|^2 = \frac{1}{d}$  for all  $1 \le i, j \le d$ .

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- ► Open question: determine the maximum number M(d) of pairwise mutually unbiased bases for C<sup>d</sup>.
- If we write the prime decomposition of  $d = p_1^{n_1} \dots p_k^{n_k}$  such that  $p_i^{n_i} \leq p_{i+1}^{n_{i+1}}$  then  $p_1^{n_1} \leq \mathfrak{M}(d) \leq d+1$ .

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Important features for us:

- $\mathfrak{M}(d) \geq 2$  for  $d \geq 1$ .
- Without loss of generality, can assume one of the bases to be the standard basis.

Example

$$\left\{ \begin{pmatrix} 1\\0 \end{pmatrix}, \begin{pmatrix} 0\\1 \end{pmatrix} \right\}, \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\-1 \end{pmatrix} \right\}, \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\+i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1\\-i \end{pmatrix} \right\}$$

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Maximum pairwise traceless set construction

- Let  $|e_1\rangle ... |e_d\rangle$  be an orthonormal basis of  $\mathbb{C}^d$  that is mutually unbiased with the standard basis.
- Define  $I_i = \sqrt{d} \operatorname{diag}(|e_i\rangle)$  for  $1 \le i \le d$ .



Properties of unitary *t*-designs



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- ► Consider the cyclic permutation group of order *d*, represented as *d*-by-*d* matrices: C<sup>1</sup>...C<sup>d</sup> where C<sup>d</sup> = C<sup>0</sup> = I.
- Define  $C_i^m = C^m I_i$



Properties of unitary *t*-designs

School of Computer Science October 7, 2009 24 / 28 Maximum pairwise traceless set construction

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• Define 
$$C_i^m = C^m I_i$$

For any tuple  $1 \leq i, j, m, n \leq d$  we have:

$$tr((C_i^m)^*C_j^n) = tr(I_i^*C^{d-m+n}I_j) = \begin{cases} d & \text{if } i = j \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

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Evaluating the average commutator over U(d)

### Theorem

For any  $V \in U(d)$  and  $[U, V] = U^*V^*UV$  we have:

$$\langle [\cdot, V] \rangle = \frac{tr(V^*)}{d}V$$

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#### Evaluating $\langle [ \cdot , V ] \rangle$

# Proof of EAC

Consider the diagonalization of  $V^*$ , i.e.  $V^* = P^*DP$ , with  $D = \text{diag}(\lambda_1, ..., \lambda_d)$ .

$$\int_{U(d)} U^* V^* U V \ dU = \left[ \int_{U(d)} U^* V^* U \ dU \right] V = \left[ \int_{U(d)} U^* P^* D P U \ dU \right] V$$

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But we know a symmetry that allows substituting  $PU \rightarrow U$  without changing the average.

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• Let  $f(U) = U^* D U$ .

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• Let  $f(U) = U^*DU$ .

• Look at the elements of the design:  $f(C_i^m) = I_i^*(C^m)^* DC^m I_i$ .

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Thus,  $\langle f 
angle = (\lambda_1 + ... + \lambda_d) I$ 

### t-designs are non-commuting

### Definition

 $X \subset U(d)$  is a *non-commuting* if there is some  $U, V \in X$  such that  $[U, V] \neq I$ .

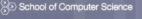
### Theorem

For all  $d \ge 2$  if  $X \subset U(d)$  is a t-design then X is non-commuting.

Supports our intuition that designs must be well 'spread out'.



Properties of unitary *t*-designs



## Outline

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Trace double sum inequality

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### Concluding remarks

- Introduces 3 definitions of unitary t-designs
- Proved the trace double sum inequality: Σ(X) ≥ ⟨|tr(U)|<sup>2t</sup>⟩ with equality if and if X is a t-design



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- Proved the trace double sum inequality: Σ(X) ≥ ⟨|tr(U)|<sup>2t</sup>⟩ with equality if and if X is a t-design
- Discussed symmetries of designs: phase,  $X^*$ , MX, and XM.
- Classified minimal designs: a t-design is minimal if and only if it has a unique proper weight function.

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Thank you for listening!

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