Properties of unitary t-designs

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Outline

Introduction

Trace double sum inequality

Symmetries and minimal designs

Greedy algorithms

Ouroboros application

Conclusion

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Preliminaries: $U(d)$

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- We can introduce the Haar measure and use it to integrate functions f of $U \in U(d)$ to find their averages:

$$\langle f \rangle = \int_{U(d)} f(U)\, dU.$$

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- For convenience we normalize integration by assuming that $\int_{U(d)} dU = 1$.
- The goal of unitary t-designs is to evaluate averages of polynomials via a finite sum.

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$$U, V \mapsto U^* V^* UV \in Hom(2, 2)$$

$$U \mapsto U^* V^* UV \in Hom(1, 1)$$

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Examples

$$\begin{aligned} U, V &\mapsto U^* V^* UV && \in Hom(2,2) \\ U &\mapsto U^* V^* UV && \in Hom(1,1) \\ U &\mapsto \frac{tr(U^* U)}{d} && \in Hom(1,1) \end{aligned}$$

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$$
\begin{array}{rcll}
U, V & \mapsto & U^* V^* U V & \in Hom(2,2) \\
U & \mapsto & U^* V^* U V & \in Hom(1,1) \\
U & \mapsto & \dfrac{tr(U^* U)}{d} & \in Hom(1,1) \\
U, V & \mapsto & tr(U^* V) U^2 + V U^* V U & \in Hom(3,1) \\
U & \mapsto & \underbrace{tr(U^* V) U^2}_{Hom(2,1)} + \underbrace{V U^* V U}_{Hom(1,1)} & \notin Hom(2,1)
\end{array}
$$

Functional definition of unitary t-designs

Definition

A function $w : X \to (0,1]$ is a weight function on X if for all $U \in X$ we have $w(U) > 0$ and $\sum_{U \in X} w(U) = 1$

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A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

$$\sum_{U \in X} w(U) f(U) = \int_{U(d)} f(U)\, dU$$

for all $f \in \mathrm{Hom}(t,t)$.

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Definition

A finite $X \subset U(d)$ is an unweighted t-design if it is a unitary t-design with a uniform weight function (i.e. $w(U) = \frac{1}{|X|}$ for all $U \in X$).

Functional definition is general enough

Proposition

Every t-design is a $(t-1)$-design.

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Proposition

For any $f \in \mathrm{Hom}(r, s)$ with $r \neq s$

$$\int_{U(d)} f(U)\, dU = 0$$

Lemma

For any $f \in \mathrm{Hom}(r, s)$, $U \in U(d)$, and $c \in \mathbb{C}$ we have $f(cU) = c^r \bar{c}^s f(U)$

Strengths and shortcomings of the functional definition

Strengths:

- Average of any polynomial with degrees in U and U^* less than t can be evaluated one summand at a time.
- Multi-variable polynomials can be evaluated:

$$\int \cdots \int_{U(d)} f(U_1, ..., U_n) dU_1 ... dU_n$$

$$= \sum_{U_1 \in X} ... \sum_{U_n \in X} w(U_1) ... w(U_n) f(U_1, ..., U_n).$$

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$$= \sum_{U_1 \in X} ... \sum_{U_n \in X} w(U_1)...w(U_n) f(U_1, ..., U_n).$$

Shortcomings:

- Not clear how to test if a given (X, w) is a t-design.
- If (X, w) is not a design, then how far away is it?

Tensor product definition of unitary t-designs

Definition

A tuple (X,w) with finite $X \subset U(d)$ and weight function w on X is a unitary t-design if

$$\sum_{U \in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t} = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU$$

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- More tractable for checking if an arbitrary (X, w) is a t-design.
- Literature has explicit formula for the RHS for many choices of d and t [Col03, CS06].
- Still not metric.

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Theorem

For all finite $X \subset U(d)$ we have

$$\sum_{U,V \in X} w(U)w(V)|tr(U^*V)|^{2t} \geq \int_{U(d)} |tr(U)|^{2t}\, dU$$

With equality if and only if X is a t-design.

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With equality if and only if X is a t-design.

- Proved earlier by Scott [Sco08].
- RHS is the number of permutations of $\{1, ..., t\}$ with no increasing subsequences of order greater than d [DS94, Rai98].
- If $d \geq t$ then RHS is $t!$.
- We will call the RHS σ.

Proof of TDSI, Part 1

Consider an arbitrary finite $X \subset U(d)$ with a weight function w, define matrices S and Σ as:

$$S = \sum_{U \in X} w(U) U^{\otimes t} \otimes (U^*)^{\otimes t}$$

$$\Sigma = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} dU$$

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Consider the matrix $D = S - \Sigma$:

$$\begin{aligned} tr(D^* D) &= tr((S^* - \Sigma^*)(S - \Sigma)) \\ &= tr(S^* S) - tr(\Sigma^* S) - tr(S^* \Sigma) + tr(\Sigma^* \Sigma) \end{aligned}$$

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Trace is linear, thus can be brought past the integrals, summations and weights.

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$$\begin{aligned} tr((U^{\otimes t} \otimes (U^*)^{\otimes t})^*(V^{\otimes t} \otimes (V^*)^{\otimes t})) \\ = tr((U^*V)^{\otimes t} \otimes (UV^*)^{\otimes t}) \\ = tr(U^*V)^t tr(UV^*)^t \\ = |tr(U^*V)|^{2t} \end{aligned}$$

Proof of TDSI, Part 3

Consider the fourth summand $tr(\Sigma^*\Sigma)$:

$$tr(\Sigma^*\Sigma) = \int_{U(d)} \int_{U(d)} |tr(U^*V)|^{2t} dVdU$$

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Let $f(U) = \int_{U(d)} |tr(U^*V)|^{2t} dV$ be the inner integral.

$$tr(\Sigma^*\Sigma) = \int_{U(d)} f(U) dU = \int_{U(d)} f(I) dU = \int_{U(d)} |tr(V)|^{2t} dV$$

$\square$

Metric definition of unitary t-designs

Definition

A weight function w is a proper weight function on X if for all other choices of weight function w' on X, we have:

$$\sum_{U,V\in X} w(U)w(V)|tr(U^*V)|^{2t} \leq \sum_{U,V\in X} w'(U)w'(V)|tr(U^*V)|^{2t}.$$

The trace double sum is a function Σ defined for finite $X \subset U(d)$ as:

$$\Sigma(X) = \sum_{U,V\in X} w(U)w(V)|tr(U^*V)|^{2t},$$

Definition

A finite $X \subset U(d)$ is a unitary t-design if

$$\Sigma(X) = \langle |tr(U)|^{2t} \rangle$$

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Four symmetries of t-designs

Proposition

If $X = \{U_1, ..., U_n\}$ is a t-design then $Y = \{e^{i\phi_1}U_1, ..., e^{i\phi_n}U_n\}$ is also a t-design for all $\phi_1, ..., \phi_n \in [0, 2\pi]$.

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Proposition

If $X \subset U(d)$ is a t-design then $\forall M \in U(d)$, $MX = \{MU : U \in X\}$ and $XM = \{UM : U \in X\}$ are also a t-design.

Minimal designs

Lemma

If X, Y are two t-designs then so is $X \cup Y$.

- Designs can be arbitrarily large

Minimal designs

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If X, Y are two t-designs then so is $X \cup Y$.

- Designs can be arbitrarily large
- We are interested in smaller designs

Definition

A minimal (unweighted) t-design X is a t-design such that all $Y \subset X$ are not (unweighted) t-designs.

Characterization of minimal t-designs

Theorem

A t-design X is minimal if and only if it has a unique proper weight function w.

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- Useful tool for proving minimality.
- Sadly, minimal designs are not necessarily minimum.
- Currently working on finding correspondences between minimal and minimum designs.

Proof of CMD, Part 1

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$$\alpha = \min_{U \in X} \frac{w'(U)}{w(U)}.$$

Let $Y = X - \{U \in X : w'(U) - \alpha w(U) = 0\}$, with weight function

$$w'' = \frac{w' - \alpha w}{1 - \alpha}$$

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Let $\langle f \rangle_X^w$ be the average of $f \in \mathrm{Hom}(t, t)$ over X with weight function w:

$$\langle f \rangle_Y^{w''} = \frac{\langle f \rangle_X^{w'} - \alpha \langle f \rangle_X^w}{1 - \alpha} = \frac{\langle f \rangle - \alpha \langle f \rangle}{1 - \alpha} = \langle f \rangle$$

Proof of CMD, Part 2

($\Leftarrow$) Consider a strengthened contrapositive: if $(X, w), (Y, w')$ are t-designs such that $Y \subset X$ then there are infinitely many proper weight functions on X.

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Assuming that $w'(U) = 0$ for $U \notin Y$, let $w'' = pw + (1 - p)w'$ for any choice of $p \in (0, 1)$. $\square$

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Introducing greedy 'algorithms'

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Definition

The contribution of U to X is a function S defined as $S(U; X) = \sum_{V \in X} w(V)|tr(U^* V)|^{2t}$

- $\Sigma(X) = \sum_{U \in X} w(U) S(U; X)$.
- Total amount $U \in X$ contributes to $\Sigma(X)$ is $d^{2t} + 2S(U; X - \{U\})$.

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2. Let $X' = X \cup \{U\}$ with $w'(U) = p$ and for $V \in X$ $w'(V) = (1-p)w(V)$
3. Repeat with $X \leftarrow X'$ until we have a t-design

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If we adjust only p and our choice of U, then the new trace double sum is:

$$\Sigma(X') = (1-p)^2\Sigma(X) + p^2 d^{2t} + 2p(1-p)S(U; X)$$

Which is minimized by:

$$p = \frac{\Sigma(X) - S(U; X)}{\Sigma(X) - 2S(U; X) + d^{2t}}$$

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Dangers:

- Weight function w' might not be proper weight function on X'.
- Might be able to lower the contribution of $S(U; X)$ at the expense of small increase in $\Sigma(X)$

A lower bound on the size of t-designs

- Note that $S(U; X) \geq 0$ for any U and X.

A lower bound on the size of t-designs

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- If we assume that $S(U,X) = 0$ at each time step, then the p-adjustment algorithm produces a proper weight function w'.
- Use this observation to find lower bounds.

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If $X \subset U(d)$ is a t-design then $|X| \geq \frac{d^{2t}}{\sigma}$.

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- Best known bounds are by Roy and Scott [RS08]: $|X| \geq \binom{d^2+t-1}{t}$
- Asymptotically, for large d and fixed t, both bounds are $\Theta(d^{2t})$

Proof of LB

Consider our algorithm with the best case of $S(U_k; X_k) = 0$ for every time step k:

$$\Sigma(X_{k+1}) = \frac{d^{2t}\Sigma(X_k)}{\Sigma(X_k) + d^{2t}}$$

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Making some changes of variable, we obtain the recurrence $x(1) = 1$ and:

$$x(k+1) = \frac{x(k)}{x(k)+1}$$

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Recurrence is solved by $x(k) = \frac{1}{k}$
Until $d^{2t}x(k)$ falls below the value σ we know that there is no possible way to construct a t-design X with $|X| \leq k$. $\square$

Limitations of greedy algorithms

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For an unweighted $X \subset U(d)$*, and all elements* $U, V \in X$*,*
$S(U;X) = S(V;X) \geq \sigma$ *with equality if and only if* X *is a t-design.*

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Lemma

For $X \subset U(d)$ *with proper weight function* w*, and any pair of elements* $U, V \in X$*, if* $w(U) \geq w(V)$ *then* $S(U; X - \{U, V\}) \leq S(V; X - \{U, V\})$*.*

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Orthonormal bases for $\mathbb{C}^{d\times d}$

Goal: find an orthonormal basis $|E_1\rangle, \ldots, |E_{d^2}\rangle$ of $\mathbb{C}^{d\times d}$ such that each $E_i \in U(d)$

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Definition

$X \subset U(d)$ is pairwise traceless if for every $U, V \in X$ with $U \neq V$ we have $tr(U^* V) = 0$.
A pairwise traceless $X \subset U(d)$ is maximum pairwise traceless if $|X| = d^2$.

Orthonormal bases of unitaries for $\mathbb{C}^{d\times d}$ are maximum pairwise traceless sets.

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Orthonormal bases of unitaries for $\mathbb{C}^{d\times d}$ are maximum pairwise traceless sets.

Proposition

For any $X \subset U(d)$, X is maximum pairwise traceless if and only if X is a minimum unweighted 1-design.

Very brief introduction to MUBs

Definition

Two orthonormal bases $\{|e_i\rangle : 1 \leq i \leq d\}$ and $\{|e'_i\rangle : 1 \leq i \leq d\}$ of $\mathbb{C}^d$ are mutually unbiased if $|\langle e_i|e'_j\rangle|^2 = \frac{1}{d}$ for all $1 \leq i, j \leq d$.

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- Open question: determine the maximum number $\mathfrak{M}(d)$ of pairwise mutually unbiased bases for $\mathbb{C}^d$.

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- Open question: determine the maximum number $\mathfrak{M}(d)$ of pairwise mutually unbiased bases for $\mathbb{C}^d$.
- If we write the prime decomposition of $d = p_1^{n_1} \ldots p_k^{n_k}$ such that $p_i^{n_i} \leq p_{i+1}^{n_{i+1}}$ then $p_1^{n_1} \leq \mathfrak{M}(d) \leq d + 1$.

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- Open question: determine the maximum number $\mathfrak{M}(d)$ of pairwise mutually unbiased bases for $\mathbb{C}^d$.
- If we write the prime decomposition of $d = p_1^{n_1} \ldots p_k^{n_k}$ such that $p_i^{n_i} \leq p_{i+1}^{n_{i+1}}$ then $p_1^{n_1} \leq \mathfrak{M}(d) \leq d + 1$.

Important features for us:

- $\mathfrak{M}(d) \geq 2$ for $d \geq 1$.
- Without loss of generality, can assume one of the bases to be the standard basis.

Example

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}, \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ +i \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$$

Maximum pairwise traceless set construction

- Let $|e_1\rangle ... |e_d\rangle$ be an orthonormal basis of $\mathbb{C}^d$ that is mutually unbiased with the standard basis.
- Define $I_i = \sqrt{d}\mathrm{diag}(|e_i\rangle)$ for $1 \leq i \leq d$.

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For any tuple $1 \leq i, j, m, n \leq d$ we have:

$$tr((C_i^m)^* C_j^n) = tr(I_i^* C^{d-m+n} I_j) = \begin{cases} d & \text{if } i = j \text{ and } m = n \\ 0 & \text{otherwise} \end{cases}$$

Evaluating the average commutator over $U(d)$

Theorem

For any $V \in U(d)$ and $[U,V] = U^* V^* U V$ we have:

$$\langle[\,\cdot\,,V]\rangle = \frac{tr(V^*)}{d} V$$

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^* D P$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^* V^* U V \, dU = \Big[\int_{U(d)} U^* V^* U \, dU \Big] V = \Big[\int_{U(d)} U^* P^* D P U \, dU \Big] V$$

Proof of EAC

Consider the diagonalization of V^*, i.e. $V^* = P^*DP$, with $D = \mathrm{diag}(\lambda_1, ..., \lambda_d)$.

$$\int_{U(d)} U^*V^*UV\, dU = \Big[\int_{U(d)} U^*V^*U\, dU\Big]V = \Big[\int_{U(d)} U^*P^*DPU\, dU\Big]V$$

But we know a symmetry that allows substituting $PU \to U$ without changing the average.

$$\int_{U(d)} U^*P^*DPU\, dU = \int_{U(d)} U^*DU\, dU$$

- Let $f(U) = U^*DU$.

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- Let $f(U) = U^*DU$.
- Look at the elements of the design: $f(C_i^m) = I_i^*(C^m)^*DC^mI_i$.
- $(C^m)^*DC^m = \mathrm{diag}(\lambda_{c^m(1)}, ..., \lambda_{c^m(d)})$

Thus, $\langle f \rangle = (\lambda_1 + ... + \lambda_d)I$ $\square$

t-designs are non-commuting

Definition

$X \subset U(d)$ is a *non-commuting* if there is some $U, V \in X$ such that $[U, V] \neq I$.

Theorem

For all $d \geq 2$ if $X \subset U(d)$ is a t-design then X is non-commuting.

Supports our intuition that designs must be well 'spread out'.

Outline

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Trace double sum inequality

Symmetries and minimal designs

Greedy algorithms

Ouroboros application

Conclusion

Concluding remarks

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- Proved the trace double sum inequality: $\Sigma(X) \geq \langle |tr(U)|^{2t} \rangle$ with equality if and if X is a t-design

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- Discussed symmetries of designs: phase, X^*, MX, and XM.
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Thank you for listening!

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